1. $I \subset S = k[x_0, \dots, x_n]$ homog.

$f \in S$ non-constant

$f(P) = 0 \quad \forall\, P \in Z^n(I)$.

Show: $f^q \in I$ for some $q > 0$.

homog. radical ideals $\neq S_+$ $\longleftrightarrow$ closed subsets of $\mathbb{P}^n$

$\sqrt{I} \longleftrightarrow Z^n(I)$

$$f \in \underset{\sqrt{I}}{\underset{\|}{I(Z^n(I))}} \qquad \square$$

comes from Nullstellensatz: $I(Z(I)) = \sqrt{I}$
for $I \subset k[x_1, \dots, x_n]$

2. $Z \subset \mathbb{P}^n(k)$ closed, every cpt of dimension $n-1$.
Show: $Z = Z^n(f)$ for some $f \in S$ homog.

error in problem

$$\begin{array}{ccc} \mathbb{A}^{n+1} \setminus 0 & \xrightarrow{\pi} & \mathbb{P}^n \\ \cup & & \cup \\ \pi^{-1}(Z) & & Z \end{array}$$

$\dim \pi^{-1}(Z) = n$ (justify?) ✓

$\pi^{-1}(Z)$ $\qquad$ Z $\qquad$ $\pi^{-1}(Z) = \pi^{-1}(Z) \cup \{0\}$

$\overline{\pi^{-1}(Z)} \subset \mathbb{A}^{n+1}$

// LECTURE

$Z(f)$

Then f is k^*-invariant

$\therefore$ homog.

Claim: $Z^n(f) = Z$

So $\{\text{closed subsets of } \mathbb{P}^n\} \hookrightarrow \{k^*\text{-inv. closed subsets in } \mathbb{A}^{n+1}\ni 0\}$

Alternatively: $Z \subset \mathbb{P}^n$ assume irred. $\longleftrightarrow$ homog. prime ideal I of S

$\operatorname{ht} I = \operatorname{codim} Z = 1$

S UFD $\Rightarrow$ I principal

$I = (f)$

$\ldots\ Z = Z(f)\ \ldots$

In general $Z = Z_1 \cup \ldots \cup Z_r$, Z_i irred,

$Z_i = Z^n(f_i)$

$Z = Z^n(f_1 \cdots f_r)$.

3. $I \lhd S$ homog. & radical

Show: $\underbrace{Z^n(I) \cap \mathbb{A}^n}_{?}$ $\leftarrow$ one of std. emb. into $\mathbb{P}^n$

has coordinate ring $\left(S/I\,[x_0^{-1}]\right)_0$.

viewed as closed subset of $\mathbb{A}^n$

NB: coherent sol^{n}s of Ex. 3 & 4 at end.

$$Z = Z^n(I) \subset \mathbb{P}^n \xleftarrow{\pi} \mathbb{A}^{n+1}\setminus 0$$

$$\overline{\pi^{-1}(Z)} = Z(I)$$

$$k[Z(I)] = S/I$$

$\mathbb{A}^n = \{(x_0:\dots:x_n) \mid x_0 \neq 0\} \subset \mathbb{P}^n$

$$Z(I)\setminus\{x_0=0\} \subset \mathbb{A}^{n+1}$$

not closed

$$\left(\subset \mathbb{A}^{n+2} \quad x_0 x_{n+1} = 1\right)$$

$$k\left[Z(I)\setminus\{x_0=0\}\right] = S/I\,[x_0^{-1}]$$

$$\downarrow \pi \qquad\qquad \Updownarrow ?$$

$$Z^n(I)\cap\mathbb{A}^n \qquad \left(S/I\,[x_0^{-1}]\right)_0$$

Claim: $Z(I)\setminus\{x_0=0\} = \pi^{-1}\left(\mathbb{A}^n \cap Z^n(I)\right)$

$\mathbb{A}^n$ = open complement of $\mathbb{P}^{n-1} \subset \mathbb{P}^n$

$\{x_0=0\}$

$$\pi^{-1}(\mathbb{A}^n) = \{x_0 \neq 0\}$$

$\therefore$ claim is proved

$$\pi^{-1}(Z^n(I)) \cup \{0\} = Z(I)$$

$$\pi^{-1}(\underset{\{x_0=0\}}{\mathbb{P}^{n-1}}) \cup \{0\} = \underset{\{x_0=0\}}{\mathbb{A}^n}$$

$$\mathbb{P}^{n-1} = \{(x_0 : \dots : x_n) \mid x_0 = 0\}$$

$$\pi^{-1}(\underbrace{Z^n(I) \setminus \mathbb{P}^{n-1}}) = \pi^{-1}(Z^n(I)) \setminus \pi^{-1}(\mathbb{P}^{n-1})$$

$$Z^n(I) \cap \mathbb{A}^n = Z(I) \setminus \{x_0 = 0\}$$

Sidetrack: what is $\left(S/I\,[x_0^{-1}]\right)_0$

$$= \left(\frac{S[x_0^{-1}]}{I[x_0^{-1}]}\right)_0 = \frac{S[x_0^{-1}]_0}{I[x_0^{-1}]_0} \overset{?}{=} \frac{k[x_1, \dots, x_n]}{I(\dots)}$$

$$S[x_0^{-1}]_0 \xrightarrow[\varphi]{x_0 \mapsto 1} k[x_1, \dots, x_1]$$

typical elt. of S homog. of deg d

$$x_1^d + a x_0 x_1 x_2^{d-2} + \dots$$

typical elt of $S[x_0^{-1}]$

$$x_1^d + a x_1 x_2^{d-1} \ \{\text{deg. } d-e$$

$$\overline{\quad x_0 e \quad}\ )$$

Define $\psi : k[x_1, \dots, x_n] \longrightarrow S[x_0^{-1}]_0$

$$f \text{ of deg. } d \longmapsto f x_0^{-d}$$

Check that ψ is inverse to φ.

Need to show: $I(\widehat{Z^h(I) \cap A^n}) \subset k[x_1, \dots, x_n]$

$$\| \qquad\qquad \text{IIS}\ \psi$$

$$(I[x_0^{-1}])_0 \qquad (S[x_0^{-1}])_0$$

$$J = \varphi(I) = \langle f(1, x_1, \dots, x_n) \mid f \in I \rangle$$

Q: Is $J = I(X)$?

C: $g = f(1, x_1, \dots, x_n) \in J$

$x \in X \quad g(x) = 0$

$\| $

$Z^h(I) \cap A^n \qquad g(x) = f(x) = 0$

$x = (1 ; x_1 ; \dots ; x_n)$

$\therefore J \subset I(X)$

$J \subset \overline{I(X)}$

$Z(J) \qquad I(\overline{Z(J)})$

$\hat{A}^n$ $= \langle$ homogenization of $g \mid g \in J \rangle$

$= \langle$ homogenizations of $f(1, x_1, \ldots, x_n) \mid f \in I \rangle$

homogenization of $f(1, x_1, \ldots, x_n) \cdot X_0^{?} = f$

Suppose $I = \langle f_1, \ldots, f_r \rangle$ and no f_i is divisible by X_0

then $I(\overline{Z(J)}) = I$

$Z(J) \supset Z(I) \cap A^n$

$\overline{Z(J)} = \overline{Z^n(I) \cap A^n} = Z^n(I)$

$\therefore J = I$.

$$Z \subset \mathbb{P}^n \longrightarrow Z \cap A^n \subset A^n$$

$$\overline{Z \cap A^n} \subset \mathbb{P}^n \longleftarrow$$

not id

$Z = Z_1 \cup Z_2$ $\quad Z_2 \subset \mathbb{P}^{n-1}$

Z_1 has no cpt $\subset \mathbb{P}^{n-1}$

$Z_2 \cap A^n = \emptyset$

$\overline{Z_1 \cap A^n} = Z_1$

$\therefore \overline{Z \cap A^n}$ = only cpts of Z meeting but contained in $\mathbb{P}^{n-1}$

Easier way to finish this pf:

$$f \in I(Z^n(I) \cap \mathbb{A}^n)$$

$\Rightarrow x_0 \cdot f$ vanishes on $Z^n(I)$

$\Rightarrow f \in J$, i.e. $I(Z^n(I) \cap \mathbb{A}^n) \subset J$.

4. $S_+ \neq I \subset S$ homog. radical. Show: $\dim Z^n(I) + 1 = \dim Z(I)$.

$\dim Z^n(I) = k \leq n$

$Z_0 \subset Z_1 \subset \dots \subset Z_k \subset Z^n(I)$

closed irred. pieces

pull back to $\mathbb{A}^{n+1} \setminus 0$

$\{x\} \subset \pi^{-1}(Z_0) \subset \dots \subset \pi^{-1}(Z_k)$

$\therefore \dim Z(I) \geq k+1$

$$\sup \dim \mathbb{A}^n \cap Z^n(I) = \dim Z^n(I)$$

$$\dim \left(S/I[x_0^{-1}] \right)_0 + 1 = \dim S/I[x_0^{-1}]$$

$S/I[x_0^{-1}] / x_0 = 1$ $\to$ prev. problem

$(t, tx_1, \dots, tx_n) \mapsto Z(I) \setminus \{x_0 = 0\}$

$\cong \mathbb{G}_m \times [\{x_0 = 1\} \cap Z(I)]$

$\downarrow$

$(t, (1, x_1, \dots, x_n))$ $\quad Z^n(I) \cap \mathbb{A}^n$

$Z(I) \cap \{x_0 = 1\}$

$$\begin{aligned}\therefore \dim Z(I) \setminus \{x_0 = 0\} \\ = \dim \left(G_m \times \{x_0 = 1\} \cap Z(I)\right) \\ = 1 + \dim \left(\left(S/_I [x_0^{-1}]\right)_0\right)\end{aligned}$$

$$\downarrow \cong$$

$$Z^n(I) \cap \mathbb{A}^n$$

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3. $$\left(S/_I [x_0^{-1}]\right)_0 = \frac{S[x_0^{-1}]_0}{I[x_0^{-1}]_0}$$

$$S[x_0^{-1}]_0 \cong k[x_1, \ldots, x_n]$$

$$f \longmapsto f(1, x_1, \ldots, x_n)$$

$$\frac{g(x_1, \ldots, x_n)}{x_0^d} \longleftarrow g \text{ homog. of deg. } d.$$

Under this correspondence $I \mapsto (f(1, x_1, \ldots, x_n) \mid f \in I) =: I'$.

Consider $S \xrightarrow{\psi} S/_{x_0 - 1} \cong k[x_1, \ldots, x_n]$.

Then $\psi(I) = I'$.

$\therefore \left(S/_I [x_0^{-1}]\right)_0$ is the coordinate ring of $Z(I) \cap \{X_0 = 1\} =: Z'$.

But Z' is "the same" as $Z^n(I) \cap \mathbb{A}^n$:

$$\begin{array}{ccccc} Z^n \cap \mathbb{A}^n & \subset & \mathbb{A}^n & \subset & \mathbb{P}^n \\ \updownarrow & & \wr\wr & & \\ Z' & \subset & \{x_0 = 1\} & \subset & \mathbb{A}^{n+1} \end{array}$$

these correspond

$\square$

4. $\dim(Z^n(I) \cap \mathbb{A}^n) = \dim(Z')$, $Z := Z(I) \subset \mathbb{A}^{n+1}$

Claim: $\dim Z' = \dim(Z \setminus \{x_0 = 0\}) - 1$

Assuming this, result follows since $\dim Z^n(I)$

$$= \sup_i (\dim Z^n(I) \cap U_i),$$

$U_0, \ldots, U_n \subset \mathbb{P}^n$ std. cover by affine opens

prev. tutorial

$$= \sup_i \dim(Z \setminus \{x_i = 0\}) - 1$$

$$= \dim Z - 1$$

To prove the claim, consider

$$\mathbb{G}_m \times Z' \longrightarrow Z \setminus \{x_0 = 0\}$$

$$(t, (1, x_1, \ldots, x_n)) \longmapsto (t, tx_1, \ldots, tx_n).$$

Check this is a homeom^m.

Now $\dim(\mathbb{G}_m \times Z') = \dim k[Z'][T, T^{-1}]$

$$= \text{tr. deg. } k(Z')(T)$$

$$= \text{tr. deg. } k(Z') + 1$$

$$= \dim Z' + 1.$$

$\square$