1. $Z = Z(X^2 - YZ, XZ - X) \subset \mathbb{A}^3(k)$.

- What are the irred. cpts?
- What are their prime ideals?

$a, b, c \in Z$ then $0 = ac - a = a(c-1)$

either $a = 0$ or $c = 1$

$bc = 0$ i.e. $b = 0$ or $c = 0$ | $a^2 = b$

$Z_1 = \{a = 0, b = 0\}$

$Z_2 = \{a = 0, c = 0\}$

$Z_3 = \{c = 1, b = a^2\}$

$Z = Z_1 \cup Z_2 \cup Z_3$

show they are prime: $k[X,Y,Z]/_{X,Y} \cong k[Z]$ integ. domain ✓

$k[X,Y,Z]/_{X,Z} \cong k[Y]$ integ. dom. ✓

$$\frac{k[X,Y,Z]}{(Z-1, Y-X^2)} \xrightarrow{\cong} k[T] \quad ✓$$

$X \longmapsto T$

$Y \longmapsto T^2$

$Z \longmapsto 1$

2. (1) $Z \subset \mathbb{A}^n(k)$ closed, irred., $H \subset \mathbb{A}^n(k)$ a hypersurf.

Show: every irred. cpt of $Z \cap H$ has dimension $\dim Z - 1$.

(2) $\dim Z(f_1, \dots, f_r) \geq n - r$.

$\underset{\mathbb{A}^n(k)}{\cap}$

(1) Say $H = Z(f)$ $\quad f \in k[x_1, \ldots, x_n]$

$Z = Z(I)$ $\quad I$

$Z \subsetneq H, \quad f \notin I.$

Claim: $ht(I + (f)) = ht(I) + 1.$

If f irred.: $\quad I \supset P_0 \supset \cdots \supset P_r \quad P_r : ht\, I$

$P_{-1} = P_0 + (f)$ prime? $\Big\}$ later

$\dim Z = \dim \mathbb{A}^n - ht\, I$

$\dim Z \cap Z(f) = \dim \mathbb{A}^n - ht\,(I + (f))$ $\Big\}$ $\therefore$ done.

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try 2: $A = k[x_1, \ldots, x_n]/I$

$fA = (f) + I \qquad f \in A$ not a zero-divisor

Principal ideal theorem: $fA \subset P \quad P$ prime ideal, min. among those of f

then $ht\, P \leq 1$. if f is not a zero-div $<$

irred. cpts of $Z(f) \cap Z \longleftrightarrow$ minimal primes

$\therefore$ dim of every cpt is $\dim Z - 1$

$\dim Z$

Recall: if X irred. $Y \subset X$ closed

$\dim X = \dim Y < \infty$

then $X = Y$

$\therefore \dim Z \cap Z(f) = \dim Z \Rightarrow Z = Z \cap Z(f)$

$\Rightarrow Z \subset Z(f)$. $\times$.

(2) $I = (f_1, \dots, f_r)$ $\quad f_i \in k[x_1, \dots, x_n]$

$Z(f_1) = Z(\pi_1) \cup \dots \cup Z(\pi_s)$ $\quad \pi_i$ the prime facts of f_1

idea works

$Z(f_1, f_2) = Z(f_1) \cap Z(f_2)$

either: $Z(f_2) \supset Z(f_1) \rightsquigarrow$ dim stays $n-1$

or: not $\overset{(1)}{\rightsquigarrow}$ dim $= n-2$

Claim: $Z \subset \mathbb{A}^n(k)$ closed, every irred. cpt. has dim $\geq n-r$.

Then every irred. cpt. of $Z \cap Z(f)$ has dim $\geq n-r-1$.

Proof: $Z = Z_1 \cup \dots \cup Z_s$ irred. cpts.

either: $Z_i \subset Z(f)$, then $\dim(Z_i \cap Z(f)) = \dim(Z_i) \geq n-r$

or: $Z_i \not\subset Z(f)$, then $\geq n-r-1$.

for every cpt C of $Z_i \cap Z(f)$ have by (1) that $\dim C = \dim Z_i - 1 \geq n-r-1$.

Irred. cpts of $Z \cap Z(f)$ are irred cpts of $\{Z_i \cap Z(f) \mid i\}$. $\square$

Claim: every cpt. of $Z(f_1, \dots, f_r)$ has dim $\geq n-r$.

Proof: ind^n on r. $r=0$: $Z(\emptyset) = \mathbb{A}^n$ ✓

$$Z(f_1, \ldots, f_r) = \underbrace{Z(f_1, \ldots, f_{r-1})}_{\substack{\text{every cpt. has dim} \\ \geq n-r+1 \text{ by ind}^n}} \cap Z(f_r)$$

$\therefore$ by claim 1, Claim 2 follows.

3. $Y \subset \mathbb{A}^n(k)$ closed. TFAE:

(1) Y is $k^\times$-invariant

(2) $f \in I(Y) \Rightarrow f_t \in I(Y)$, $f_t(x) := f(tx)$
$t \in k^\times$

(3) $I(Y)$ is homogeneous

(3) $\Rightarrow$ (1) $\quad I = (f_1, \ldots, f_r)$ $\quad f_i$ homogeneous of degree d_i.

Let $y \in Y$, $t \in k^\times$.

Need to show: $ty \in Y$.

i.e.: if $f \in I(Y)$ then $f(t \cdot y) = 0$.

If f is homogeneous of degree d:

then $f(ty) = t^d \cdot f(y) = 0$.

In general $f = \sum_i g_i \cdot \underbrace{f_i}_{\text{homog. gen.}}$ $\quad$ i.e. $I = \langle f_i \rangle_i$

$$f(ty) = \sum_i g_i(ty) \cdot \underbrace{f_i(ty)}_{0}$$

$\therefore ty \in Z(I(Y)) = Y$

$(1) \Rightarrow (2)$: Y is $k^\times$-invariant

$$f \in I(Y) \qquad f_t \in I(Y).$$

I.e. $\forall y \in Y$, $\underbrace{f_t(y)}_{\overset{\shortparallel}{f(\underbrace{t\cdot y}_{\in Y})}} \overset{?}{=} 0$

$$f(\underbrace{t\cdot y}_{\in Y}) \overset{!}{=} 0.$$

$(2) \Rightarrow (3)$ $f \in I(Y)$ $\qquad k^\times$ infinite b/c k alg. cl.

Fix $x \in Y$. $\quad g(T) = f_T(x) \qquad f = \sum_i f_i$ ← homog. cpt. of deg. i

$$= \sum_i T^i \cdot f_i(x)$$

$$g(t) = 0 \quad \forall t \in k^\times$$

$$\therefore g \equiv 0$$

$$\therefore f_i(x) = 0 \ \forall i$$

$\therefore f_i \in I(Y)$, b/c $x \in Y$ was arbitrary

$I(Y) = \langle f_i \mid f \in I(Y) \rangle$ is homogeneous.

4. Show that $\mathbb{P}^n(k)$ is Noeth. of dim n.

$$\mathbb{P}^n(k) \twoheadleftarrow \mathbb{A}^{n+1}(k) \setminus \{0\}$$

$\qquad$ cb $\qquad\qquad$ Noeth.

i.e. images of Noeth. spaces are Noeth.

$\{\text{closed subsets of } \mathbb{P}^n\} \longleftrightarrow \{\text{closed subsets of } \mathbb{A}^{n+1}(k) \setminus \{0\}$ invariant under $k^\times\}$

$Z_i = \{(x_0, \ldots, x_i, 0, \ldots, 0)\}$

shows that $\dim \geq n$.

Conversely if $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d$ is a chain of $k^\times$-invariant subsets then

$|Z_0| > 1$.

b/c if $x \in Z_0$ then $t \cdot x \in Z_0$

$\neq x$ if $t \neq 1$

b/c $x \neq 0$.

$\{x\} \subsetneq Z_0 \subsetneq \cdots \subsetneq Z_d$ is another chain in $\mathbb{A}^{n+1} \setminus 0$

$\therefore d < n+1$. //

Alternatively: $\{(x_0 : \cdots : x_n) \mid x_i \neq 0\} = U_i \subset \mathbb{P}^n(k)$

open cover, & $U_i \cong \mathbb{A}^n$.

$\therefore \dim \mathbb{P}^n(k) = \sup_i \dim U_i = n$.