Def^n X a top. space.

X has Krull dimension $\geq n$ if $\exists$

$$Z_0 \subset Z_1 \subset \dots \subset Z_n \subset X$$

$Z_i \subset X$ closed & irreducible, $Z_i \neq Z_{i+1}$.

X has Krull dimension $= n$ if it has dim $\geq n$ but not $\geq n+1$.

NB: $\mathbb{R}^n$ + euclidean top. has dim 0.

Ex 1 X top. space, $U \subset X$ subspace, $Y \subset U$ closed.

Show: $\bar{Y} \cap U = Y$

STP: $\bar{Y} \cap U \subset Y$

$\exists A \subset X$ closed st. $Y = A \cap U$

Then $\bar{Y} \subset A$.

$\therefore \bar{Y} \cap U \subset A \cap U = Y$ $\square$

Ex 2 $X \in \mathrm{Top}$

$Y \subset X$.

Show: $\dim Y \leq \dim X$.

| may use: $Z \subset X$ irred. iff $\bar{Z}$ is irred.

?

$Z_0 \subsetneq \cdots \subsetneq Z_n \subset Y$
closed, irred.

$\overline{Z_0} \subsetneq \cdots \subsetneq \overline{Z_n} \subset X$

irred. by /
inclusions are proper b/c $\overline{Z_i} \cap Y = Z_i$ by (1)

Ex 3 $X \in \mathrm{Top}$.

(1) X Noeth. $\Leftrightarrow$ all subsets are cpct

(2) X Noeth. $\Rightarrow$ subspaces are Noeth.

X Noeth. $\overset{\text{obv.}}{\Leftrightarrow}$ ascending chains of opens stabilize

(1) $\Rightarrow$: $A \subset X$. $\{U_i\}_{i \in I}$ an open cover of A

$$\{U_{i_1} \cup \cdots \cup U_{i_n} \mid i_j \in I\} \subset P(X)$$

claim: this has a max. elt M, cf. A

by Zorn's Lemma

s.t. if not, then $\exists\, a \in A \setminus M$
$a \in U_i$ for some i
$\therefore M \cup U_i \in P(X)$
bigger than M. ↯

$\Leftarrow$: $Z_1 \supset Z_2 \supset \cdots$ closed in X

$A = \bigcap_i Z_i$

$U = X \setminus A = \bigcup_i \overbrace{(X \setminus Z_i)}^{\text{open}}$

U cpct, so $\exists$ finite subcover

$U = (X \setminus Z_1) \cup \cdots \cup (X \setminus Z_n)$

$$\therefore A = \bigcap_{i=1}^{n} Z_i = Z_n$$

$$\subset Z_m \quad \forall m > n \qquad \therefore Z_n = Z_{n+1} = \dots$$

$\square$

(2) X Noeth. $\overset{(1)}{\Rightarrow}$ $\forall$ subsub cpct

$\Rightarrow$ for $Z \subset X$, all subsp. of Z cpct

$\overset{(1)}{\Rightarrow}$ Z Noeth.

<u>Ex 4</u> X Noeth.

Show: $X = \bigcup_{i=1}^{N} X_i$, $X_i \subset X$ closed & irred.

X irred. $\Rightarrow$ done.

else: $X = X_1 \cup X_2$, $X_i \subsetneq X$ closed

X_1, X_2 irred $\Rightarrow$ done

X_1 reducible: $X_1 = X_{11} \cup X_{12}$

$\rightsquigarrow$

$$\begin{tikzcd}[column sep=small]
 & & & X \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & X_1 \arrow[dl, dash] \arrow[d, dash] & & & & X_2 \arrow[d, dash] \arrow[dr, dash] & \\
X_{11} & X_{12} & & & & X_{21} & X_{22} \\
 & & & \vdots & & &
\end{tikzcd}$$

$\Big\}$ has to stop b/c Noetherian

$\leftarrow$ leaves irreducible.

& form cover of X.

Alternative: $\mathcal{P} \subset P(X)$ subset of those closed subsets which does not admit decomp.

ordered by reverse inclusion.

Noeth. + Zorn's lemma $\Rightarrow$ $\mathcal{P}$ has max. elt. A

Then $A \in \mathcal{P}$, so A reducible (else $\times$)

$\therefore A = A_1 \cup A_2$ proper subsets

$A_i > A$ $\therefore$ not in $\mathcal{P}$

$\therefore A_i = A_i^1 \cup \ldots \cup A_i^{n_i}$

$\therefore A = \bigcup_{i,j} A_i^j$ $\cancel{\times}$

I.e.: any family of closed subsets of X has a minimal element.

$\therefore$ proving some property P holds for all closed subsets, WMA it holds for any proper subset of X.

"Noetherian induction"

<u>Extra 1</u> Noeth. + Hausdorff $\Rightarrow$ finite discrete

$$\begin{array}{ccc} \text{Noeth.} & & \text{Hausdorff} \\ \Downarrow & & \Downarrow \\ \forall \text{ subsets cpct} & & \text{cpct. subsets are closed} \end{array}$$

$$\Downarrow$$

$$X \text{ discrete} \xRightarrow{X \text{ cpct}} \text{finite}$$

<u>Extra 2</u> (a) $X = \bigcup_{i \in I} U_i$, $U_i \subset X$ open

$\Rightarrow \dim X = \sup \dim U_i$

(b) $Y \subset X$ closed, X irreducible, $\dim X < \infty$,

$\dim X = \dim Y \Rightarrow Y = X$

(c) Example of noetherian ∞-dim^l space?

(d) Ex. of $U \subset X$ open dense, $\dim X > \dim U$

(a) $\dim U_i \leq \dim X$ by prev. work

$\therefore \sup_i \dim U_i < \dim X$

Conversely, suppose $\dim X \geq n$.

$\exists\ Z_0 \subsetneq \cdots \subsetneq Z_n \subset X$ $\quad Z_i$ closed irred.

$\exists\ i \in I$ s.t. $Z_0 \cap U_i \neq \emptyset$

claim: $\underbrace{Z_0 \cap U_i}_{\substack{\text{irred.} \\ \text{closed} \\ \text{in } U_i}} \subsetneq \cdots \subsetneq \underbrace{Z_n \cap U_i}\subset U_i$

$\emptyset \neq Z_n \cap U_i \subset Z_n$ open, irred. by ex. 3(1) for Z_A

dense in Z_n

claim: $\overline{Z_n \cap U_i} = Z_n$

$\subset$: clear

$\supset$: ✓

$\therefore Z_n \cap U_i \subsetneq Z_{n+1} \cap U_i$

o/w else take closures in X. $\quad \therefore \dim U_i \geq n$

$\square$

(b) $Y \subsetneq X \Rightarrow \dim Y < \dim X$ } contrapositive.

& $\dim X < \infty$

Take $Z_0 \subsetneq \cdots \subsetneq Z_n \subset Y$ maximal length.

then $Z_0 \subsetneq \cdots \subsetneq Z_n \subsetneq Z_{n+1} = X$ $\subset X$

$\therefore \dim X \geq \dim Y + 1$.

(d) X Noeth., $U \subset X$ open & dense, $\dim U < \dim X$.

A local noeth. ring $\qquad k[t]_{(t)}$

$\qquad\qquad\qquad\qquad\qquad \mathbb{Z}_{(p)}$

$x \in \operatorname{Spec} A$ = prime spectrum.

$\uparrow$ $\dim \operatorname{Spec} A = \dim A$

unique closed pt.

& $\dim \left(\operatorname{Spec} A \setminus \{x\}\right) = \dim A - 1.$

$\operatorname{Spec} \mathbb{Z}_{(p)} = \left\{ \underset{\eta}{(0)}, \underset{x}{(p)} \right\}$ $\qquad$ open subsets $\{\eta\}$ $\quad$ $\{\eta, x\}$

gen. pt. $\quad$ closed pt

(c) FACT: $\exists$ Noeth. ring A of infinite dimension

$\therefore \operatorname{Spec} A$ does the trick

FACT: $X = \mathbb{N}$ $\quad$ closed subsets are

$\{0, 1, \dots, n\}$, $\emptyset$

also works