1. $f : X \to Y \in Sch$

X, Y integral

f finite type + dominant.

$K(Y) \xrightarrow{\cong} K(X)$.

Show: $\exists\, U \subset X$ open, $f(U) \subset Y$ open

& $f : U \xrightarrow{\cong} f(U)$.

Claim: if $\underset{\text{Spec} B}{U} \subset X$ affine open, $\underset{\text{Spec} A}{V} \subset Y$ affine open
recall.

$f(U) \subset V$ then $f^\# : A \to B$ is of finite type.

Part of def^n/basic property of "locally of finite type"

$\therefore$ also holds for f.t.

Also: very easy if X, Y noeth.

Note that if $U \neq \emptyset$ then

$$\overline{f(U)} = \overline{f(\bar{U})} = \overline{f(X)} = Y$$

b/c X integral, b/c f dominant.

$\therefore f : U \to V$ also dominant

of course U, V integral, $K(X) = K(U)$
$K(Y) = K(V)$

$\therefore$ WMA $X = \operatorname{Spec} B$.
$Y = \operatorname{Spec} A$

$\varphi : A \longrightarrow B$ f.t. $\Rightarrow$ $B = A[b_1, \dots, b_n] \subset B$

integral domains

$K(A) \xrightarrow{\cong} K(B) \implies b_i = \frac{a_i}{a_i'}$ for some $a_i, a_i' \in A$

$a = \prod_{i=1}^{n} a_i'$

Then $b_i \in A\left[\frac{1}{a}\right]$

$\therefore A_a \xrightarrow{\cong} B_a$

$U = f^{-1}(D(a)) \subset X$
$V = D(a) \subset Y$ works. $\square$

2. $X \in \mathrm{Sch}$.

X noeth. $\iff$ X q.c. & $\operatorname{Spec} A \subset X$ open aff.
$\Rightarrow A$ noeth. ring.

i.e. $X = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$, A_i noeth. $\Longleftarrow$ obv.

$\Longrightarrow$ $|X|$ is noeth. in part. q.c.

$\operatorname{Spec} A \subset X$ affine open.

$$\operatorname{Spec} A = \bigcup_{i=1}^{n} \Big[\underbrace{\operatorname{Spec} A \cap \operatorname{Spec} A_i}\Big]$$

$$\subset \operatorname{Spec} A_i$$

$$\therefore = \bigcup_{j=1}^{N_i} \underbrace{\operatorname{Spec} A_i\left[\tfrac{1}{a_{ij}}\right]}_{\text{noeth. since } A_i \text{ is}}$$

$\therefore$ $\operatorname{Spec} A$ is noetherian. Conclude by

Lemma $\operatorname{Spec} A$ noeth. $\to$ A noeth.

Pf $I_1 \supset I_2 \supset \dots$ chain of ideals in A

$\operatorname{Spec} A = \bigcup_{i=1}^{n} U_i$, $U_i = \operatorname{Spec} A_i$ $\leftarrow$ noeth.

$I_i|_{U_i}$ is an ideal in A_i

$\therefore$ for $n > N_i$, $I_n|_{U_i} = I_{N_i}|_{U_i}$

$\therefore$ for $n > N$, $I_n|_{U_i} = I_N|_{U_i}$ $\forall i$

$\therefore$ $I_n = I_N$ for $n > N$. $\square$

A ring $\quad M$ an A-module

Sheaf $\tilde{M}$ on $\operatorname{Spec} A$ corresponding to M

$$\tilde{M}(\operatorname{Spec} A) = M$$

$$\tilde{M}(D(a)) = M_a$$

quasi-coherent sheaves

$$\mathrm{QCoh}(X) \subset \mathcal{O}_X\text{-Mod}.$$

$$X = \operatorname{Spec} A \Rightarrow \mathrm{QCoh}(X) \simeq A\text{-Mod}$$

Other(?) proof: $\operatorname{Spec} A = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$

$$\operatorname{Spec} A_i = \bigcup_{j=1}^{n_i} D(f_{ij})$$

$\therefore$ $(f_{ij}) = A$

$A[1/f_{ij}]$ is noeth. $\rightsquigarrow \ldots$ $\square$

3. X a scheme

$x \in X$ $\Omega_x = \frac{m_x}{m_x^2}$.

- Show this is a $k(x)$ - ~~vector~~ space.

$a \in m_x \Rightarrow a \cdot m_x \subset m_x^2$

$$m_x \cdot \Omega_x = 0$$

$\therefore$ Ω_x is an $\underbrace{\frac{\mathcal{O}_{X,x}}{m_x}}_{k(x)}$ - module

- $T_x X = (\Omega_x X)^\vee$. Now $X \in Sch_k$.

$$\mathrm{Hom}_{Sch_k}\left(\mathrm{Spec}\, \frac{k[\varepsilon]}{\varepsilon^2}, X\right) = \coprod_{x \in X(k)} T_x X$$

$$\left(\frac{k[\varepsilon]}{\varepsilon^2}\right)_{red} = \frac{k[\varepsilon]}{\varepsilon} = k$$

$$\left|\mathrm{Spec}\, \frac{k[\varepsilon]}{\varepsilon^2}\right| = \{*\}$$

$\therefore$ If $f : \mathrm{Spec}\, \frac{k[\varepsilon]}{\varepsilon^2} \longrightarrow X$

$f(*) \in U \subset X$ open

Then f factors through U:

$$\operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2} \longrightarrow U \hookrightarrow X.$$

$\therefore$ WMA $X = \operatorname{Spec} A$. $A \in k$-alg.

We need to determine k-alg. maps from

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi_1"'] & k[\varepsilon]/\varepsilon^2 \arrow[d] \\ & k \end{tikzcd}$$

$\varphi_1 \longleftrightarrow x \in \operatorname{Spec} A$, $\kappa(x) = k$.

$k[\varepsilon]/_{\varepsilon^2}$ local $\rightsquigarrow$ factor φ as

$$A \to A_m \to k[\varepsilon]/_{\varepsilon^2}.$$

$$\therefore \operatorname{Hom}_{Sch_k}(\operatorname{Spec} k[\varepsilon]/_{\varepsilon^2}, X) = \coprod_{x \in X(k)} \operatorname{Hom}_{\substack{\text{local} \\ k\text{-alg.}}}(\mathcal{O}_{X,x}, k[\varepsilon]/_{\varepsilon^2})$$

$\therefore$ RTP $\operatorname{Hom}_{\text{local}}(A, k[\varepsilon]/_{\varepsilon^2}) \cong \left(\frac{m}{m^2}\right)^\vee.$

k-alg.

any local k-algebra with $A/_m = k$.

$$A \xrightarrow{\varphi} k[\varepsilon]/_{\varepsilon^2}$$

$$\varphi(a) = \varphi_1(a) + \varphi_2(a)\cdot\varepsilon$$

$\varphi_1(a) \in k$, $\varphi_2(a) \in k$

$\varphi_1 : A \to k$ is a local k-alg. hom. $\therefore$ uniquely det'd.

$$\varphi(1) = \underset{\substack{\shortparallel \\ 1}}{\varphi_1(1)} + \varepsilon \underbrace{\varphi_2(1)}_{=0} = 1$$

$$\varphi(a+b) = \varphi_1(a+b) + \varepsilon\varphi_2(a+b)$$

$$= \varphi_1(a)+\varphi_1(b) + \varepsilon(\varphi_2(a)+\varphi_2(b))$$

$$\therefore \varphi_2(a+b) = \varphi_2(a)+\varphi_2(b)$$

$$\varphi(a\cdot b) = \varphi_1(a\cdot b) + \varepsilon\varphi_2(a\cdot b)$$

$$= \varphi(a)\varphi(b) = \varphi_1(a)\varphi_1(b) + \varepsilon(\varphi_1(a)\varphi_2(b) + \varphi_1(b)\varphi_2(a))$$

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If $a, b \in m$ then $\varphi(ab) = 0$ b/c $\varphi_1(a) = 0 = \varphi_1(b)$

$\therefore$ $\varphi_2 : m \to k$ annihilates m^2 &

induces $\overline{\varphi_2} : \frac{m}{m^2} \to k$.

which is a morphism of v.s.

$\rightsquigarrow$ $\overline{\varphi_2} \in T_x X$.

Claim: this map $\mathrm{Hom}_{\substack{\text{local}\\ \text{hom}}}(A, k[\varepsilon]/\varepsilon^2) \to T_x X$

is a bijection.

Conversely, given $\overline{\varphi_2} \in T_x X$, define

$$\varphi(a) = \varphi_1(a) + \varepsilon \cdot \overline{\varphi_2}\big(\underbrace{a - \varphi_1(a)}_{\in m}\big)$$

Verify that ⊛ holds, check that the two maps are inverses. $\square$

$$P(X + \varepsilon) = P(X) + \varepsilon \cdot P'(X')$$

4. $\mathbb{P}^1_{\mathbb{Z}} = U_1 \underset{U_{12}}{\cup} U_2$

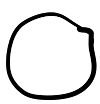
$U_1 = \operatorname{Spec} \mathbb{Z}[T]$
$U_2 = \operatorname{Spec} \mathbb{Z}[T^{-1}]$
$U_{12} = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$

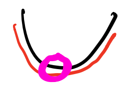
$\overline{\mathbb{A}}^1_{\mathbb{Z}} = U_1' \underset{U_{12}'}{\cup} U_2'$

$U_1' = \operatorname{Spec} \mathbb{Z}[T]$
$U_2' = \operatorname{Spec} \mathbb{Z}[T]$
$U_{12}' = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$

$\mathbb{P}^1$: $\mathbb{A}^1$

$\mathbb{A}^1$

$\rightsquigarrow$ 

$\overline{\mathbb{A}}^1$: 

$\rightsquigarrow$ 

(1) $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathbb{P}^1_{\mathbb{Z}}) = \left\{ (f \in \mathbb{Z}[T], g \in \mathbb{Z}[T^{-1}]) \;\middle|\; f = g \in \mathbb{Z}[t, t^{-1}] \right\}$

$= \mathbb{Z}$

$$\left[\begin{array}{l} f = a_0 + a_1 T + \dots \\ g = g_0 + g_1 T^{-1} + \dots \end{array}\right.$$

$f = g \iff a_i = 0 = g_i$ for $i > 0$, $a_0 = g_0$

Also: $\mathcal{O}_{\overline{\mathbb{A}}^1_{\mathbb{Z}}}(\overline{\mathbb{A}}^1_{\mathbb{Z}}) = \left\{ (f \in \mathbb{Z}[t], g \in \mathbb{Z}[t]) \;\middle|\; f = g \in \mathbb{Z}[t, t^{-1}] \right\}$

$= \mathbb{Z}[t]$

(2) $\mathbb{P}^1_{\mathbb{Z}} \not\cong \overline{\mathbb{A}}^1_{\mathbb{Z}}$ b/c $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathbb{P}^1_{\mathbb{Z}}) \not\cong \mathcal{O}_{\overline{\mathbb{A}}^1_{\mathbb{Z}}}(\overline{\mathbb{A}}^1_{\mathbb{Z}})$ as rings

(3) Show that neither is affine.

Recall: X affine $\iff$ $X \xrightarrow{\sim} \operatorname{Spec} \mathcal{O}_X(X)$.

$\mathbb{P}^1_{\mathbb{Z}} \longrightarrow \operatorname{Spec} \mathbb{Z}$

$\overline{\mathbb{A}}^1_{\mathbb{Z}} \longrightarrow \operatorname{Spec} \mathbb{Z}[t] = \mathbb{A}^1_{\mathbb{Z}}$

not injective on points