Algebraic Geometry 1

Exercises 6

Dr. Tom Bachmann Winter Semester 2020–21

Given a topological space X, we denote by $PSh(X)$ (respectively $Shv(X)$) the category of presheaves (respectively sheaves) on X.

Exercise 1. Let A be a commutative ring and $P \in \mathrm{Spec}\,(A)$. Show that P is a closed point of $\mathrm{Spec}\,(A)$ if and only if it is a maximal ideal of A.

Exercise 2. Let $*$ be a topological space with underlying set of cardinality 1.

(1) Describe the category $PSh(*)$ and exhibit an equivalence $Shv(*) \simeq Set$.

(2) Let X be a topological space, $x \in X$ and write $\pi : * \to X$ for the map with image $\{x\}$. Write
$$\pi_* : Set \simeq Shv(*) \to Shv(X)$$
for the direct image along π. Describe $\pi_*(E)$ for any set E, and compute the stalks $\pi_*(E)_y$ for any $y \in X$.

Exercise 3. Let $f : X \to Y$ be a continuous map of topological spaces, $F \in Shv(X)$, $x \in X$ and $y = f(x)$. Show that there is a canonical induced map
$$f_*(F)_y \to F_x,$$
and given an example where it is not an isomorphism.

Exercise 4. Let X be a topological space. Given $F \in Shv(X)$, denote by $(X_F \to X)$ the associated space over X constructed in the lecture.. Show that $F \mapsto (X_F \to X)$ induces an equivalence of categories between $Shv(X)$ and the category whose objects are pairs (α, E) with E a topological space and $\alpha : E \to X$ a local homeomorphism. (You should first define this category in more detail.)