# Algebraic Geometry 1

## Exercises 10

Dr. Tom Bachmann Winter Semester 2020–21

**Exercise 1.** Consider a commutative diagram of schemes

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y. \end{tikzcd}$$

Suppose that the right hand square is cartesian. Show that the left hand square is cartesian if and only if the outer rectangle is cartesian.

**Exercise 2.** Let $k$ be a field and $X$ a $k$-scheme with $A = \mathcal{O}_X(X)$.

(1) Exhibit a bijection $\mathrm{Hom}_{k\text{-}Sch}(X, \mathbb{A}^1_k) \simeq A$.

(2) Show that $f : X \to \mathbb{A}^1$ is dominant if and only if $f \in A$ is transcendental over $k$.

**Exercise 3.** Let $k$ be a field and $X$ an integral $k$-scheme. Exhibit an injection

$$\{\varphi : X \to \mathbb{P}^1_k \text{ dominant } k\text{-morphism}\} \hookrightarrow \{\varphi \in K(X) \text{ transcendental over } k\}.$$

**Exercise 4.** Put $\mathbb{A}^2_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[T, S]$ and let $n \in \mathbb{Z} \setminus 0$. Let

$$X = V(T^2 - nS) \subset \mathbb{A}^2_{\mathbb{Z}}.$$

(1) Let $p$ be a prime not dividing $n$. Show that the fiber of $X \to \operatorname{Spec} \mathbb{Z}$ over $(p)$ is isomorphic to $\mathbb{A}^1_{\mathbb{Z}}$.

(2) For $p$ dividing $n$, show that the fiber is not reduced.

(3) Show that the fiber over the generic point $(0)$ is isomorphic to $\mathbb{A}^1_{\mathbb{Q}}$.