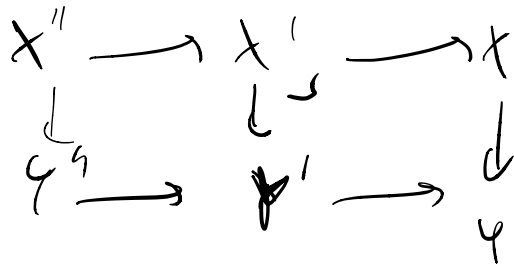


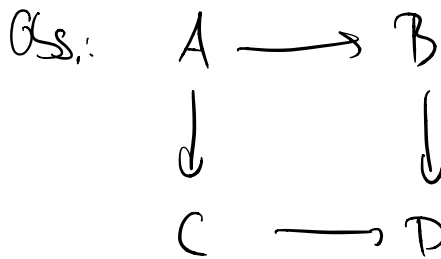
1. Commutative diag. of ~~sets~~ sets



Show bh. square cat. \Leftrightarrow outer rect.

Square is cartesian $\Leftrightarrow \forall T \in \text{Set}, \text{Hom}(T, -)$
 produces a cartesian square of sets.

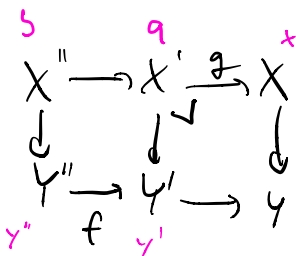
\therefore this will hold in any cat. \Leftrightarrow holds in cat. of sets.



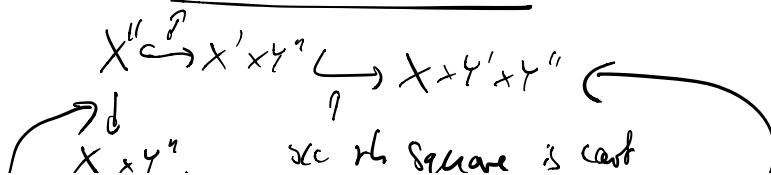
$\in \text{Set}$ is cartesian

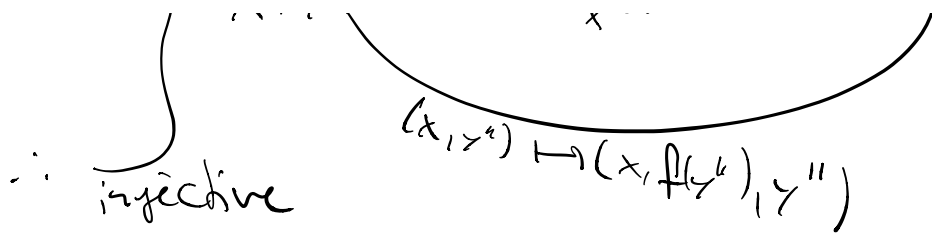
$$\Leftrightarrow A \hookrightarrow B \times C$$

\hookrightarrow image of $A \rightarrow B \times C$ contains all (b,c) with some image in A .



assume left cart.





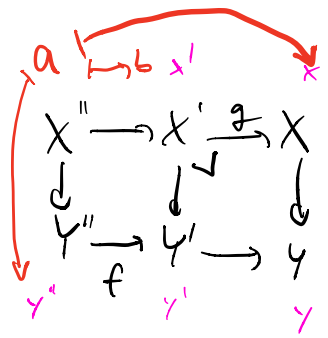
Other half: suppose $(y'', x) \in Y'' \times X$ with one image in Y .

$$a = (f(y''), x) \in Y' \times X = X'$$

$$(x'', a) \in X'' \times Y'' \text{ have image in } f(Y'')$$

$$\Leftrightarrow S \in X'' = X' \times_{Y'} Y''$$

which is what we wanted.



Outer rect. cart

$$X'' \rightarrow Y'' \times X' \rightarrow Y'' \times X$$

inj. $\&c$ outer rect. cart.

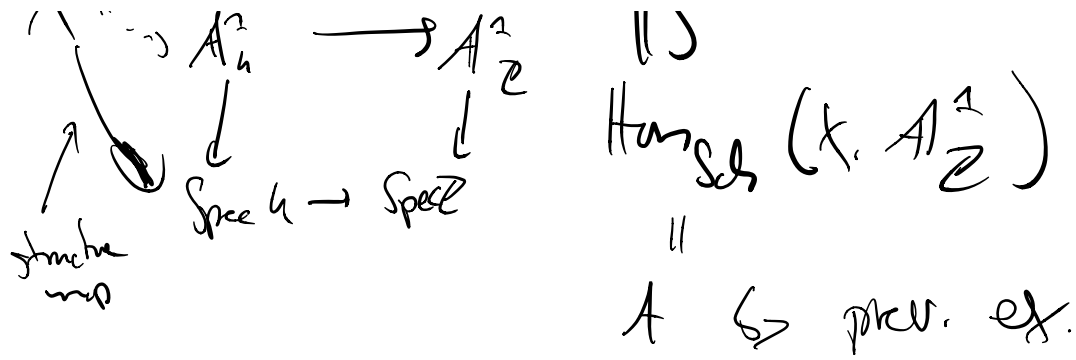
$b = x'$
 $\&c$ $X' \hookrightarrow Y' \times X$
 \therefore done

\therefore injective



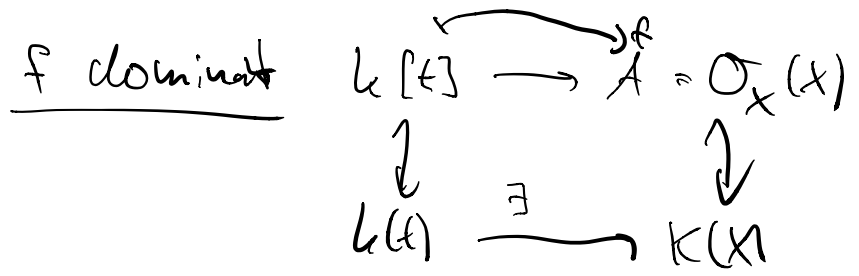
2. $X \in \text{Sch}_k$, k a field

(1) Show $\text{Hom}_{k\text{-alg}}(X, A_k^2) \cong \mathcal{O}_X(X) =: A$
 $X \rightarrow \mathbb{A}^2_k$



(2) Show $f: X \rightarrow A^1$ dominant $\Leftrightarrow f \in A$ is transcendental over k .

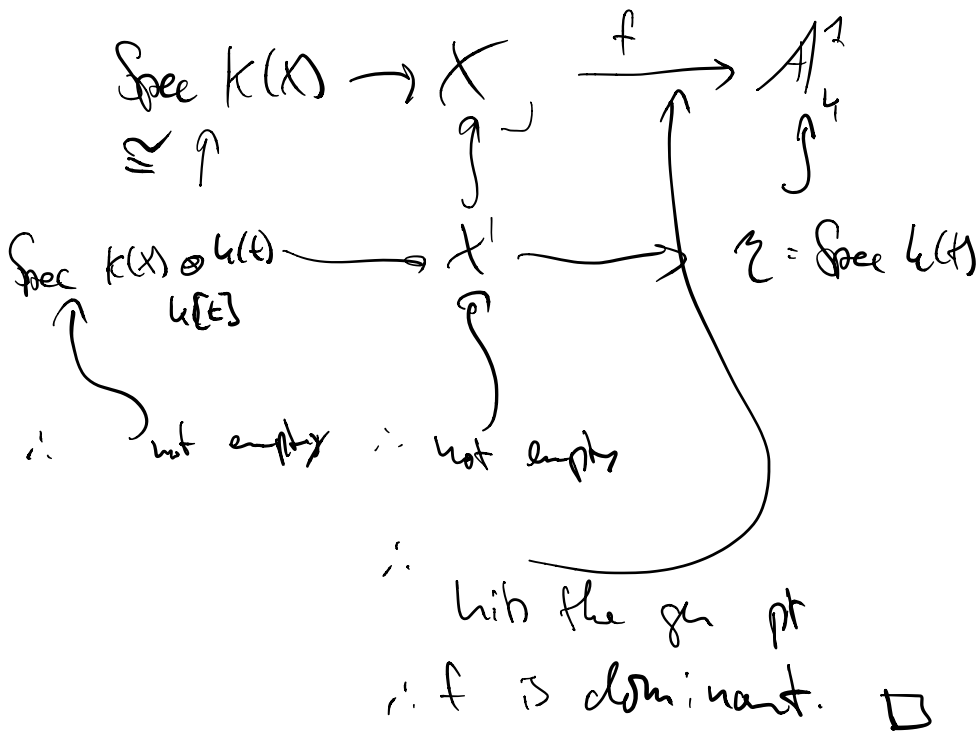
Assume X integral for the problem to make sense.



$\therefore k(f) \subset k(X)$
 \uparrow
 subfield gen. by f over k
 \uparrow
 $k(t) = k(A_k^1)$
 \rightarrow transcendental over k .

f transcendental

$$k(X) \supset k(f) \cong k(t)$$



$$\begin{aligned}
 & k(X) \otimes_{k[t]} k[t] && k(t) = S^{-1} k[t] \\
 & \cong S^{-1} k(X) && S = k[t] \setminus \{0\} \\
 & S^{-1} = \{ p(t) \mid p \in k[t] \setminus \{0\} \} \\
 & \cong k(X) \quad \text{since } S^{-1} \text{ consists of units}
 \end{aligned}$$

3. X/k integral - Exhibit an injection

$$\{ \varphi: X \rightarrow \mathbb{P}_k^1 \text{ dominant } k\text{-irr.} \}$$

$\downarrow \alpha$

$$\{ \varphi \in k(X) \text{ transcendental} \}$$

Construct a map: Given $f: X \rightarrow \mathbb{P}_k^1$ dominant

$$\rightsquigarrow f^*: k(\mathbb{P}_k^1) \rightarrow k(X)$$

\cong

$$k(t)$$

$$t \longmapsto \alpha(f)$$

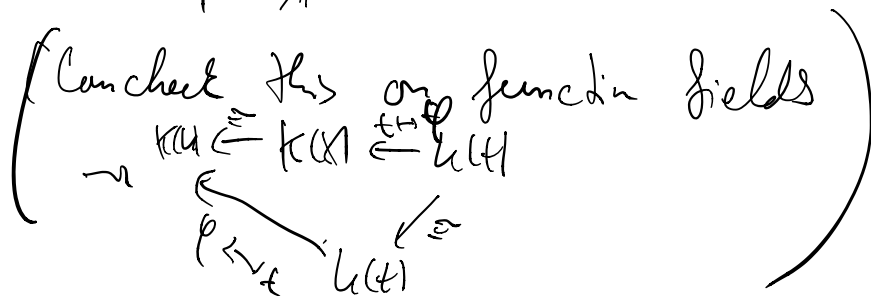
α injective Suppose $\emptyset \neq U \subset X$, $\varphi \in k(X)$
 φ regular on U . $\begin{matrix} \downarrow \alpha \\ \mathbb{P}_k^1 \\ \downarrow \alpha \\ k(t) \end{matrix}$

$$\text{OBS.: } U \rightarrow X \xrightarrow{f} \mathbb{P}_k^1$$

$$\downarrow \varphi$$

$$\mathbb{A}^2$$

commutes.

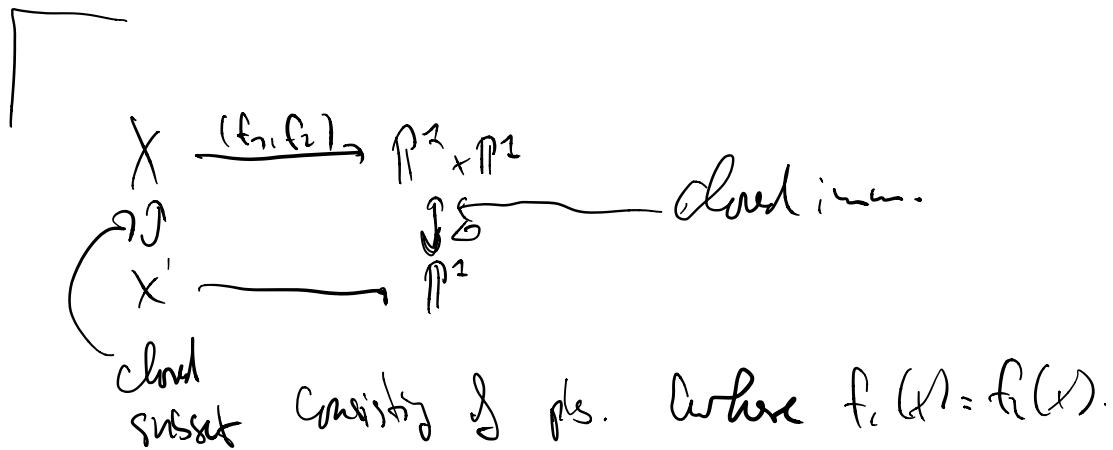


Say $f_1, f_2: X \rightarrow \mathbb{P}^2$, $\alpha(f_1) = \alpha(f_2)$.

Pick $\emptyset \neq U \subset X$ s.t. $\alpha(f_1)$ is regular on U .

Then $f_2|_U = f_1|_U$. (By def.)

Since $U \subset X$ is dense & \mathbb{P}^2 separated,
 $f_1 = f_2$. \square



$$\therefore \tilde{U} \subset X'$$

$$\parallel$$

$$X$$

So why is α not a diffeomorphism?

Ex: $X = \mathbb{A}^2$ $\varphi = \frac{x}{y}$ is not in the image

$$\mathbb{A}^2 \setminus \{0\} \xrightarrow{\varphi} \mathbb{P}^1$$

$$(x, y) \longmapsto (x:y)$$

Depending on a curve in A^2 converging to 0, get different limits under f .

$$\begin{aligned} f(0, x) &= 0 \in A^1 \subset P^1 \\ f(y, x) &= 1 \end{aligned}$$

Pr: If X = smooth curve the bijection holds.

$$4. A^2 = \text{Spec } \mathbb{Z}[T, S] \quad u \in \mathbb{Z} \setminus 0$$

$$X = V(T^2 - uS) \subset A^2$$

$$\begin{array}{ccc} & \searrow \varphi & \downarrow \\ & & \text{Spec } \mathbb{Z} \end{array}$$

Fiber of φ ?

$$\begin{aligned} (1) \ p \neq u : \quad \varphi^{-1}(p) &= \text{Spec } \mathbb{Z}[T, S] / \langle T^2 - uS \rangle \\ &\cong \mathbb{Z}[T] \end{aligned}$$

$$u \neq 0 \in \mathbb{F}_p : \quad \mathbb{F}_p[T, S] / \langle T^2 - uS \rangle \cong \mathbb{F}_p[T]$$

$$\cong \text{Spec } \mathbb{F}_p[T] = \mathbb{A}_{\mathbb{F}_p}^1$$

(2) $p \nmid 4$

$$\varphi^{-1}(p) = \text{Spec } \mathbb{F}_p[T, S]$$

not reduced, $\mathbb{A}_{\mathbb{F}_p}^2 = \text{Spec } \mathbb{F}_p[S]$
 ~~$T^2 - 4S$~~
 reduced closed subscheme
 not asked

(3)

$$\varphi^{-1}(p) = \text{Spec } \mathbb{Z}[T, S] / T^2 - 4S$$

$$S = \mathbb{Z} \setminus \{0\}$$

$$\cong \text{Spec } \mathbb{Q}[T, S]$$

$$/ T^2 - 4S$$

$$\cong \text{Spec } \mathbb{Q}(T) = \mathbb{A}_{\mathbb{Q}}^1$$