1. Commutative diag. of ~~schemes~~ sets

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

Show lft. square cart. $\Leftrightarrow$ outer rect.

Square is cartesian $\Leftrightarrow$ $\forall T \in Sch$, $\mathrm{Hom}(T, -)$ produces a cartesian square of sets.

$\therefore$ this will hold in any cat. $\Leftrightarrow$ holds in cat. of sets.

Obs.:
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \in Set$$
is cartesian

$\Leftrightarrow A \hookrightarrow B \times C$

& image of $A \to B \times C$ contains all (b, c) with same image in D.

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r, "f"'] & Y' \arrow[r] & Y \end{tikzcd}$$

s, q, x, y'', y'

assume left cart.

$X'' \overset{?}{\hookrightarrow} X' \times Y'' \hookrightarrow X \times Y' \times Y''$

$\overset{?}{\downarrow}$

$X \times Y''$.

since rh square is cart

$(x, y'') \mapsto (x, f(y''), y'')$

$\therefore$ injective

Other half: suppose $(y'', x) \in Y'' \times X$ with same image in Y.

$a = (f(y''), x) \in Y' \times X = X'$

$(y'', a) \in X' \times Y''$ have same image $f(y'')$ in Y'

$\Longrightarrow \exists\, b \in X'' = X' \times_{Y'} Y''$

which is what we wanted.

$a \mapsto b \quad x' \quad x$

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\checkmark"] & X \arrow[d] \\ Y'' \arrow[r, "f"] & Y' \arrow[r] & Y \end{tikzcd}$$

$y'' \quad y' \quad y$

Outer rect. cart

$$X'' \longrightarrow Y'' \times X' \longrightarrow Y'' \times X$$

inj. b/c outer rect. cart.

$\therefore$ injective

$b = x'$

b/c $x' \hookrightarrow Y' \times X$

$\therefore$ done

$\square$

2. $X \in Sch_k$, k a field

(1) Show $\mathrm{Hom}_{k\text{-}Sch}(X, \mathbb{A}^1_k) \cong \mathcal{O}_X(X) =: A$

X

$$\begin{tikzcd} \arrow[r, dashed] \arrow[dr] & \mathbb{A}^1_k \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathbb{Z}} \arrow[d] \\ & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

structure map

$$\| \circlearrowright$$

$$\mathrm{Hom}_{Sch}(X, \mathbb{A}^1_{\mathbb{Z}})$$
$$\|$$
A by prev. ex.

(2) Show $f : X \to \mathbb{A}^1$ dominant $\Leftrightarrow$ $f \in A$ is transcendental $/k$.

Assume X integral for the problem to make sense.

<u>f dominant</u>

$$\begin{tikzcd} k[t] \arrow[r] \arrow[d, hook] \arrow[r, bend left, "f"] & A = \mathcal{O}_X(X) \arrow[d, hook] \\ k(t) \arrow[r, "\exists"] & K(X) \end{tikzcd}$$

$\therefore$ $k(f) \subset K(X)$

$\|$

subfield gen. by f over k

$\cong$

$k(t) = K(\mathbb{A}^1_k)$

$\rightsquigarrow$ transcendental over k.

<u>f transcendental</u> $\quad K(X) \supset k(f) \cong k(t)$

$$\begin{tikzcd}
\operatorname{Spec} K(X) \arrow[r] & X \arrow[r, "f"] & \mathbb{A}^1 \\
\operatorname{Spec} K(X) \otimes_{k[t]} k(t) \arrow[u, "\cong"] \arrow[r] & X' \arrow[u, hook] \arrow[r] \arrow[ur] & \eta = \operatorname{Spec} k(t) \arrow[u, hook]
\end{tikzcd}$$

$\therefore$ not empty $\quad \therefore$ not empty

$\therefore$ hits the gen pt

$\therefore f$ is dominant. $\square$

$$K(X) \otimes_{k[t]} k(t) \cong S'^{-1} K(X)$$

$k(t) = S^{-1} k[t]$

$S = k[t] \setminus 0$

$S' = \{ p(f) \mid p \in k[T] \setminus 0 \}$

$\not\ni 0$

$\cong K(X)$ b/c S' consist of units

3. X/k integral. Exhibit an injection

$$\{\varphi : X \dashrightarrow \mathbb{P}^1_k \text{ dominant } k\text{-mor.}\}$$
$$\Big\downarrow \alpha$$
$$\{\varphi \in K(X) \text{ transcendental}\}$$

Construct a map: Given $f : X \dashrightarrow \mathbb{P}^1_k$ dominant

$$\rightsquigarrow f^* : K(\mathbb{P}^1_k) \longrightarrow K(X)$$
$$\cong$$
$$k(t)$$
$$t \longmapsto \alpha(f)$$

α injective Suppose $\emptyset \neq U \subset X$, $\varphi \in K(X)$

$$\begin{array}{cc} \downarrow f & \| \\ \mathbb{P}^1 & \alpha(f) \end{array}$$

φ regular on U.

Oss.:
$$\begin{tikzcd} U \arrow[r] \arrow[dr, "\varphi"'] & X \arrow[r, "f"] & \mathbb{P}^1 \\ & \mathbb{A}^1 \arrow[ur, hook] & \end{tikzcd}$$
commutes.

(Can check this on function fields

$$\rightsquigarrow \begin{tikzcd} K(U) & K(X) \arrow[l, "\cong"'] & k(t) \arrow[l, "t \mapsto \varphi"'] \\ & k(t) \arrow[ul, "\varphi \leftarrow t"] \arrow[ur, "\cong"'] & \end{tikzcd}$$
)

Say $f_1, f_2 : X \to \mathbb{P}^1$, $\alpha(f_1) = \alpha(f_2)$.

Pick $\emptyset \neq U \subset X$ s.t. $\alpha(f_1)$ is regular on U.

Then $f_1\big|_U = f_2\big|_U$. (By defs.)

Since $U \subset X$ is dense & $\mathbb{P}^1$ separated,

$f_1 = f_2$. $\square$

$$\begin{tikzcd} X \arrow[r, "(f_1, f_2)"] & \mathbb{P}^1 \times \mathbb{P}^1 \\ X' \arrow[u, hook] \arrow[r] & \mathbb{P}^1 \arrow[u, hook, "\delta"'] \end{tikzcd}$$

closed imm.

closed subset consisting of pts. where $f_1(x) = f_2(x)$.

$\therefore \tilde{U} \subset X'$
$\quad \shortparallel$
$\quad X$

So why is α not a bijection?

Ex: $X = \mathbb{A}^2$ $\varphi = \frac{x}{y}$ is not in the image

$$\mathbb{A}^2 \setminus 0 \xrightarrow{f} \mathbb{P}^1$$

$$(x,y) \longmapsto (x:y).$$

Depending on a curve in $\mathbb{A}^2 \setminus 0$ converging to 0, get different limits under f.

$$f(0,t) = 0 \qquad f(t,t) = 1 \qquad \in \mathbb{A}^2 \subset \mathbb{P}^2.$$

Btw: If X = smooth curve then bijectivity holds.

4. $\mathbb{A}^2_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[T,S]$ $\qquad n \in \mathbb{Z} \setminus 0$

$$\begin{tikzcd} X = V(T^2 - nS) \arrow[drr, "\varphi"'] & \subset & \mathbb{A}^2_{\mathbb{Z}} \arrow[d] \\ & & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

Fibers of φ?

(1) $p \nmid n$: $\varphi^{-1}(p) = \operatorname{Spec} \mathbb{Z}[T,S] / \underbrace{(T^2 - nS, p)}$

$n \neq 0 \in \mathbb{F}_p$: $\qquad \overset{\|}{\mathbb{F}_p[T,S]} / \quad \cong \mathbb{F}_p[T]$

$$\Big/ T^2 - 4S$$

$$\cong \operatorname{Spec} \mathbb{F}_p[T] = \mathbb{A}^1_{\mathbb{F}_p}$$

(2) $p \mid 4$ $\qquad \varphi^{-1}(p) = \operatorname{Spec} \mathbb{F}_p[T,S] \Big/ T^2 - \cancel{4S}$

not reduced, $\boxed{\text{reduced closed subscheme is } \mathbb{A}^2_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[S]}$

not asked

(3) $\qquad \varphi^{-1}((0)) = \operatorname{Spec} S^{-1}\mathbb{Z}[T,S] \Big/ T^2 - 4S$

$S = \mathbb{Z} \setminus 0$

$$\cong \operatorname{Spec} \mathbb{Q}[T,S] \Big/ T^2 - 4S$$

unit

$$\cong \operatorname{Spec} \mathbb{Q}[T] = \mathbb{A}^1_{\mathbb{Q}}.$$