1. $f: X \longrightarrow Y \in Sch$

$U \subset X$, $V \subset Y$ open subsch.

$f(U) \subset V$.

Show: $\exists ! f': U \longrightarrow V$ making the following commute:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

$$\boxed{\begin{tikzcd} \mathcal{O}_X(f^{-1}(T)) \arrow[r] & \mathcal{O}_U(f'^{-1}(T \cap V)) \\ \mathcal{O}_Y(T) \arrow[r] \arrow[u, "f^\#"] \arrow[ur, phantom, "G"] & \mathcal{O}_V(T \cap V) \arrow[u, "f'^\#"'] \end{tikzcd}}$$

$f': |U| \longrightarrow |V|$ the restriction is the only poss. choice.

$f'^\#: \mathcal{O}_V \longrightarrow f'_* \mathcal{O}_U$

$T \underset{op.}{\subset} V$ is open in Y

$f^\#: \mathcal{O}_V(T) = \mathcal{O}_Y(T) \longrightarrow \mathcal{O}_X(f^{-1}(T))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathcal{O}_U(f'^{-1}(T))$

is also the only choice.

2. $f: X \longrightarrow Y \in Sch$. TFAE:

- f open imm. $\frown f$

- f factors as $X \xrightarrow{\cong} U \hookrightarrow Y$.

open subsch.

top. space

A scheme is a pair $(X, \mathcal{O}_X)$.

sheaf on X

Sometimes I write X for the sch.
$|X|$ for the top. space.
$X = (|X|, \mathcal{O}_X)$

$\Uparrow$: super dev.

$\Downarrow$: $f^\#: \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ is an iso b/c it induces iso on stalks.

By ass. $|f|$ is homeom onto $U \subset Y$.

$\therefore$ $\mathcal{O}_X \text{"="} \mathcal{O}_Y|_U$, i.e. the result.

3. **Def^n** $f: X \to Y \in Sch.$ is quasi-cpct if $\exists$ affine open cover $\{U_i\}_{i\in I}$ of Y s.t. each $f^{-1}(U_i)$ is a quasi-cpct.

i.e. Any open covering of $f^{-1}(U_i)$ has a finite sub cover.

Show: f is q.c. $\Leftrightarrow$ $\forall U \subset Y$ open aff.,
$f^{-1}(U)$ is q.c.

$\Leftarrow$: true b/c every scheme has open affine cover

$\Rightarrow$: $\{V_i\}_i$ open cover, $f^{-1}(V_i)$ qc

$V \subset Y$ open aff.

$\{V \cap V_i\}$ cover V $\qquad V \cap V_i = \bigcup_{j=1}^{n} D(f_j)$

$\subset V = \operatorname{Spec} A$

$V = \bigcup V \cap V_i$

$= \bigcup_{i,j} D(f_{ij})$ $\quad \therefore (f_{ij}) = A$

b/c V is q.c.

$f^{-1}(V \cap V_i)$

$= \bigcup \underbrace{f^{-1}(D(f_i))}_{\text{need not be affine}}$

$\rightsquigarrow \exists$ finite subcover $V = \bigcup_{i=1} V \cap V_i$

$= f^{-1}(V)_{f_i}$

$$f^{-1}(V) = \bigcup_{i=1}^{n} f^{-1}(V \cap V_i)$$

$$\subset \underbrace{\bigcup_{i=1}^{n} f^{-1}(V_i)}_{\text{q.c.}}$$

Now goal: if $X \in \mathrm{Sch}$, X q.c.

$U \subset X$ open $\Rightarrow$ U q.c.

I.e. given $\{U_i\} \subset X$ open

$\bigcup_i U_i \supset U$

show that $U \subset U_{i_1} \cup \dots \cup U_{i_n}$.

X sch. $\Rightarrow$ $\exists$ affine open cover

X q.c. $\Rightarrow$ $X = U_1 \cup \dots \cup U_n$, U_i affine open

$\therefore U = \bigcup_{i=1}^{n} (U_i \cap U)$

$\therefore$ STP $U_i \cap U \subset \underset{\text{affine}}{U_i}$ q.c.

$\therefore$ WMA $\underset{\substack{\cup \\ U}}{X} = \operatorname{Spec} A$ is affine.

Suppose A is Noeth. (I.e. working with loc. noeth. Sch.)

Then this is easy: $U = X \setminus V(\underset{\substack{\| \\ (f_1, \dots, f_n)}}{I})$

$= \bigcup_{i=1}^{n} D(f_i)$ q.c.

$U \subset Y$ open aff.

$Y = \bigcup_{i \in I} U_i$ $\quad U_i$ open aff.

$f^{-1}(U_i)$ q.c.

$\underset{\substack{\| \\ \bigcup_j D(a_{ij})}}{U \cap U_i} \subset U_i$, $a_{ij} \in \mathcal{O}(U_i)$

U q.c. $\to$ $U = \bigcup D(a_{ij})$

$$f^{-1}(U) = \bigcup_{\substack{i \in I \\ \text{finitely many}}} D(a_{i}) \quad \begin{matrix}\text{many} \\ \text{many } i\in I\end{matrix}.$$

$\therefore$ STP: If $V \subset Y$ affine, $f^{-1}(V)$ q.c.

$a \in \mathcal{O}(V)$ $\quad f^{-1}(D(a))$ q.c.

$$\overset{\shortparallel}{f^{-1}(V)} \setminus V(a \circ f)$$

$$f^{-1}(V) = \bigcup_{i=1}^{n} V_i \leftarrow \text{open aff.}$$

$$f^{-1}(V) \setminus V(a) = \bigcup_{i=1}^{n} V_i \setminus V(a)$$

$V_i = \operatorname{Spec} A_i \qquad a \text{ "}\in\text{" } A_i$

$V_i \setminus V(a) = \operatorname{Spec} A_i[\tfrac{1}{a}]$ is itself affine.

$\therefore$ q.c.

X q.c. sch

$a \in \mathcal{O}(X)$

$X \setminus V(a) = X_a$ q.c.

b/c $X = \bigcup_{i=1}^{n} \underbrace{\operatorname{Spec} A_i}_{V_i} \qquad a|_{V_i} \in \mathcal{O}(V_i) = A_i$

$$X_a \cap V_i = D(a|_{V_i}) = \operatorname{Spec} A_i[\tfrac{1}{a|_{V_i}}]$$

$\therefore X_a$ is finite union of affine opens $\therefore$ q.c.

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4. X reduced scheme $\qquad \mathcal{O}_X \to \{\text{functions}\}$

$Z \subset X$ closed $\rightsquigarrow I_Z \subset \mathcal{O}_X$ subsheaf of functions vanishing on Z

Show: $\{\text{closed subsets of } Z\}$

$\longrightarrow \{\text{subsheaves of } \mathcal{O}_X\}$

$Z \longmapsto I_Z$ is injective.

Image?

Reduce to X affine: $X = \bigcup_i \underbrace{\operatorname{Spec} A_i}_{U_i}$.

$Z \subset X$ closed, $Z \cap \operatorname{Spec} A_i$ closed $\subset \operatorname{Spec} A_i$

$Z = Z' \iff Z \cap U_i = Z' \cap U_i \quad \forall i$

$I_1, I_2 \subset \mathcal{O}_X$: $I_1 = I_2 \iff I_1|_{U_i} = I_2|_{U_i} \ \forall i$.

$\therefore$ WMA $X = \operatorname{Spec} A$. $Z \subset \operatorname{Spec} A$

$Z = V(J)$, some ideal $J \subset A$.

$f \in I_Z(X) \iff f \in \bigcap P = \sqrt{J}$

$P \in Z$

$\therefore \; I_Z = I_{Z'}$

$Z = V(\mathcal{J}) \quad Z' = V(\mathcal{J}') \Rightarrow \sqrt{\mathcal{J}} = \sqrt{\mathcal{J}'} \Rightarrow Z = Z'$

$\mathcal{O}_X \longrightarrow \{$functions valued in $\kappa(x)\}$
is injective only if X is reduced.

$\{$closed subsets of $X\} \longrightarrow \{$subsheaves of $\mathcal{O}_X\}$

$\shortparallel$

$\{$reduced closed subschemes$\}$ a: quasi-coherent radical sheaves of ideals

ex: $X = \operatorname{Spec} A$

$\{$closed subsets$\}$ $\quad\quad$ $\{$ideal sheaves$\}$

$\updownarrow$

$\{$ideals$\}$ $\longrightarrow$ not bijection