1. $f \in A$, a comm. ring.

Show: $\operatorname{Spec} A \setminus V(f) \cong \operatorname{Spec} A_f$ as locally ringed spaces.

loc. ringed space = $(X, \mathcal{O})$ — $X \in \mathrm{Top}$, $\mathcal{O} \in \mathrm{Sh}(X)$ sheaf of comm. rings, $\mathcal{O}_x$ a local ring $\forall x \in X$

$\mathfrak{p} \in \operatorname{Spec} A \setminus V(f) \iff f \notin \mathfrak{p}$

$\hookrightarrow \operatorname{Spec} A_f$

via standard correspondence

$(A_f)_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ $\quad X = \operatorname{Spec} A \quad X_f = \operatorname{Spec} A_f$

$U \subset \operatorname{Spec} A \setminus V(f)$ $\quad \mathcal{O}_X(U) \ni \alpha \quad \alpha : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$

$$\begin{array}{ccc} \mathcal{O}_X(U) & & \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \\ \updownarrow & & \parallel \\ \mathcal{O}_{X_f}(U) & & \coprod_{\mathfrak{p} \in U} A_{f\mathfrak{p}} \end{array}$$

$\operatorname{Spec} A_f$ identifies with an open subset of $\operatorname{Spec} A$.

$\mathcal{O}_X$ a sheaf on $\operatorname{Spec} A = X$

Consider $\mathcal{O}_X|_{\operatorname{Spec} A_f}$, defines a sheaf on $\operatorname{Spec} A_f \subset X$.

Need to show: $\mathcal{O}_X|_{X_f} \cong \mathcal{O}_{X_f}$.

Sheaf $\mathcal{O}_X$ is determined by: $\mathcal{O}_X(D(g)) = A_g$

$$\begin{array}{ccc} \mathcal{O}_X(D(g)) & = & A_g \\ \downarrow & & \downarrow \\ \mathcal{O}_X(D(gh)) & = & A_{gh} \end{array}$$

Find that if $g \in A_f$, $g = \frac{g_0}{f^n}$

$$\mathcal{O}_X|_{X_f}(D(g)) = \mathcal{O}_X(D(g_0 \cdot f)) = A_{g_0 \cdot f} \cong (A_f)_g = \mathcal{O}_{X_f}(D(g))$$

$\therefore \mathcal{O}_X|_{\operatorname{Spec} A_f} \cong \mathcal{O}_{\operatorname{Spec} A_f}$.

$D(f) = \operatorname{Spec} A \setminus V(f)$

$\cong$

$\operatorname{Spec} A_f$

$\operatorname{Spec} A_f \xrightarrow{i} \operatorname{Spec} A$ mor. of schemes

i.e. $|\operatorname{Spec} A_f| \hookrightarrow |\operatorname{Spec} A|$

$+ \; i^\#: \mathcal{O}_{\operatorname{Spec} A} \to i_* \mathcal{O}_{\operatorname{Spec} A_f}$

$\mathrm{Spec}: \mathrm{Ring}^{op} \longrightarrow \mathrm{LRS}$

$A \xrightarrow{i^\#} A_f \rightsquigarrow \mathrm{Spec}\, A_f \xrightarrow{i} \mathrm{Spec}\, A.$

2. X a scheme. TFAE:

(1) $\mathcal{O}_X(U)$ reduced $\forall U \subset X$

(2) $\mathcal{O}_X(U_i)$ reduced $\forall i$, $\{U_i\}$ an affine open cover of X

(3) $\mathcal{O}_{X,x}$ reduced $\forall x \in X$.

(1) $\Rightarrow$ (2): obv.

(2) $\Rightarrow$ (3): $x \in X$ $\exists i$, $x \in U_i$.

open nbd. of x contained in U_i are cofinal

$\therefore$ STP that $\mathcal{O}_X(V)$ is reduced for $V \subset U_i$ open.

b/c filtered colimits of reduced rings are reduced

$$\mathcal{O}_{X,x} = (\mathcal{O}_X|_{U_i})_x = (A_i)_x.$$

$x \in \mathrm{Spec}\, A_i = U_i$

STP: localizations of reduced rings are reduced

$S \subset A$ mult. closed

$\frac{a}{s} \in S^{-1}A$, $\left(\frac{a}{s}\right)^n = 0$

$\Rightarrow y a^n = 0$

$\cdot \; \cdot \; ^n \; -$

$\Rightarrow (ya) = 0$

$\Rightarrow ya = 0$

$\Rightarrow \frac{a}{s} = 0 \in S^{-1}A$. □

(5) $\Rightarrow$ (1): $a \in \mathcal{O}_X(U)$

$a^n = 0$

[$\mathcal{O}_{X,x}$ reduced $\Rightarrow \exists$ open nbhd. U of x s.t. $\mathcal{O}_X|_U$ is reduced.

$a \in \mathcal{O}_X(U)$

$a^n = 0 \Rightarrow (a_x)^n = 0 \Rightarrow a_x = 0$

$\Rightarrow \exists U' \subset U$ s.t. $a|_{U'} = 0$.]

$\Rightarrow (\sigma_x)^n = 0 \quad \forall x \in U$

$\Rightarrow \sigma_x = 0$ b/c $\mathcal{O}_{X,x}$ reduced

$\Rightarrow \sigma = 0$ by sheaf cond^n.

3. Y reduced, $f: Y \to X$, $\exists!\, g: Y \dashrightarrow X_{red}$, $X_{red} \hookrightarrow X$

| $\mathcal{O}_{X_{red}}(U) := \mathcal{O}_X(U)_{red}$

Show: $(X, \mathcal{O}_{X_{red}})$ is a scheme

$\quad\quad !!$

$\quad\quad X_{red}$.

Let A be a commutative ring.

$N = N_A = \{a \in A \mid a^n = 0 \text{ for some } n\}$ "nilradical"

$A_{red} := A/N$ (this is reduced)

$X_{red} := (X, \mathcal{O}_{X_{red}})$ $\quad X_{red} \xrightarrow{i} X$

i.e.: $|i| : |X_{red}| \xrightarrow{\cong} |X|$

$+ \; i^\# : \mathcal{O}_X \to |i|_* \mathcal{O}_{X_{red}}$

$\qquad \qquad \qquad \qquad \quad \| $

$\qquad \qquad \qquad \qquad \mathcal{O}_{X_{red}}$.

$X = (|X|, \mathcal{O}_X)$

$X \in \text{Top}$

$|X|$ underlying set

$X \in \text{Sch}$

$|X|$ underlying top space

Want to show: $(X, \mathcal{O}_{X_{red}})$ is a scheme.

I.e.: $\exists$ open cover $\{U_i\}$ of X s.t. $(U_i, \mathcal{O}_{U_i, red})$ is iso to affine scheme.

B/c X is a scheme, then exists cover s.t. $U_i \cong \operatorname{Spec} A_i$.

$\mathcal{O}_{X_{red}}|_{U_i} = \mathcal{O}_{U_i, red}$ $\therefore$ WMA $X = \operatorname{Spec} A$.

Guess: X_{red} is also affine, namely $X_{red} \cong \operatorname{Spec} A_{red}$.

$\operatorname{Spec} A_{red} \xrightarrow{\cong} \operatorname{Spec} A$ as top. spaces

b/c every prime ideal contains N.

$\mathcal{O}_{A_{red}}$, $\mathcal{O}_{X_{red}}$ viewed as sheaves on the same top. space (i.e. X).

Want to prove: $\mathcal{O}_{A_{red}}(U) \cong \mathcal{O}_{X_{red}}(U) = \mathcal{O}_X(U)_{red}$.

More explicitly: $f \in A$ $\quad \mathcal{O}_{X_{red}}(D(f)) = \mathcal{O}_X(D(f))_{red}$

$$= (A_f)_{red}$$

$$\mathcal{O}_{Spec(A_{red})}(D(f)) = (A_{red})_f$$

So need to prove: $(A_{red})_f \cong (A_f)_{red}$.

$$(A_{red})_f = \left(A/N_A\right)_f \cong \frac{A_f}{(N_A)_f}$$

$$(A_f)_{red} = \frac{A_f}{N_{A_f}} \quad \therefore \text{ need } (N_A)_f = N_{A_f}.$$

$$\frac{a}{s} \in N_{A_f} \Leftrightarrow \left(\frac{a}{s}\right)^n = 0 \in A_f$$

$$\Leftrightarrow \underbrace{f^m}_{y} a^n = 0$$

$$\Rightarrow (ya)^n = 0$$

$$\Rightarrow ya \in N_A$$

$$\Rightarrow \frac{a}{s} \in (N_A)_f.$$

$$\frac{a}{f^m} \in (N_A)_f \Rightarrow a^n = 0$$

$$\Rightarrow \left(\frac{a}{f^m}\right)^n = 0$$

$$\Rightarrow \frac{a}{f^m} \in N_{A_f}$$

$\therefore (N_A)_f = N_{A_f}$.

Part 2: Y reduced

$$\begin{array}{ccc} & & Y \\ & & \downarrow f \\ X_{red} & \overset{i}{\hookrightarrow} & X \end{array}$$

have $|f|: |Y| \longrightarrow |X|$

$$\begin{array}{ccc} f^\#: \mathcal{O}_X & \longrightarrow & f_* \mathcal{O}_Y \\ \downarrow & & \| \\ \mathcal{O}_{X_{red}} & \xrightarrow{???} & f_* \mathcal{O}_Y \end{array}$$

Put $|g| = |f|$

want $g^\#: \mathcal{O}_{X_{red}} \xrightarrow{???} f_* \mathcal{O}_Y$

$(f_* \mathcal{O}_Y)(U)$ is reduced

Since $\mathcal{O}_{X_{red}}(U) = \mathcal{O}_X(U)_{red}$, $f^\#(U)$ factors uniquely as displayed.

- check $g^\#$ is a morphism of sheaves:

$V \subset U$

$$\begin{array}{ccccc} & & \mathcal{O}_X(U) & \twoheadrightarrow & \mathcal{O}_{X_{red}}(U) \\ & \swarrow & \downarrow & & \downarrow \\ f_*\mathcal{O}_Y(U) & & \mathcal{O}_X(V) & \twoheadrightarrow & \mathcal{O}_{X_{red}}(V) \\ \downarrow & \swarrow & & & \\ f_* \mathcal{O}_Y(V) & & & & \end{array}$$

⇝ easy diagram chase

- need to check f is mor. of loc. ringed spaces

i.e. given $y \in Y$, $x = f(y)$, need to prove that

$$\begin{array}{ccc} \mathcal{O}_{X_{red},x} & \xrightarrow{\alpha_{red}} & \mathcal{O}_{Y,y} \\ \| & & \uparrow \alpha \\ (\mathcal{O}_{X,x})_{red} & \twoheadleftarrow & \mathcal{O}_{X,x} \\ & & \cup \\ & & \mathfrak{m}_{X,x} \end{array}$$

is a local hom^m

← this is a local hom^m

$\alpha(\mathfrak{m}_{X,x}) \subset \mathfrak{m}_{Y,y}$

$\mathfrak{m}_{X_{red},x} \overset{\alpha}{\twoheadleftarrow} \mathfrak{m}_{X,x}$

$\therefore \alpha_{red}(\mathfrak{m}_{X_{red},x}) = \alpha(\mathfrak{m}_{X,x}) \subset \mathfrak{m}_{Y,y}$.

i.e.: $A \to B$ local hom^m

$\Rightarrow A_{red} \to B_{red}$ local hom^m.