1. Give ex. of top. space X, sheaves F, G cat, mor. $\alpha : F \to G$ s.t.

1) α is an epi in cat. of sheaves
2) $\alpha(X) : F(X) \to G(X)$ is <u>not</u> an epimorphism of sets
[or: not epi of presheaves]

$X = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

$\mathbb{C} \to \mathbb{C}^\times$
$t \mapsto e^t$

$F = \mathcal{O}$ = holo. functions
$G = \mathcal{O}^\times$ = non-zero holo functions

$\exp : \mathcal{O} \to \mathcal{O}^\times$
$f \mapsto e^f$

Analysis: surjective on stalks: locally choose a branch of log.

$\therefore F \to G$ epi of sheaves

Show: $F(\mathbb{C}^\times) \to G(\mathbb{C}^\times)$ is not surjective
$\quad\quad\quad\quad\quad\quad\quad\quad\quad \ni$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad 1/z \; z$

Claim: $\nexists f : \mathbb{C}^\times \to \mathbb{C}$ s.t. $e^f = \frac{1}{z}$.

would need $f = \log\left(\frac{1}{z}\right)$ z but this does <u>not</u> admit a cb ext^n to all of $\mathbb{C}^\times$.

Or: $X = \mathbb{R}$

G = const. sheaf $C(\mathbb{R},\mathbb{R})$

$F = C(-,\mathbb{R})$

$G \longrightarrow F$ Claim: $G \to F$ is not epi of presh.

$f \longmapsto f|_U$ but: epi of sheaves

$\frac{1}{x-1}$ on $(-\infty, 1)$ cannot be extended to $\mathbb{R}$

Any sheaf F

presheaf $G_0(U) = F(X)$ $\forall U$

$G_0 \longrightarrow F$

$aG_0 \longrightarrow F$ is epi $\Leftrightarrow$

$X = (0,1)$

F_1 = const. sheaf on $C(\mathbb{R},\mathbb{R})$ } sheaves of a.s. sps.

$F_2 = C(-,\mathbb{R})$

K kernel

$\cap$

$F_1 \twoheadrightarrow F_2$.

$H^1(*, \dots) = 0$

$\parallel$

$H^1((0,1), \text{const. sheaf})$

$\parallel$

If $0 \to K \to F_2 \to F_2 \to 0 \in AS(X)$.

Then $0 \to K(X) \to F_1(X) \to F_2(X) \longrightarrow H^1(X,K) \longrightarrow H^1(X,F_2) \to \dots$

$0 \neq$ $H^1(\sigma, -) = 0$

Ex 2: Show that $\mathrm{Shv}(X) \to \mathrm{PSh}(X)$ does not pres $\sqcup$ in general.

$X = \{1, 2\}$ discrete top.

$F \in \mathrm{PSh}(X)$

$$\begin{tikzcd}
 & F(X) \arrow[dl] \arrow[dr] \arrow[dd] & \\
F(\{1\}) \arrow[dr] & & F(\{2\}) \arrow[dl] \\
 & F(\emptyset) &
\end{tikzcd}$$

When is F a sheaf?

$F(X) = F(\{1\}) \times F(\{2\})$

$F(\emptyset) = *$

$F_1(\{1\}) = A$ $\qquad$ $F_2(\{1\}) = *$

$F_1(\{2\}) = *$ $\qquad$ $F_2(\{2\}) = A$

$F_1(\emptyset) = \emptyset$ $\qquad$ $F_2(\emptyset) = \emptyset$

$F_1(X) = A$ $\qquad$ $F_2(X) = A$

$$F_1 \sqcup^{\text{shef}} F_2 = \begin{array}{l} \{1\} : A \sqcup * \\ \{2\} : * \sqcup A \\ \{1,2\} : \underbrace{(A \sqcup *) \times (* \sqcup A)}_{\neq \ A \sqcup A \text{ (in pr.)}} \\ \emptyset : * \end{array}$$

$$F_1 \sqcup^{\text{presh}} F_2 = \begin{array}{l} \{1\} : A \sqcup * \\ \{2\} : * \sqcup A \\ \{1,2\} : A \sqcup A \\ \emptyset : \emptyset \end{array}$$

$$(F_1 \sqcup^p F_2)(\{1\}) = F_1(\{1\}) \sqcup F_2(\{1\}) = A \sqcup *$$

$F(\{1\})$ = stalk of F at $\{1\}$

$= \operatorname{colim} \left(F(X) \to F(\{x\}) \right)$

3. X an aff. alg. var. over $k = \bar{k}$.

Show: $\operatorname{Mor}(X, \mathbb{A}^1_k) = \mathcal{O}_X(X)$.

sheaf of regular functions

Put $A = A[X]$.

$$= \frac{k[X_1, \ldots, X_n]}{I(X)}$$

Def^n $\operatorname{Mor}(X, \mathbb{A}^1) = \{ f : X \xrightarrow{cts} \mathbb{A}^1 \mid \exists P \in k[X_1, \ldots, X_n] \text{ s.t. } P|_X = f \}$

$= A$

Def^n $\mathcal{O}_X(U) = \{ g : U \to k \text{ s.t. locally } g = \frac{P}{Q},\ P, Q \in k[X_1, \ldots X_n] \}$

Want: $\mathcal{O}_X(X) = A[X] \hookrightarrow \operatorname{Frac}(A[X])$

Clear: $A[X] \subset \mathcal{O}_X(X)$.

Conversely: $g \in \mathcal{O}_X(X)$

$$J = \left\{ h \in \frac{k[X_1, \ldots X_n]}{I} \;\middle|\; h \cdot g \in A[X] \right\}$$

Will show: $1 \in \mathcal{J}$.

$\mathcal{J} \lhd A[X]$ is an ideal.

$1 \in \mathcal{J} \iff Z(\mathcal{J}) = \emptyset$.

I.e. show: $\forall x \in X$, $x \notin Z(\mathcal{J})$.

Let $x \in X$. Since $g \in \mathcal{O}_X(X)$ $\exists\ x \in U \subset X$

s.t. $g|_U = \frac{g_1}{g_2}$, $g_1, g_2 \in A[X]$

$\therefore g_1 = g \cdot g_2$ on U, i.e. $g_2 \in \mathcal{J}$

$g_2(x) \neq 0$

$\therefore \overline{U} = X$ b/c X irred.

$\therefore x \notin Z(\mathcal{J})$. $\square$

4. Same problem but X not necessarily affine.

Examples to keep in mind: $X = \mathbb{P}^1$ $\quad k[\mathbb{P}^1] = k$

$X = \mathbb{A}^2 \setminus 0$. $\quad k[\mathbb{A}^2 \setminus 0] = k[\mathbb{A}^2]$

$\varphi \in \mathrm{Mor}(X, \mathbb{A}^1) \rightsquigarrow \varphi^*(t) \in \mathcal{O}_X(X)$.

$$\mathrm{Mor}(-, \mathbb{A}^1) \xrightarrow{\alpha} \mathcal{O} \in \mathrm{PSh}(X) \quad \Big| \quad \alpha(\varphi) = \varphi^*(t)$$

Obs.: both of these are sheaves.

- clear for $\mathcal{O}_X$

- check it for $\mathrm{Mor}(-, \mathbb{A}^1)$

$f: X \to \mathbb{A}^1$ is a mor. $\Longleftrightarrow$ f^*(reg. fcn) is regular

$\Longleftrightarrow$ loc. f is a quotient of reg. f^ns.

X has a basis of open nbds $U \subset X$ st. U is affine.

By prev. prop. $\alpha(U)$ is iso.

$\therefore \alpha$ is iso $\square$