Algebraic Geometry 1

Exercises Tutorium 3

Dr. Tom Bachmann Winter Semester 2020–21

Throughout k is an algebraically closed field.

Exercise 1. Let $Z = Z(x^2 - yz, xz - x) \subset \mathbb{A}^3(k)$. Show that Z is a union of three irreducible components, describe them, and determine their prime ideals.

Exercise 2.

(1) Let $Z \subset \mathbb{A}^n(k)$ be closed and irreducible, and $H \subset \mathbb{A}^n(k)$ a hypersurface. Suppose that $Z \not\subset H$. Show that every irreducible component of $Z \cap H$ has dimension $\dim Z - 1$.

(2) Let $I \subset k[\mathbb{A}^n]$ be generated by r elements. Show that every irreducible component of $Z(I)$ has dimension $\geq n - r$.

[*Hint:* You may use Krull's principal ideal theorem.]

Exercise 3. Let Y be a closed subset of $\mathbb{A}^{n+1}(k) \setminus \{0\}$. Show that the following statements are equivalent.

(1) Y is $k^\times$-invariant.

(2) For each $f \in I(Y) \subset k[X_0, \dots, X_n]$ and $t \in k^\times$ we have $f(tX_0, \dots, tX_n) \in I(Y)$.

(3) $I(Y)$ is a homogeneous ideal.

Exercise 4. Show that $\mathbb{P}^n(k)$ is a noetherian topological space of dimension n.

Exercise 5. *Extra problem*: Let $n, d > 0$ and let $M_0, \dots, M_N$ be all the monomials of degree d in $n+1$ variables $x_0, \dots, x_n$. Show that the assignment

$$(x_0 : \cdots : x_n) \mapsto (M_0 : \cdots : M_N)$$

defines a morphism $\mathbb{P}^n(k) \to \mathbb{P}^N(k)$ which is a homeomorphism onto a closed subset, and identify its ideal.

Exercise 6. *Extra problem*: Let $Y \subset \mathbb{A}^3(k)$ be the curve given parametrically by (t^3, t^4, t^5). Show that $I(Y)$ is a prime ideal of height 2 that cannot be generated by 2 elements.