X integral scheme

$\forall P \in X \quad \mathrm{Frac}(\mathcal{O}_{X,P}) = K(X) = k(\underset{\text{gen. pt.}}{\eta})$

$\exists \operatorname{Spec} A \subset X$ open affine nbd. P

$P \longleftrightarrow P \subset A$ prime ideal

A integral domain

$K(X) = \mathrm{Frac}(A)$

Algebra problem: A integral domain, $P \subset A$ prime ideal

$$\mathrm{Frac}(A) \cong \mathrm{Frac}(A_P).$$

$$\|$$

$$S^{-1}(A) \qquad S = A \setminus 0$$

$$A_P = S_P^{-1} A \qquad S_P = A \setminus P$$

$$\mathrm{Frac}(A_P) = (A_P \setminus 0)^{-1} A_P$$

$$\begin{tikzcd} A \arrow[r] & A_P \arrow[r] \arrow[d] & \mathrm{Frac}(A) \\ & \mathrm{Frac}(A_P) \arrow[ur] & \end{tikzcd}$$

exists by univ property

Surjective: $x \in \mathrm{Frac}\, A$

$x = \frac{a}{b} \quad b \neq 0 \quad a, b \in A$

$\frac{a}{b} \in \mathrm{Frac}(A_P)$ since $b \neq 0$ in A_P

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s.t. $A_p \hookrightarrow \mathrm{Frac}(A)$
s/c integral domain

A integral domain

$$\begin{array}{ccc} A & \hookrightarrow & K \leftarrow \text{any field} \\ \downarrow & \nearrow & \\ \mathrm{Frac}(A) & & \end{array}$$

$\mathrm{Frac}(A)$ = smallest subfield of K cdg A

X noeth. $\quad R = X \setminus \mathrm{Supp}\, N_X$ ← ideal sheaf of nilradicals

$\underbrace{\{p \in X \mid \mathcal{O}_{X,p} \text{ reduced}\}}_{R} \subset X$ is open.

$V \subset X$ is open $\iff$ $V \cap U_i \subset U_i$ is open, $\forall i$
where $\{U_i\}$ is some open cover of X

$\therefore$ WMA $X = \mathrm{Spec}\, A$ A noeth. ring

STP. if A_p is reduced then $\exists f \in A \setminus p$ s.t.
A_f is reduced.

Indeed then $D(f) \subset R$, so R a union of opens
$\therefore$ open

N_A = nilradical = set of all nilpotent elts
$\subset A$ is an ideal, f.g. SC A noeth.

$N_A = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$

$$\left.\begin{array}{l} N_{S^{-1}A} = S^{-1} N_A \\ A \text{ reduced} \Leftrightarrow N_A = 0 \end{array}\right\} \text{facts}$$

A_p reduced $\Leftrightarrow (N_A)_p = 0$

i.e. $\exists\, f_1, \dots, f_n \in A \setminus p$

s.t. $\varepsilon_i \cdot f_i = 0$

$f = \prod_i f_i$ Then $\varepsilon_i \cdot f = 0$

$\therefore (N_A)_f = 0 \quad \therefore A_f$ is reduced.

$\square$