1. $S \in \mathrm{Sch}$.

$$\mathbb{P}^1_S := \mathbb{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S.$$

Show: $\mathbb{P}^1_S = \mathbb{A}^1_S \cup \mathbb{A}^1_S$

$$\begin{tikzcd}
Y^{\pm} \arrow[r] \arrow[d, phantom, "\cap"] & \operatorname{Spec}\mathbb{Z}[T^{\pm 1}] \arrow[d, phantom, "\cap"] \\
Y \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{Z}} \arrow[d] \\
S \arrow[r] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}
\rightsquigarrow \quad Y^{\pm} \longrightarrow \mathbb{A}^1_S$$

$$\begin{tikzcd}
Y^+ \cap Y^- \arrow[r, hook] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & Y^+ \arrow[d] \\
Y^- \arrow[r] & \mathbb{A}^1 \underset{\mathbb{A}^1\setminus 0}{\cup_S} \mathbb{A}^1
\end{tikzcd}
\quad \therefore \quad
\begin{tikzcd}
Y \arrow[d, equal] \\
Y^+ \cup_{Y^{\pm}} Y^- \arrow[d] \\
\left(\mathbb{A}^1 \cup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1\right)_S
\end{tikzcd}.$$

$\therefore$ $\mathbb{A}^1_S \cup_{(\mathbb{A}^1\setminus 0)_S} \mathbb{A}^1_S$ satisfies univ. pr. of $\mathbb{P}^1_S$

$\therefore$ canonically iso. $\square$

$$\mathbb{A}^1_S = S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$$

$$\begin{tikzcd}
\mathbb{P}^1_{\mathbb{Z}} = & \mathbb{A}^1_{\mathbb{Z}} \underset{\mathbb{A}^1\setminus 0}{\cup} & \mathbb{A}^1_{\mathbb{Z}} \\
U_1 \arrow[ur] & & U_2 \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}^1_S \arrow[d, phantom, "\cup"] & \supset & \mathbb{A}^1_S \arrow[r, phantom, "\cong"] & U_1 \times_{\mathbb{Z}} S \\
\mathbb{A}^1_S \arrow[d, equal] & & \text{inters.} \arrow[ll] \arrow[u] \arrow[r, phantom, "\overset{(?)}{\cong}"] & U_{12} \times_{\mathbb{Z}} S. \\
U_2 \times_{\mathbb{Z}} S & & &
\end{tikzcd}$$

X any scheme. $X = U_1 \cup U_2$, $U_i \subset X$ open

Then $X = U_1 \bigcup_{U_1 \cap U_2} U_2$.

$\therefore$ need only find $V_1, V_2 \subset \mathbb{P}^1_S$ s.t. $\mathbb{P}^1_S = V_1 \cup V_2$

$V_i \cong \mathbb{A}^1_S$.

"Fancy" $U \hookrightarrow X$ open imm.

$\Rightarrow U \times_S S' \hookrightarrow X \times_S S'$ open imm. $\qquad V_i = U_i \times_{\mathbb{Z}} S$

NB: $V_1 \cap V_2 = V_1 \times_{\mathbb{P}^1_S} V_2 \overset{\text{formal stuff}}{=} (U_1 \times_{\mathbb{P}^1_{\mathbb{Z}}} U_2) \times_{\mathbb{Z}} S$

$= (U_1 \cap U_2) \times_{\mathbb{Z}} S$

$X \xrightarrow{f} Y$

$f^{-1}(U_i)$ $\quad$ $\{U_i\}$

$\therefore$ open cover $\quad$ open cover

$$\mathbb{P}^1_S \xrightarrow{f} \mathbb{P}^1_{\mathbb{Z}}$$

$\cup$

U_1, U_2

$f^{-1}(U_1) = \mathbb{P}^1_S \times_{\mathbb{P}^1_{\mathbb{Z}}} U_1 = U_1 \times_{\mathbb{Z}} S$.

2. $k = \bar{k}$

$S = \operatorname{Spec} k(t)$

Describe $S \times_k S$.

$\qquad \qquad \shortparallel$

$\operatorname{Spec} k(t) \otimes_k k(s)$

$$\overbrace{\qquad}^{k} \neq k(t,s)$$
!!

Know: $k[t] \underset{k}{\otimes} k[s] = k[t,s]$ $\qquad$ $k[t] \underset{k}{\otimes} A = A[t]$

$$R^{-1}M \otimes N = R^{-1}(M \otimes N)$$

$k(t) = R^{-1}k[t]$ $\qquad$ $R = \langle t-a \mid a \in k \rangle$

$$k(t) \underset{k}{\otimes} k(s) = R^{-1}\underbrace{k[t] \underset{k}{\otimes} k(s)}_{k(s)[t]} = R^{-1}k(s)[t]$$

What is Spec $R^{-1}k(s)[t]$?

$\cap$

Spec $k(s)[t]$ = {gen. pt (0), closed pts (p) | $p \in k(s)[t]$ irred. monic}

({gen. pt. (0), closed pts (p) | $p \in k(s)[t]$ irred., monic, $(p) \not\supseteq (t-a)$ $\forall a \in k$, i.e. p has no roots in k}

ex. $(t-s)$ closed pt is $S \times_k S$ different from gen pt.

$k \to k(s)$

$$\begin{tikzcd}
 & k(s,t) & & \frac{1}{s+t} \arrow[ll] \arrow[r, mapsto] & \frac{1}{S+T} \\
k(t) \arrow[ur] \arrow[rr, squiggly] & & A \arrow[ul, "??"] & s \arrow[d, mapsto] & \\
t \arrow[rr, mapsto] & & T & S &
\end{tikzcd}$$

Set also should $S+T \in A^\times$??

$k(s,t) \neq k[s,t,\frac{1}{s},\frac{1}{t}]$?

$k(s) \neq k[s,s^{-1}]$

3. S/k f.t.

Show closed pts are dense.

Also: give eg. of $S \in Sch$ s.t. closed pts are not dense.

Try to reduce to S affine.

$U \subset S$ open $x \in U$ is closed in U $\Rightarrow$ x closed in S.

discuss later.

Let $S_0 \subset S$ be the set of closed pts.

$\Leftrightarrow$ $S_0 \cap U = U_0$

$\overline{U_0} = U \implies \overline{S_0} = S.$

$\therefore$ WMA $S = \operatorname{Spec} A$, A/k f.t.

closed pts of Spec A $\leftrightarrow$ max. ideals.

[illegible]

STP: $\emptyset \neq D(f)$ contains a closed pt $\forall f$.

Let $f \in A$, $D(f)$ does not contain any max. ideal.

i.e. $f \in m$ $\forall$ max. ideals

$\therefore$ $f \in \underbrace{\bigcap m}_{\substack{= \text{nilradical fact} \\ (\text{Nullstellensatz!})}}$

$\therefore$ f is nilpotent $\therefore D(f) = \emptyset$ are needed.

Example: A s.t. nilrad. $\neq$ Jacobson rad.

any local ring with more than one prime

e.g. $\mathbb{Z}_{(p)}$, $k[[t]]$, ...

Fact X/k f.t. $x \in X$. Then x is closed $\Leftrightarrow k(x)/k$ is finite over k.

Proof x closed, $\operatorname{Spec} A \ni x$ open aff. nbd.

$x \leftrightarrow m_x \subset A$, A/m is finite$/k$ (NST)

$k(x)/k$ finite, $\operatorname{Spec} A \ni x$, show that p_x is maximal.

$$A/p \hookrightarrow k(x)$$

$A/\mathfrak{p}$ is a field b/c

Lemma K/k finite field ext^n,

$k \subset A \subset K$ subalgebra $\Rightarrow A$ a field.

Pf mult^n by $a \neq 0 \in A$ is inj b/c K domain

$\therefore$ surj b/c A is f.d. k-v.s.

$\therefore A$ field. $\square$

Back to original problem: $\emptyset \neq U \subset S$.

RTP that U contains a closed pt of S.

i.e. any pt finite residue field ext^n irrelevant

4. $T \xrightarrow{f} S$ f.t.

S noeth.

Show: T is noeth.

Suppose $S = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$, A_i noeth

f f.t. means that if $\operatorname{Spec} A \subset S$

then $f^{-1}(\operatorname{Spec} A) = \bigcup_{i=1}^{n} B_i$

$A \to B_i$, B_i f.g. as A-algebra.

For A noeth., B_i also is (Hilbert Basis)

∴ $T = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \operatorname{Spec} B_i$ is noeth.

5. Show that $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is not closed.

What ~~bout~~ $\mathbb{P}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$?

$$\begin{tikzcd} Y \arrow[d, phantom, "\parallel"] & & \\ V(x^2+p) \arrow[r, phantom, "\subset"] & \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "f"'] \arrow[r, phantom, "="] & \operatorname{Spec}\mathbb{Z}[X] \arrow[d, phantom, "\cup"] \\ & \operatorname{Spec}\mathbb{Z} & Q = (f, \underset{\text{prime}}{q}) \ni (x^2+p) \\ x \arrow[ur, phantom, "\in" sloped] & & \end{tikzcd}$$

x in the img of f $\iff Y_x \neq \emptyset$

$\iff X^2+p=0$ has a sol^n in $k(x)$.

$x =$ gen.pt. $k(x) = \mathbb{Q}$ ⇝ no sol^n

$x = (q)$ ⇝ sol^n iff $-p$ is a quad. res. mod q

will happen ∞ many times but not always

∴ not closed

$V(5X-1)$ ⇝ same argument but easier