Algebraic Geometry 1

Exercises Tutorium 10

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Exercise 1. Let $\bar{k}$ be an algebraically closed field. Exhibit an equivalence of categories between integral quasi-projective $\bar{k}$-schemes and the category of (quasi-projective) $\bar{k}$-varieties from the first half of the course.

Exercise 2. Let S be a scheme. Show that he category Sch_S admits binary products.

Exercise 3. Let $X \to S \leftarrow Y \in Sch$. Show that there is a canonical map

$$\alpha : |X \times_S Y| \to |X| \times_{|S|} |Y|.$$

For $x \in X, y \in Y$ with common image $s \in S$, show that $\alpha^{-1}(x, y)$ can be identified with

$$|\operatorname{Spec} \kappa(x) \otimes_{\kappa(s)} \kappa(y)|.$$

Exercise 4. Call $f : X \to Y \in Sch$ *locally of finite presentation* if there exists an affine open cover $\{U_i\}_{i \in I}$ of Y and for each $i \in I$ an affine open cover $\{V_{ij}\}_{j \in J_i}$ such that $\mathcal{O}_X(U_i) \to \mathcal{O}_Y(V_{ij})$ is a ring map of finite presentation. (A morphism $A \to B$ of rings is called *of finite presentation* if there exist an isomorphism of A-algebras $B \simeq A[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$ for some $m, n \geq 0$ and $f_i \in A[T_1, \ldots, T_n]$.) Show that the following are equivalent.

(1) $f : X \to Y$ is of locally of finite presentation.
(2) For all open subschemes $V \subset X, U \subset Y$ with $f(V) \subset U$, the induced morphism $V \to U$ is locally of finite presentation.
(3) For all affine open subschemes $V \subset X, U \subset Y$ with $f(V) \subset U$, the induced ring map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is of finite presentation.

[Hint: you may use that if $A \to B$ is a ring homomorphism and $b_1, \ldots, b_n \in B$ with $(b_1, \ldots, b_n) = B$ and each homomorphism $A \to B_{b_i}$ of finite presentation, then $A \to B$ is of finite presentation.]