

Def<sup>y</sup>  $\mathbb{P}^n_{\mathbb{Z}}$  know this

$$\mathbb{P}^n_k = \text{Spec } k \times_{\text{Spec } \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$$

$$\mathbb{P}^n_{\mathbb{Z}} = \bigcup_{i=0}^n A^n_{\mathbb{Z}} \leftarrow \text{approximately glued}$$

$$\mathbb{P}^n_k = \bigcup_{i=0}^n A^n_k$$

$$A^n_{\mathbb{Z}} = \text{Spec } \mathbb{Z}[t_0, \dots, t_n]$$

$$A^n_k = \text{Spec } k[x_0, \dots, x_n] = \text{Spec } k[t_0, \dots, t_n]$$

$X \in \text{Sch}_k$  is called quasi-projective if  
 $\exists$  (loc. closed) immersion  $X \hookrightarrow \mathbb{P}^n_k$  for some  $n$



i.e.  $X$  is open subset of some closed subset of some  $\mathbb{P}^n_k$

$$\mathbb{P}^1 = A^1 \sqcup_{A^1_{(0)}} A^1$$

$$\begin{array}{ccc} \mathbb{Z}[T] & & \mathbb{Z}[U] \\ \downarrow T \mapsto X & & \uparrow U \mapsto Y \\ \mathbb{Z}[X, X^{-1}] & \xrightarrow{\quad} & \mathbb{Z}[Y, Y^{-1}] \end{array}$$

$$\rightsquigarrow \mathbb{P}^2$$

$$\mathbb{P}^n \text{ cover } U_0, U_1, \dots, U_n$$

$$U_i = A^n \text{ coordinates } \mathbb{Z} \left[ \frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \dots, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right]$$

$$A^n_{(0)} \times A^{n-1} \cong U_i \cap U_j \subset U_i$$

$$D\left(\frac{x_j}{x_i}\right) \cap U_i$$

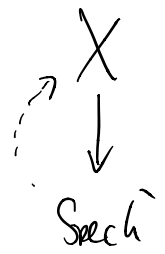
$$\begin{array}{l}
 U_0 = A^{\sim} \leftarrow U_{01} = \dots \\
 U_1 = A^{\sim} \leftarrow U_{12} \\
 \vdots \\
 U_n = A^{\sim}
 \end{array}$$

this yields a string data.

Ex 1.  $\bar{k}$  alg. closed field.

Show:  $(\bar{k}\text{-varieties}) \cong (\text{integral } \mathfrak{p} \text{ in } \bar{k}[x])$

build functor  $F$ .



$$\begin{aligned}
 F(X) &= \text{Hom}_{\text{Spec } \bar{k}}(\text{Spec } \bar{k}, X) \\
 &= \{ (x, \alpha) \mid x \in X, \alpha: R(x) \hookrightarrow \bar{k} \} \\
 &= \{ x \in X \mid R(x) = \bar{k} \} \\
 &\subset X \text{ a subset.}
 \end{aligned}$$

Give this the subset topology.

$$\begin{array}{ccc}
 f: X \rightarrow Y & \text{induces a ch map} & F(X) \xrightarrow{F(f)} F(Y) \\
 \text{So if } x \in X & & \downarrow \cong \\
 R(x) = \bar{k} & & F(R(x)) \cong k(x)
 \end{array}$$

- Need to check:
- $F(X)$  is a  $\bar{k}$ -var. ✓
  - $F(f)$  is a map of  $\bar{k}$ -var. ✓

- $F$  is fully faithful
- $F$  is ess. surj.

Other functor:  $X \leftrightarrow U \hookrightarrow \mathbb{P}_k^4$

$$\downarrow$$

$$u$$

$\mathbb{P}_k^4$  (scheme)  $\neq$   $\mathbb{P}_k^4$  (variety) do not have the same underlying set.

Ex  $\text{Spec } \widehat{k}[T] = \{ (\overline{T-t}, (0)) \mid t \in \widehat{k} \}$

$\uparrow$   
gen. pt

old variety  $A_k^1 \rightarrow \text{MSpec } \widehat{k}[T] = \{ (\overline{T-t}) \mid t \in \widehat{k} \}$

In general: the points of the variety corresponding to  $\widehat{k}[T_1, \dots, T_n] / (f_1, \dots, f_n) = A$   
 $\leftrightarrow$  max. ideals of  $A$

$$\text{MSpec } A \hookrightarrow \text{Spec } A$$

Subspace of int<sup>e</sup> points / closed pts

Another way to define  $F$ : for every integral g.f.  $\widehat{k}$ -sch  $X$ ,

Choose an embedding  $X \hookrightarrow \mathbb{P}^n$   
 Let's call this choice of data  $\mathbb{H}$ .

Define  $F^{\mathbb{H}}: (\text{q.p. int. } \bar{U} \text{ -sch.}) \rightarrow (\bar{U} \text{-varieties})$   
 $X \longmapsto (\text{image of } X \text{ in } \mathbb{P}^n)_{\text{closed points}}$

Now  $F^{\mathbb{H}}(X)$  is ~~not~~ indeed a variety.

Claim:  $X$  irred.  $\Rightarrow F(X)$  irred.

$f: X \rightarrow Y \in \text{Sch}_{\bar{U}}$ . Claim:  $F^{\mathbb{H}}(X) \rightarrow F^{\mathbb{H}}(Y)$  is a mor. of var.

Suppose  $X, Y$  are affine.

$$X = \text{Spec } A$$

$$Y = \text{Spec } B$$

Then  $\text{Hom}_{\bar{U}}(X, Y)$

$$= \text{Hom}_{\bar{U}\text{-alg}}(B, A)$$

$$= \text{Hom}_{\bar{U}\text{-var.}}(F^{\mathbb{H}}(X), F^{\mathbb{H}}(Y))$$

Recall/claim: If  $\bar{X}, \bar{Y}$  are  $\bar{U}$ -varieties then

~~not~~

ch. map  $\bar{f}: \bar{X} \rightarrow \bar{Y}$  is a mor. of varieties

$\Leftrightarrow \exists$  affine cover  $\bar{Y} = \cup \bar{V}_i$ ,  $\bar{f}^{-1}(\bar{V}_i) = \cup \bar{U}_{i,j}$   
 s.t.

$\bar{f}: \bar{U}_{i,j} \rightarrow \bar{V}_i$  is a mor. of var.

Now give  $f: X \rightarrow Y$ , pick affine open cover

$$Y = \cup U_i$$

$$f^{-1}(U_i) = \cup U_{i,j}$$

$$\text{th } F^\#(U_i) = \cup_j F^\#(U_{i,j})$$

$$F^\#(f^{-1}(U_i)) = \cup_j F^\#(U_{i,j})$$

$$F^\#(U_{i,j}) \rightarrow F^\#(U_i)$$

is a mor. of varieties by  $\otimes$

$\therefore F^\#(f)$  is " — by  $\otimes$ .

ess. surj:  $\bar{X} \subset F(\mathbb{P}^n)$

Want:  $X \subset \mathbb{P}^n$

$$X \subset \widehat{\mathbb{P}^n}$$

$$\text{s.t. } X \cap F(\mathbb{P}^n) = \bar{X}.$$

$$\bar{X} = \bar{V}(f_1, \dots, f_r) \cap \bar{V}(g_1, \dots, g_m)$$

$$X = V(f_1, \dots, f_r) \cap V(g_1, \dots, g_m)$$

check this works ( $f_i$  gen. a radical ideal)

fully faithful:  $\text{Hom}_{\bar{K}\text{-var}}(X, Y) \xrightarrow[\text{full}]{\text{faithful}} \text{Hom}_{\bar{K}\text{-var}}(FX, FY)$

Reduce to the affine case as already done.  $\square$

2. Show that  $\text{Sch}_S$  admits binary products.

Given  $X, Y \in \text{Sch}_S$ , want  $P \in \text{Sch}_S$

i.e.  $\begin{array}{c} X \\ \downarrow \\ S \end{array}, \begin{array}{c} Y \\ \downarrow \\ S \end{array} \in \text{Sch}$     i.e.  $\begin{array}{c} P \\ \downarrow \\ S \end{array} \in \text{Sch}$

s.t. ...

Guess:  $P = X \times_S Y \rightarrow Y$     with it canonical map to  $S$ .

$$\begin{array}{ccc} X \times_S Y & \rightarrow & Y \\ \downarrow & & \downarrow \\ X & \rightarrow & S \end{array}$$

Suppose also  $Z \in \text{Sch}_S$ ,  $\begin{array}{ccc} Z & \rightarrow & X \\ \downarrow & \nearrow^{X+Y} & \uparrow \\ Y & \leftarrow & S \end{array} \in \text{Sch}_S$

Let that  $\begin{array}{ccc} Z & \rightarrow & X \\ \downarrow & \nearrow^{X+Y} & \uparrow \\ Y & \leftarrow & S \end{array}$  commutes, so diagram is  $\text{Sch}_S$

$\therefore Z \xrightarrow{\exists!} X \times_S Y$ , check that this is a morphism of  $S$ - $\text{Sch}$ .  $\square$

↑  
of  $S$ - $\text{Sch}$

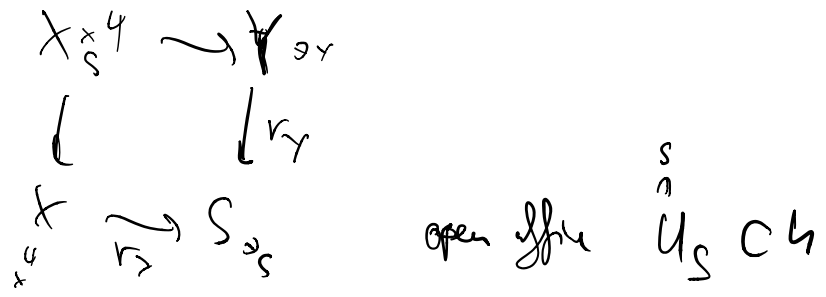
3. Let  $X \rightarrow S \leftarrow Y \in \text{Sch}$ .

Construct  $\left( X \times_S Y \right) \xrightarrow{\alpha} \left( |X| \times_S |Y| \right)$

*fiber prod. of  $\text{sch}$ .*      *means pullback / fiber prod. of top spaces*



First thought:  $X_S$  obtained by gluing together  $\mathcal{O}$  of rings.

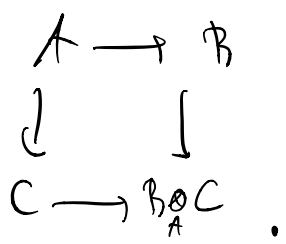


$$\begin{array}{ll}
 x \in U_X \subset X & r \in U_Y \subset Y \\
 r_X(U_X) \subset U_S & r_Y(U_Y) \subset U_S
 \end{array}$$

then  $U_X \times_{U_S} U_Y \subset X_S^x \times Y$   
 is open affine subsh.

contains all points mapping to  $(x, \gamma)$ .

$\therefore$  WMA  $S = \text{Spec } A, X = \text{Spec } B, Y = \text{Spec } C$



Important fact:  $X \leftarrow X \times_Y \text{Spec } k(\gamma) = X_\gamma$

$$\begin{array}{ccc}
 \text{Spec } k(\gamma) & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow \\
 \gamma \in Y & \longleftarrow & \text{Spec } k(\gamma)
 \end{array}$$

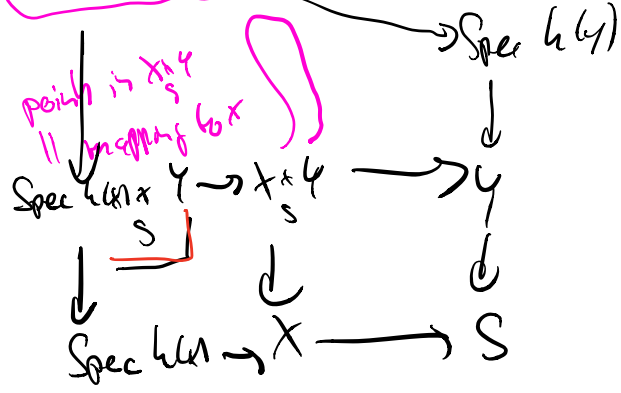
We know that  $|X_\gamma| \subset |X|$  is precisely the fiber over  $\gamma$ .

J.e.  $|X \times_Y \text{Spec } k(\gamma)| = |X| \times | \text{Spec } k(\gamma) |$



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\*

$\text{Spec } k(y) \times_S \text{Spec } k(x)$



point in  $X \times_S Y$   
|| mapping to  $X$

this is the  
Spec that we  
want!  
 $S = \text{Spec } A$

$$\left( \begin{matrix} A & \otimes & C & \otimes & D \\ B & & C & & B \end{matrix} \right) = \begin{matrix} A & \otimes & D \\ B & & B \end{matrix}$$

$$\alpha^{-1}(x, y) = \text{Spec}(k(y) \otimes k(x))$$

$$k(s) \left\{ \begin{matrix} A \\ \mathbb{I} \end{matrix} \right.$$

$$\begin{aligned} \{ t \in X \times_S Y \mid \rho_X(t) = X \} &= \cancel{X \times_S Y} \times \text{Spec } k(x) \\ &= Y \times_S \text{Spec } k(x) \end{aligned}$$