Def^n $\mathbb{P}^n_{\mathbb{Z}}$ know this

$$\mathbb{P}^n_k = \operatorname{Spec} k \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$$

$$\left[\ \mathbb{P}^n_{\mathbb{Z}} = \bigcup_{i=0}^{n} \mathbb{A}^n_{\mathbb{Z}} \leftarrow \text{appropriately glued}\right.$$

$$\left.\mathbb{P}^n_k = \bigcup_{i=0}^{n} \mathbb{A}^n_k\ \right]$$

$$\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t_1,\dots,t_n]$$

$$\mathbb{A}^n_k = \operatorname{Spec} k \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} k[t_1,\dots,t_n]$$

$X \in \mathrm{Sch}_k$ is called quasi-projective if
∃ (loc. closed) immersion $X \hookrightarrow \mathbb{P}^n_k$ for some n

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr] & & \mathbb{P}^n_k \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

i.e. X is open subsch. of some closed subsch. of some $\mathbb{P}^n_k$

$$\mathbb{P}^1 = \mathbb{A}^1 \coprod_{\mathbb{A}^1\setminus 0} \mathbb{A}^1$$

$$\begin{tikzcd} \mathbb{Z}[T] \arrow[dr, "T \mapsto x"'] & & \mathbb{Z}[U] \arrow[dl, "U \mapsto x^{-1}"] \\ & \mathbb{Z}[x,x^{-1}] & \end{tikzcd}$$

$\rightsquigarrow \mathbb{P}^1$

$\mathbb{P}^n$ cover $U_0, U_1, \dots, U_n$

$$U_i = \mathbb{A}^n \text{ coordinates } \mathbb{Z}\left[\frac{x_0}{x_i},\dots,\cancel{\frac{x_i}{x_i}},\dots,\frac{x_n}{x_i}\right]$$

$$\begin{tikzcd} \mathbb{A}^1\setminus 0\times\mathbb{A}^{n-1} \cong U_i\cap U_j \arrow[r, phantom, "\subset" , "D(\frac{x_j}{x_i})"'] \arrow[d, phantom, "\cap", "D(\frac{x_i}{x_j})"'] & U_i \\ U_j & \end{tikzcd}$$

$$\begin{tikzcd}
U_0 = \mathbb{A}^n & \\
 & U_{01} = \dots \arrow[ul] \arrow[dl] \\
U_1 = \mathbb{A}^n & \\
 & U_{12} \arrow[ul] \arrow[dl] \\
\vdots & \\
U_n = \mathbb{A}^n &
\end{tikzcd}$$

this yields a gluing datum.

Ex 1. $\bar{k}$ alg. closed field.

Show: ($\bar{k}$-varieties) $\simeq$ (integral q.-p. $\bar{k}$-sch).

$$\begin{tikzcd}
(\bar{k}\text{-varieties}) & (\text{integral q.-p. } \bar{k}\text{-sch}) \arrow[l, bend left, "F"]
\end{tikzcd}$$

build functor F.

$$\begin{tikzcd}
X \arrow[d] \\
\operatorname{Spec} \bar{k} \arrow[u, dashed, bend left]
\end{tikzcd}$$

$F(X) = \operatorname{Hom}_{Sch_{\bar{k}}}(\operatorname{Spec} \bar{k}, X)$

$= \{(x, \alpha) \mid x \in X,\ \alpha : k(x) \hookrightarrow \bar{k}\}$ with $\overset{\uparrow \simeq}{\bar{k}}$

$= \{x \in X \mid k(x) = \bar{k}\}$

$\subset X$ a subset.

Give this the subset topology.

$f: X \to Y$ induces a cts map $F(X) \xrightarrow{F(f)} F(Y)$.

b/c if $x \in X$, $k(x) = \bar{k}$: $\bar{k} \downarrow k(f(x)) \Rightarrow k(x)$

Need to check:
- $F(X)$ is a $\bar{k}$-var. ✓
- $F(f)$ is a mor of $\bar{k}$-var. ✓

- F is fully faithful
- F is ess. surj.

Other functor:
$$\begin{tikzcd} X \arrow[r, mapsto] & U \arrow[r, hook] & \mathbb{P}^n_k \arrow[d] \\ & & k \end{tikzcd}$$

$\mathbb{P}^n_k \neq \mathbb{P}^n_k$ do not have the same underlying sets.

scheme variety

Ex $\operatorname{Spec} \bar{k}[T] = \{ (T-t), (0) \mid t \in \bar{k} \}$

$\nearrow$ gen. pt

old variety $\mathbb{A}^1_k$ $\rightarrow$ $\operatorname{MSpec} \bar{k}[T] = \{ (T-t) \mid t \in \bar{k} \}$

In general: the points of the variety corresponding to $\bar{k}[T_1, \dots, T_n]/(f_1, \dots, f_m) = A$

$\longleftrightarrow$ max. ideals of A

$\operatorname{MSpec} A \hookrightarrow \operatorname{Spec} A$

subspace of ratl points/closed pts

Another way to define F: for every integral q.p. $\bar{k}$-scheme X, choose an embedding $X \overset{i_X}{\hookrightarrow} \mathbb{P}^{n_X}$.

Let's call this choice of data $\oplus$.

Define $F^{\oplus}$: (q.p. int. $\bar{k}$-sch.) $\longrightarrow$ ($\bar{k}$-varieties)

$$X \longmapsto \left(\text{image of } X \text{ in } \mathbb{P}^{n_X}\right) \cap \text{closed points}$$

Know $F^{\oplus}(X)$ is* indeed a variety.

Claim: X irred. $\Rightarrow$ $F(X)$ irred.

$f: X \to Y \in Sch_{\bar{k}}$. Claim: $F^{\oplus}(X) \to F^{\oplus}(Y)$ is a mor. of var.

Suppose X, Y are affine.

$X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Then

$$\left.\begin{aligned} &\operatorname{Hom}_{\bar{k}}(X, Y) \\ &= \operatorname{Hom}_{\bar{k}\text{-alg}}(B, A) \\ &= \operatorname{Hom}_{\bar{k}\text{-var.}}(F^{\oplus}(X), F^{\oplus}(Y)) \end{aligned}\right\} ⊛$$

Recall/claim: If $\bar{X}, \bar{Y}$ are $\bar{k}$-varieties then

(**) ch. map $\bar{f}: \bar{X} \to \bar{Y}$ is a mor. of varieties

$\Leftrightarrow$ $\exists$ affine cover $\bar{Y} = \bigcup \bar{V}_i$, $\bar{f}^{-1}(\bar{V}_i) = \bigcup \bar{U}_{ij}$ s.t.

$\bar{f}: \bar{U}_{ij} \to \bar{V}_i$ is a mor. of var.

Now fix $f: X \longrightarrow Y$, pick affine open cover

$$Y = \bigcup U_i$$

$$f^{-1}(U_i) = \bigcup U_{ij}$$

then

$$F^{\textcircled{\#}}(Y) = \bigcup_i F^{\textcircled{\#}}(U_i)$$

$$F^{\textcircled{\#}}(f^{-1}(U_i)) = \bigcup_j F^{\textcircled{\#}}(U_{ij})$$

$$F^{\textcircled{\#}}(U_{ij}) \longrightarrow F^{\textcircled{\#}}(U_i)$$

is a mor. of varieties by $\circledast$

$\therefore$ $F^{\textcircled{\#}}(f)$ is " — by $\textcircled{**}$.

---

ess. surj: $\bar{X} \subset F(\mathbb{P}^n)$ Want: $X \subset \mathbb{P}^n$

$$\begin{array}{ccc} & & F(\mathbb{P}^n) \\ & & \cap \\ X & \subset & \mathbb{P}^n \end{array}$$

s.t. $X \cap F(\mathbb{P}^n) = \bar{X}$.

$$\bar{X} = \bar{V}(f_1, \dots, f_n) \setminus \bar{V}(g_1, \dots, g_m)$$

$$X = V(f_1, \dots, f_n) \setminus V(g_1, \dots, g_m)$$

check this works (f_i gen. a radical ideal)

fully faithful: $\mathrm{Hom}_{k\text{-Sch}}(X, Y) \underset{\text{full}}{\overset{\text{faithful}}{\hookrightarrow}} \mathrm{Hom}_{k\text{-Var}}(FX, FY)$

Reduce to the affine case as already done. "$\square$"

2. Show that Sch_S admits binary products.

Given $X, Y \in \mathrm{Sch}_S$, want $P \in \mathrm{Sch}_S$

i.e. $\begin{matrix} X \\ \downarrow \\ S \end{matrix}, \begin{matrix} Y \\ \downarrow \\ S \end{matrix} \in \mathrm{Sch}$ i.e. $\begin{matrix} P \\ \downarrow \\ S \end{matrix} \in \mathrm{Sch}$

s.t. ....

Guess:

$$\begin{tikzcd} P = X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

with its canonical map to S.

Suppose given $Z \in \mathrm{Sch}_S$,

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, dashed, "\exists!?"] & X \\ Y & X \times Y \arrow[l] \arrow[u] \end{tikzcd} \in \mathrm{Sch}_S$$

know that

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d] \\ Y \arrow[r] & S \\ & X \times_S Y \arrow[ul] \arrow[uu] \end{tikzcd}$$

commutes, so a diagram in Sch_S

$\therefore$ $Z \xrightarrow{\exists!} X \times_S Y$, check that this is a morphism of S-sch.

of schemes

$\square$

3. Let $X \longrightarrow S \longleftarrow Y \in \mathrm{Sch}$.

Construct $|X \times_S Y| \xrightarrow{\alpha} |X| \times_{|S|} |Y|$

fiber prod. of sch.

means pullback / fiber prod. of top. spaces

& show that for (x,y) we have

$$\alpha^{-1}(x,y) \cong \left|\operatorname{Spec} k(x) \otimes_{k(s)} k(y)\right| .$$

$\subset X \times Y$

$X \times_S Y$ fibr over s is $X_s \times Y_s$

$$\begin{tikzcd} X \times_S Y \arrow[d] \\ S \ni s \end{tikzcd}$$

Part 0: If $X \xrightarrow{f} S \xleftarrow{g} Y \in \mathrm{Top}$

then $X \times_S Y \hookrightarrow X \times Y$ prod. top.

$\quad\quad \| $

$\{(x,y) \mid f(x) = g(y)\}$

Part 1: $s \in |\overbrace{X \times_S Y}^{\Pi}|$

$$\begin{tikzcd} {|X \times_S Y|} \arrow[r, "|p_X|"] \arrow[d, "|p_Y|"'] \arrow[dr, "\exists! \alpha"] & {|X|} \arrow[d] \\ {|Y|} \arrow[r] & {|S| \ni t} \\ & {|X| \times_{|S|} |Y|} \end{tikzcd}$$

$|X| \times_{|S|} |Y| \xrightarrow{q_X} |X|$, $\quad |X| \times_{|S|} |Y| \xrightarrow{q_Y} |Y|$, $\quad s \mapsto t$

Part 2: $q_X \alpha(s) = p_X(s)$ $\quad$ Similarly $p_Y(s) = y$.

$\quad\quad \|$

$\quad\quad x$

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "p_X^\#"] & \mathcal{O}_{\Pi,s} \\ \mathcal{O}_{S,t} \arrow[u] \arrow[r] & \mathcal{O}_{Y,y} \arrow[u, "p_Y^\#"'] \end{tikzcd}$$

$$\begin{tikzcd} k(x) \arrow[r] \arrow[dr] & k(s) \\ & k(x) \otimes_{k(t)} k(y) \arrow[u, dashed, "\exists! ?\beta"'] \\ k(t) \arrow[uu] \arrow[r] & k(y) \arrow[uu] \arrow[u] \end{tikzcd}$$

will not work b/c

$k(x) \otimes_{k(t)} k(y)$ need not be a field

First thought: $\times_S$ obtained by gluing together $\otimes$ of rings.

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \ni y \arrow[d, "r_Y"] \\ x \in X \arrow[r, "r_X"'] & S \ni s \end{tikzcd}$$

open affine $\overset{s}{\overset{\cap}{U_S}} \subset Y$

$x \in U_X \subset X$ $\quad$ $y \in U_Y \subset Y$

$r_X(U_X) \subset U_S$ $\quad$ $r_Y(U_Y) \subset U_S$

then $U_X \times_{U_S} U_Y \subset X \times_S Y$

is open affine subsch.,

contains all points mapping to (x, y).

$\therefore$ WMA $\quad S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$

$Y = \operatorname{Spec} C$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \otimes_A C \end{tikzcd}$$

.

Important fact:

$$\begin{tikzcd} X \arrow[d] & X \times_Y \operatorname{Spec} k(y) = X_y \arrow[l] \arrow[d] \\ y \in Y & \operatorname{Spec} k(y) \arrow[l] \end{tikzcd}$$

$\mathrm{Sch} \ni$

we know that $|X_y| \subset |X|$ is precisely the fiber over y.

I.e. $|X \times_Y \operatorname{Spec} k(y)| = |X| \times |\operatorname{Spec} k(y)|$

$|Y|$ *

$$\begin{tikzcd}
\operatorname{Spec} k(y) \times_S \operatorname{Spec} k(x) \arrow[rr] \arrow[d] & & \operatorname{Spec} k(y) \arrow[d] \\
\operatorname{Spec} k(x) \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
\operatorname{Spec} k(x) \arrow[r] & X \arrow[r] & S
\end{tikzcd}$$

points in $X \times_S Y$

$\|$ mapping to x

this is the Spec that we want!

$S = \operatorname{Spec} A$

$\alpha^{-1}(x,y) = |\operatorname{Spec}(k(y) \otimes k(x))|$

$(A \otimes_B C \otimes_C D = A \otimes_B D)$

$k(s) \left\{ \begin{matrix} A/I \\ \\ \end{matrix} \right.$

$$\{t \in X \times_S Y \mid p_X(t) = x\} = (\cancel{X} \times_S Y) \times_{\cancel{X}} \operatorname{Spec} k(x)$$

$$= Y \times_S \operatorname{Spec} k(x)$$