Ex 1

$\mathbb{A}^n(\mathbb{C})$ — class. top. / — Zariski top. } when the same?

$n=0$: true
$n>0$: no

$\left[\begin{array}{ll} U \subset \mathbb{A}^{n-1}(\mathbb{C}) & \text{open in std., not in Zar.} \\ \Rightarrow U \times \mathbb{C} \subset \mathbb{A}^n(\mathbb{C}) & \text{— '' —} \end{array}\right]$

[open Ball of radius 1 $\subset \mathbb{A}^n(\mathbb{C})$
claim: not Zariski open]

$Z(x_2, \dots, x_n) = Z \subset \mathbb{A}^n(\mathbb{C})$

$\mathbb{A}^1(\mathbb{C}) \cong Z = \{(x_1, 0, \dots, 0)\}$

for either std. or Zar. top.

$\therefore$ $\mathbb{A}^n(\mathbb{C})$ zar = std.
$\Downarrow$
$\mathbb{A}^1(\mathbb{C})$ zar = std

$U \subset \mathbb{A}^n(\mathbb{C}) \xleftarrow{i} \mathbb{A}^1(\mathbb{C})$: $(x_1, \dots, x_n) \mapsto x_1$

$i^{-1}(U)$

$\downarrow p$

$p(U) \subset \mathbb{A}^1(\mathbb{C})$

Extra question: Z alg set
zar. top. = class. $\iff Z$ finite
(Z infinite $\Rightarrow \exists$ infinite closed subset with cofinite topology)

Ex 2

Is product top. on $\mathbb{A}^n(\mathbb{C}) \times \mathbb{A}^m(\mathbb{C})$ the same as Zar. top. on $\mathbb{A}^{n+m}(\mathbb{C})$?

Ans: no(!)

$\left[\begin{array}{ll} Z \subset \mathbb{A}^{n+m} & Z = Z(f_1, \dots, f_k) \\ \text{closed} & \text{is it closed in prod. top.?} \end{array}\right]$

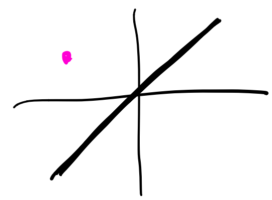
Reminder: product topology on $\mathbb{A}^n \times \mathbb{A}^m$
is the smallest topology in which
$U \times V$ is open $\forall\, U \subset \mathbb{A}^n$ both
$V \subset \mathbb{A}^m$ open.

$n = 0$ or $m = 0$: yes

else: no

$n = 1 = m$: $Z = Z(y - x)$

$U = \mathbb{A}^2 \setminus Z$



Let $U_1 \subset \mathbb{A}^1$ both open
$U_2 \subset \mathbb{A}^1$

want: $U_1 \times U_2 \subset U$

i.e.: $U_1 \cap U_2 = \emptyset$ $\left(\begin{array}{l} x \in U_1, y \in U_2 \\ (x,y) \in U \iff x \neq y \end{array} \right)$

for which U_1, U_2 is this possible?

only if $U_1 = \emptyset$ or $U_2 = \emptyset$

and then $U_1 \times U_2 = \emptyset$

$\therefore$ U is <u>not</u> open in product topology

$n, m \geq 1$: $\mathbb{A}^2 \subset \mathbb{A}^{n+m}$

$$\begin{array}{ccc} \mathbb{A}^2 & \subset & \mathbb{A}^{n+m} \\ \# & & \therefore \# \\ \mathbb{A}^1 \times \mathbb{A}^1 & \subset & \mathbb{A}^n \times \mathbb{A}^m \end{array}$$

Ex 3 Show: $Z \subset \mathbb{A}^n(k)$ algebraic

$\Longrightarrow Z$ is T_1.

$x \neq y \in \mathbb{A}^n$ $\quad f(T_1, \dots, T_n) = \prod_{\substack{i=1 \\ x_i \neq y_i}}^{n} (T_i - y_i)$

$f(x) \neq 0$

$f(y) = 0$

$U = \mathbb{A}^n \setminus Z(f) = D(f)$

$x \in$ but $y \notin U$

$\therefore \mathbb{A}^n$ is T_1

<u>LA</u> X T_1-space, $Z \subset X$ arbitrary

$\Rightarrow Z$ is T_1

<u>LA</u> If pts of X are closed, then X is T_1.

(b/c if $x \neq y \in X$, $X \setminus y \subset X$ open, sep. x, y)

<u>LA</u> Points in $\mathbb{A}^n(k)$ are closed.

$x = (x_1, \dots, x_n)$ $\quad I \langle (T_i - x_i)_i \rangle$

$Z(I) = \{x\}$.

NB: $Z \subset \mathbb{A}^n(k)$ Hausdorff $\Rightarrow Z$ finite

Ex 4 (1) top. space X Hausdorff

$\Longleftrightarrow \Delta(X) \subset X \times X$ is closed

(2) $\Delta(\mathbb{A}^n(k)) \subset \mathbb{A}^{n+n}(k)$ is closed

Is $\mathbb{A}^n(k)$ Hausdorff?

(1)

VII. X Hausdorff, $x \neq y \in X$

U_x, U_y sep. nbhds $\quad (x,y) \in X \times X$

$U_x \cap U_y = \emptyset \Rightarrow U_x \times U_y \subset X \times X \setminus \Delta \quad \therefore X \times X \setminus \Delta$ is open

$\therefore \Delta$ is closed

$\Delta(X)$ closed, $x \neq y \in X$

$(x,y) \in X \times X \setminus \Delta$

then $\exists U, V \quad U \times V \subset$

$U, V \subset X$ open $\quad \Rightarrow U \cap V = \emptyset$

$\therefore X$ Hausdorff

(2) $\Delta(\mathbb{A}^n_{(k)}) = \bigcap_{i=1}^{n} Z(X_i - X_{n+i})$

$\quad ||$ $\quad$ for $X_1, \dots, X_n, X_{n+1}, \dots, X_{2n}$ coord. on $\mathbb{A}^{2n}$

$Z(X_1 - X_{n+1}, \dots, X_{n-1} - X_{2n})$

$\not\Rightarrow \mathbb{A}^n(k)$ is Hausdorff b/c top. on $\mathbb{A}^{n+n}$ is <u>not</u> the product top.

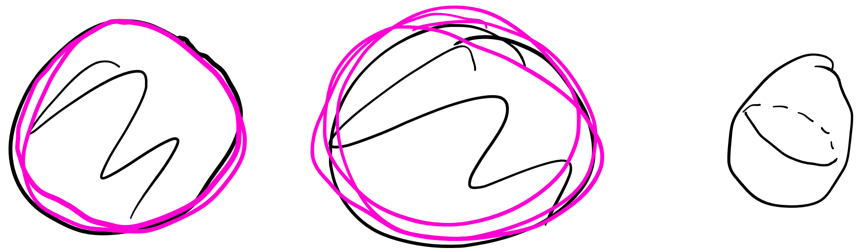
X variety (e.g. alg. set)

$X \times X$ variety (not product top.)

Say that X is separated if

$\Delta : X \to X \times X$ is closed

we have seen: $\mathbb{A}^n$ is separated



ex: affine line with double pt

not separated

Recall: Category $\mathcal{C}$ has objects $X, Y, \dots \in \mathcal{C}$

has morphisms $\mathrm{Hom}(X,Y)$ ← a set

+ composition $\forall X, Y$ obj. of $\mathcal{C}$

+ axioms

ex: Cat of sets, ob = sets
mor = maps of sets

of top. spaces, ob = top. spaces
mor = continuous maps

cat. of algebraic sets, ob = algebraic sets
$(Z, Z \hookrightarrow \mathbb{A}^n(k))$

mor = morphisms of alg. sets
$(Z \hookrightarrow \mathbb{A}^n : p_i(x_1, \dots, x_n))$
$W \hookrightarrow \mathbb{A}^m \quad i = 1, \dots, m$
$\rightsquigarrow \mathbb{A}^n \to \mathbb{A}^m$

Universal construction: binary product
$\mathcal{C}$ a category, $X, Y \in \mathcal{C}$.

Def^n A *product* of $X, Y \in \mathcal{C}$ is an object $Z \in \mathcal{C}$ together with morphisms $Z \xrightarrow{p_1} X$, $Z \xrightarrow{p_2} Y$

s.t.: for any object $T \in \mathcal{C}$ & morphisms $f_1 : T \longrightarrow X$, $f_2 : T \longrightarrow Y$

$\exists !\ f : T \longrightarrow Z$ s.t. the following commutes:

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists ! f"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & Z \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

Rmk - products may or may not exist

- if they exist, they are unique up to unique isomorphism

ex: $\mathcal{C} = \mathrm{Sets}$

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & X \times Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\ & Y & \end{tikzcd}$$

is a product of X & Y

Sets

$$\begin{tikzcd} X \times Y \times \{*\} \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

is also a product

Claim $X \hookrightarrow \mathbb{A}^n(k)$ $\quad X = Z(f_1, \dots, f_r)$

$Y \hookrightarrow \mathbb{A}^m(k)$ $\quad Y = Z(g_1, \dots, g_s)$

$X \times Y \hookrightarrow \mathbb{A}^{n+m}(k)$ is an algebraic set, the projections are maps of algebraic sets, & this yields the product in cat. of alg. sets.

$$\begin{tikzcd}
X \arrow[dr, "\exists!\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times X \arrow[r] \arrow[d] & X \\
& X &
\end{tikzcd}$$