AN EXAMPLE CONCERNING SPECIALIZATION OF TORSION SUBGROUPS OF CHOW GROUPS

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Abstract. We give examples where the specialization map between the ℓ-primary torsion subgroups of Chow groups is not injective for almost every prime ℓ.

1. Introduction

Let $S = \operatorname{Spec}(A)$ be an affine, regular, integral scheme of finite type over $\operatorname{Spec}(\mathbb{Z})$ (or a localization of such a scheme) and let $f : \mathcal{V} \to S$ be a smooth, projective morphism with geometrically connected n-dimensional fibers. Consider the diagram

$$\begin{tikzcd} V_{\overline{s}} \arrow[r] \arrow[d] & V_s \arrow[r] \arrow[d] & \mathcal{V} \arrow[d, "f"'] & V_\eta \arrow[l] \arrow[d] & V_{\overline{\eta}} \arrow[l] \arrow[d] \\ \overline{s} \arrow[r, "\nu"] & s \arrow[r, "i_s"] & S & \eta \arrow[l, "g"'] & \overline{\eta} \arrow[l, "\epsilon"'] \end{tikzcd}$$

where g is the inclusion of the generic point $\eta = \operatorname{Spec}(K)$, i_s is the inclusion of a non-generic point $s = \operatorname{Spec}(\mathbb{F})$, and the maps ν and ϵ are the base change morphisms to algebraic closures of $\mathbb{F}$ and K.

If ℓ is a prime number, there is a specialization map on ℓ-primary torsion subgroups of Chow groups

$$\sigma^r_{\overline{s}} : \mathrm{CH}^r(V_{\overline{\eta}})[\ell^\infty] \to \mathrm{CH}^r(V_{\overline{s}})[\ell^\infty],$$

for the construction of this map in case $\operatorname{codim}_S s > 1$, see [7, pg.12]. If ℓ is prime to the characteristic of $\mathbb{F}$, this map is known to be injective in codimensions $r = 1, 2, n$; this is classical for divisors, and follows in the remaining cases from work of Bloch, Roitman and Merkurjev-Suslin [7, Proposition]. In [7] Schoen has given examples where for ℓ prime to the characteristic of $\mathbb{F}$ the specialization map $\sigma^r_{\overline{s}}$ is *not* injective in the range $2 < r < n$; in these examples $\ell \in \{5, 7, 11, 13, 17\}$. Similar examples, for finitely many primes ℓ, have been obtained by Soulé-Voisin in [9]; in their examples the cycles are external products of the Kollar cycles giving counterexamples to the integral Hodge conjecture with the cycle of an ℓ-torsion line bundle. We show

Theorem 1. *For all $2 < r < n$ there are examples where the map*

$$\sigma^r_{\overline{s}} : \mathrm{CH}^r(V_{\overline{\eta}})[\ell^\infty] \to \mathrm{CH}^r(V_{\overline{s}})[\ell^\infty]$$

is not injective for all but finitely many primes ℓ prime to $\operatorname{char}(\mathbb{F})$.

We remark that our examples cover the case of mixed characteristic (as Schoen's examples in [7]) as well as the case of equal characteristic 0.

2. Proof of the Theorem

Proof. The proof is similar to the proof given in [7], using our results from [4]; to make this note self-contained, we give the complete argument.

If X is a smooth projective variety over an algebraically closed field of characteristic prime to ℓ, Bloch [1, 2.7] has constructed a map

$$\lambda^r_X : \mathrm{CH}^r(X)[\ell^\infty] \to \mathrm{H}^{2r-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)).$$

For $f : \mathcal{V} \to S$ as above, this map is compatible with the cospecialization isomorphism $c_{\overline{s}}$ in étale cohomology, i.e. we have a commutative square

$$\begin{array}{ccc}
\mathrm{CH}^r(V_{\overline{\eta}})[\ell^\infty] & \xrightarrow{\lambda^r_{V_{\overline{\eta}}}} & \mathrm{H}^{2r-1}_{\text{ét}}(V_{\overline{\eta}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \\
{\scriptstyle \sigma^r_{\overline{s}}}\downarrow & & \cong\downarrow{\scriptstyle c^{-1}_{\overline{s}}} \\
\mathrm{CH}^r(V_{\overline{s}})[\ell^\infty] & \xrightarrow{\lambda^r_{V_{\overline{s}}}} & \mathrm{H}^{2r-1}_{\text{ét}}(V_{\overline{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))
\end{array}$$

Thus to give non-trivial cycles in $\ker \sigma^r_{\overline{s}}$ one needs cycles in $\ker \lambda^r_{V_{\overline{\eta}}}$. Such cycles have been constructed using external products as follows: Let $n = 4$ and $r = 3$. There are examples of smooth projective 3-folds W, defined over an algebraically closed subfield $\overline{k} \subseteq \overline{K}$ (here $\overline{\eta} = \mathrm{Spec}(\overline{K})$ and $\mathrm{char}(\overline{k}) = 0$), such that $\mathrm{CH}^2(W_{\overline{k}}) \otimes \mathbb{Z}/\ell$ is infinite; this holds for the triple product of a general elliptic curve and the primes $\ell \in \{5, 7, 11, 13, 17\}$ [5, Theorem 0.1], and for the generic abelian 3-fold and all but finitely many primes ℓ [4, Theorem 1.2]. Given this, it follows that $\mathrm{CH}^2(W_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ has infinite corank. If Y is an elliptic curve with j-invariant $j(Y) \notin \overline{k}$, let $\overline{K} = \overline{k(j(Y))}$ and $V_{\overline{\eta}} = W_{\overline{k}} \times_{\overline{k}} Y_{\overline{K}}$. Because the external product map

$$\mathrm{CH}^2(W_{\overline{k}}) \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus 2} \cong \mathrm{CH}^2(W_{\overline{k}}) \otimes \mathrm{Pic}(Y_{\overline{\eta}})[\ell^\infty] \xrightarrow{\times} \mathrm{CH}^3(V_{\overline{\eta}})[\ell^\infty]$$

is injective [6, Proposition 0.2], it follows that $\mathrm{CH}^3(V_{\overline{\eta}})[\ell^\infty]$ has infinite corank and $\ker \lambda^3_{V_{\overline{\eta}}}$ is non-trivial. Note that by rigidity [3, 3.9] we can replace in the external product above $\mathrm{CH}^2(W_{\overline{k}})$ by $\mathrm{CH}^2(W_{\overline{\eta}})$. For $n > 4$ one gets analogous statements replacing $W_{\overline{k}}$ by $W_{\overline{k}} \times \mathbb{P}^{n-4}_{\overline{k}}$ [2, 3.3(b)].

The idea is now to take such a product cycle $z \times \tau \in \mathrm{CH}^3(W_{\overline{\eta}} \times Y_{\overline{\eta}})[\ell^\infty]$, $z \in \mathrm{CH}^2(W_{\overline{\eta}})$ and $\tau \in \mathrm{Pic}(Y_{\overline{\eta}})[\ell^\infty]$, and arrange that z specializes to a torsion point. Then the divisibility of τ will imply that $z \times \tau$ specializes to zero.

To achieve this, one needs the relative version of the above external product considered by Schoen. Let ℓ be a prime and let k_0 be a field of characteristic prime to ℓ. Assume W is a geometrically connected, smooth projective variety of dimension $n - 1$ over k_0. Let $A_0 \subset k_0$ be a regular integral domain of finite type over $\mathbb{Z}$ with quotient field k_0. By localizing we may assume that $\ell \in A_0^\times$, and that W extends to a smooth, projective morphism $h : \mathcal{W} \to S_0 = \mathrm{Spec}(A_0)$. Let $k_0 \subset K$ be a field extension of transcendence degree 1 and let Y be an elliptic curve whose j-invariant $j(F) \in K$ is transcendental over k_0. Assume Y has a torsion point ζ_K of exact order ℓ^m. Let A be a regular integral domain, flat and of finite type over A_0 whose quotient field is isomorphic to K. Localizing we may arrange that Y/K extends to an abelian scheme $\mathcal{Y} \to S = \mathrm{Spec}(A)$ with identity section e. Write ζ for the ℓ^m-torsion section extending ζ_K. The composition

$$f : \mathcal{V} := \mathcal{W} \times_{S_0} \mathcal{Y} \to \mathcal{Y} \to S$$

defines a smooth projective morphism of relative dimension n with connected fibers. If $s \in S$ is an arbitrary point and s_0 is the image of s in S_0, the fibers

$$\begin{array}{rclclcl} V_\eta & = & \mathcal{W} \times_{S_0} \mathcal{Y} \times_S \eta & \cong & \mathcal{W} \times_{S_0} \mathcal{Y}_\eta & \cong & W \times_{k_0} \mathcal{Y}_\eta \\ V_s & = & \mathcal{W} \times_{S_0} \mathcal{Y} \times_S s & \cong & \mathcal{W} \times_{S_0} \mathcal{Y}_s & \cong & \mathcal{W}_{s_0} \times_{s_0} \mathcal{Y}_s \end{array}$$

have product structures.

If $\mathcal{X}$ is a scheme over T, define $Z^p_{fl}(\mathcal{X})$ as the subgroup of the free abelian group of codimension p cycles on $\mathcal{X}$ whose components are flat over T

$$Z^p_{fl}(\mathcal{X}) = \{\sum n_i \mathcal{Z}_i \in Z^p(\mathcal{X}) \mid \text{each subscheme } \mathcal{Z}_i \text{ is flat over } T\},$$

and let $\mathrm{CH}^p_{fl}(\mathcal{X})$ be the image of $Z^p_{fl}(\mathcal{X})$ in $\mathrm{CH}^p(\mathcal{X})$. Let $\overline{k}_0$ be an algebraic closure of k_0. Schoen constructs the following commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^{r-1}(\mathcal{W}_{\overline{s}_0}) \otimes \mathrm{CH}^1(\mathcal{Y}_{\overline{s}}) & \xrightarrow{\times} & \mathrm{CH}^r(\mathcal{W}_{s_0} \times_{s_0} \mathcal{Y}_{\overline{s}}) \\ {\scriptstyle i^!_W \otimes i^!_Y}\big\uparrow & & \big\uparrow{\scriptstyle i^!_V} \\ \mathrm{CH}^{r-1}_{fl}(\mathcal{W}) \otimes \mathrm{CH}^1_{fl}(\mathcal{Y}) & \xrightarrow{\times} & \mathrm{CH}^r_{fl}(\mathcal{W} \times_{S_0} \mathcal{Y}) \\ {\scriptstyle j^*_W \otimes j^*_Y}\big\downarrow & & \big\downarrow{\scriptstyle j^*_V} \\ \mathrm{CH}^{r-1}(W_{\overline{k}_0}) \otimes \mathrm{CH}^1(\mathcal{Y}_{\overline{\eta}}) & \xrightarrow{\times} & \mathrm{CH}^r(\mathcal{W}_{\overline{k}_0} \times_{\overline{k}_0} \mathcal{Y}_{\overline{\eta}}) \end{array}$$

where the horizontal maps are the external products from [2, 1.10 and 20.2] and [6, 1.1] (the middle row requires flatness), the j^* are flat pullbacks, and the $i^!$ are intersections with a geometric closed fiber. The restriction of the bottom product map to $\mathrm{CH}^r(\mathcal{W}_{\overline{s}_0}) \otimes \mathrm{CH}^1(\mathcal{Y}_{\overline{\eta}})[\ell^\infty]$ is injective [6, 0.2].

Let $\mathcal{T} = \zeta - e$. Schoen gives the following criterion for a cycle to define a non-trivial element in $\ker \sigma^r_{\overline{s}}$ [7, Lemma 2]: Assume we are given a cycle $\mathcal{Z} \in \mathrm{CH}^{r-1}_{fl}(\mathcal{W})$ with the following two properties

(i) $j^*_W(\mathcal{Z}) \otimes 1/\ell^m$ has exact order ℓ^m in $\mathrm{CH}^{r-1}(W_{\overline{k}_0}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$, and

(ii) $i^!_W(\mathcal{Z}) \in CH^{r-1}(\mathcal{W}_{\overline{s}_0})_{tors}$.

Then the cycle $j^*_W(\mathcal{Z}) \times j^*_Y(\mathcal{T})$ has exact order ℓ^m in $\mathrm{CH}^r(V_{\overline{\eta}})[\ell^\infty]$ and its image $\sigma^r_{\overline{s}}(j^*_W(\mathcal{Z}) \times j^*_Y(\mathcal{T}))$ is trivial in $\mathrm{CH}^r(V_{\overline{s}})[\ell^\infty]$.

Here the first condition (i) implies that $j^*_W(\mathcal{Z}) \otimes j^*_Y(\mathcal{T})$ has exact order ℓ^m in $\mathrm{CH}^{r-1}(\mathcal{W}_{\overline{k}_0}) \otimes \mathrm{CH}^1(\mathcal{Y}_{\overline{\eta}})[\ell^\infty]$, so that $j^*_W(\mathcal{Z}) \times j^*_Y(\mathcal{T})$ has exact order ℓ^m because of the injectivity of the external product map quoted above.

Furthermore, we have $j_W^*(\mathcal{Z}) \times j_Y^*(\mathcal{T}) = j_V^*(\mathcal{Z} \times \mathcal{T})$ and by intersection theory $\sigma_{\overline{s}}^r(j_V^*(\mathcal{Z} \times \mathcal{T})) = i_V^!(\mathcal{Z} \times \mathcal{T})$ [7, Lemma 1]. Since the top square in the above diagram commutes, it suffices to show

$$i_W^!(\mathcal{Z}) \otimes i_Y^!(\mathcal{T}) = 0 \text{ in } \mathrm{CH}^{r-1}(\mathcal{W}_{\overline{s}_0}) \otimes \mathrm{CH}^1(\mathcal{Y}_{\overline{s}}).$$

This follows from condition (ii), if $i_W^!(\mathcal{Z})$ is a torsion cycle, $i_W^!(\mathcal{Z}) \otimes i_Y^!(\mathcal{T}) = 0$ since $\mathrm{CH}^1(\mathcal{Y}_{\overline{s}})[\ell^\infty]$ is a divisible group.

The results of [4] show that this criterion is fulfilled for all but finitely many primes ℓ for the Ceresa cycle on the Jacobian of the generic curve of genus 3 over $\mathbb{C}$. Recall that if $\mathrm{J}(C)$ is the Jacobian of a smooth projective complex curve of genus 3, the Ceresa cycle Z is defined as

$$Z = \rho_*(C) - [-1] * \rho_*(C),$$

where c_0 is a base point, $\rho : C \to \mathrm{J}(C), c \mapsto c - c_0$ is the canonical embedding and $[-1]_*$ is the morphism on cycle groups induced by the involution on $\mathrm{J}(C)$. The Ceresa cycle is homologically trivial and defines a class $[Z]$ in the Griffiths group $\mathrm{Griff}^2(\mathrm{J}(C))$ which is independent of the choice of c_0. By [4, Theorem 1.1] the image of $[Z]$ in $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)) \otimes \mathbb{Z}/\ell = \mathrm{Griff}^2(\mathrm{J}(C)) \otimes \mathbb{Z}/\ell$ is non-trivial for all but finitely many primes ℓ.

Let k_0 be a finitely generated field such that Z and C are defined over k_0, and its image in $\mathrm{CH}^2_{\mathrm{hom}}(\mathrm{J}(C)_{\overline{k}_0}) \otimes \mathbb{Z}/\ell$ is non-zero for almost every ℓ. Set $W = \mathrm{J}(C)_{k_0}$ and choose $A_0 \subseteq k_0$ as above such that C, $\mathrm{J}(C)$ and ρ extend over $S_0 = \mathrm{Spec}(A_0)$. If $\mathcal{Z} \in Z^2_{fl}(\mathcal{W})$ is the extension of Z, then $j_W^*(\mathcal{Z}) \otimes 1/\ell^m = Z \otimes 1/\ell^m$ has exact order ℓ^m in $\mathrm{CH}^2(W_{\overline{k}_0}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$, i.e. the cycle $\mathcal{Z}$ satisfies condition (i).

Let $C_{\overline{s}_0}$ be the fiber of the relative curve extending C over $\overline{s}_0$, and let $\rho_{\overline{s}_0} : C_{\overline{s}_0} \to W_{\overline{s}_0}$ be the canonical embedding. Consider the cycle $i_W^!(\mathcal{Z}) = \rho_{\overline{s}_0 *}(C_{\overline{s}_0}) - [-1]_* \rho_{\overline{s}_0 *}(C_{\overline{s}_0}) \in \mathrm{CH}^2(W_{\overline{s}_0})$. Since $W_{\overline{s}_0}$ is an abelian variety, $[-1]_*$ acts as the identity on $\mathrm{H}^4(W_{\overline{s}_0}, \mathbb{Q}_\ell(2))$ and $i_W^!(\mathcal{Z})$ is homologically trivial. Since s_0 is a closed point, $\overline{s}_0$ is the spectrum of an algebraic closure of a finite field, and by a theorem of Soulé [8, Théorème 4] the Chow group of homologically trivial 1-cycles on abelian varieties over such fields is a torsion group. Hence $i^!(\mathcal{Z}) \in \mathrm{CH}^2(W_{\overline{s}_0})_{tors}$, i.e. we have (ii) as well.

This proves the case $n = 4$ and $r = 3$. For $n > 4$ one obtains analogous examples by replacing W by $W \times \mathbb{P}^{n-4}$. □

Remark 3. For the case of equal characteristic 0 we may specialize to the Jacobian of a hyperelliptic curve and choose a Weierstrass point as a base point. Since the involution -1 on the Jacobian of such a curve restricts to the hyperinvolution on the curve, the image of the curve in the Jacobian, as a cycle, is invariant under $[-1]_*$, and specialization of the Ceresa cycle yields a trivial cycle.

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