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Rationality questions in algebraic geometry

Wednesday 2-4pm, Theresienstr. 39, B 045

A classical question in algebraic geometry is to decide whether a given variety X is rational, that is, birational to projective space. Similarly, one may ask whether X is stably rational, that is, $X \times \mathbb{P}^m$ is rational for some $m \geq 0$. While the class of rational or stably rational varieties is poorly understood, the bigger class of rationally connected varieties is much better behaved.

Recently, there has been major progress in the question which rationally connected varieties are stably rational. For example, many low degree hypersurfaces, ramified covers, conic bundles or quadric bundles have been proven to be stably non-rational. Often, even the rationality problem for those varieties was previously open.

In this seminar, we go through most of the above mentioned work. We also discuss solutions of the following two longstanding open questions, which have been established only very recently: Is a deformation or specialization of a rational variety again rational?

The program and dates are provisional and open for discussion. Please contact one of the organizers if you are willing to give a talk.

- **Introduction (Schreieder, Oct 18)**
- **Rationally connected varieties and the theorem of Graber–Harris–Starr (Oct 25)**

  Introduce rationally connected varieties and explain the main result in [8].
- **The Artin–Mumford example: A unirational variety which is not stably rational (Nov 8)**

  In [1], Artin and Mumford give the first example of a unirational variety which is not stably rational. Explain this result, following either the original source or one of the alternative proofs found by Beauville, which are somewhat easier to follow, see [2, §9] and [3, Section 6.3].
- **Decompositions of the diagonal à la Bloch–Srinivas (Nov 15)**

  Show that any rationally connected variety admits a rational decomposition of the diagonal, see for instance [4, Proposition 1] or [20, Corollary 10.21]. Discuss some consequences/applications of rational decompositions of the diagonal, see for instance [4], [20, Section 10.2] and [21, Section 3].
- **The degeneration method à la Voisin and Colliot-Thélène–Pirutka (Nov 29)**

  The degeneration method started with [22, Theorem 2.1] of Voisin. Shortly afterwards, Colliot-Thélène–Pirutka gave a generalization to more general degenerations which work also in mixed characteristic [7, Théorème 1.14]. You can find alternativ treatments in [15, Sections 4 and 5]. Explain the above results and explain the original applications of Voisin [22] and Colliot-Thélène–Pirutka [7] to quartic double solids and quartic hypersurfaces, respectively.

- **Degeneration to characteristic p method: Non-rational Hypersurfaces (Dec 13)**

  Introduce the degeneration method of Kollár [10] and explain how it is used to show that many low degree hypersurfaces are not ruled, hence not rational. The reference is the original article [10] and Chapter V.5 in the book [11].

- **Hypersurfaces that are not stably rational (Dec 20)**

  Explain the results in [18], where Totaro combines the method of Kollár with the decomposition of the diagonal method of Voisin and Colliot–Thélène–Pirutka, to show that many low degree hypersurfaces are not stably rational. Present also the application in [18, Section 4], which shows that rationality does not specialize in mildly singular families; see also [19, 14] for slightly more general results.

- **Unramified cohomology (Jan 10)**

  Introduce unramified cohomology and present the main results of [6], where the Artin–Mumford example and higher dimensional generalizations are studied from the perspective of unramified cohomology. For the basics of unramified cohomology, the survey [5] may be a useful source. The examples in [6] have been generalized to all dimensions in [16, Section 6].

- **Rationality is not a deformation invariant (Jan 17)**

  Explain the main result of [9], which provides the first examples of smooth projective families over connected bases with both, rational and irrational fibres.

- **The rationality problem for quadric bundles (Jan 24)**

  Explain the results in [16], where a generalization of the method of Voisin and Colliot-Thélène–Pirutka is introduced, which allows to apply the degeneration method to certain families where no assumption on the singularities of the special fibre is needed. Further applications of that method can be found in [17].

- **(Stable) rationality specializes in families (Jan 31)**

  Explain the main result of [12], where Kontsevich and Tschinkel show that rationality specializes in smooth projective families in characteristic zero. This result builds on [13], where Nicaise and Shinder show that stable rationality specializes in smooth projective families in characteristic zero.

Literatur

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