TORSION IN THE LICHTENBAUM CHOW GROUP OF ARITHMETIC SCHEMES

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ABSTRACT. We give an example of a smooth arithmetic scheme $\mathfrak{X} \to B$ over the spectrum of a Dedekind domain and primes p with the property that the p-primary torsion subgroup of the Lichtenbaum Chow group $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ has positive corank.

1. INTRODUCTION

Let E be an elliptic curve over $\mathbb{Q}$ with conductor N and with complex multiplication by the ring of integers of an imaginary quadratic field K, and let $X = E \times_{\mathbb{Q}} E$. Langer-Raskind [12, Theorem 0.1] have shown that if p is a prime not dividing $6N$, the p-primary torsion subgroup $\mathrm{CH}^2(X)\{p\}$ of the Chow group of codimension 2 cycles is finite. In fact, they show the following: If p is a prime number not dividing $6N$, let B be the spectrum of the ring $\mathbb{Z}$ with the primes dividing $6Np$ inverted, and let $\mathfrak{X}$ be a smooth proper model of X over B. By a result of Mildenhall [15, Theorem 5.8] the kernel of the natural map $\mathrm{CH}^2(\mathfrak{X}) \to \mathrm{CH}^2(X)$ is finite, which implies that $\mathrm{CH}^2(X)\{p\}$ is finite if and only if $\mathrm{CH}^2(\mathfrak{X})\{p\}$ is finite. Langer-Raskind use this equivalence and prove the quoted finiteness result for the Chow group of the model.

For both X and $\mathfrak{X}$ (or more generally, smooth schemes over a field or the spectrum of a Dedekind domain) Bloch's cycle complex [2] defines a complex $\mathbb{Z}(n)$ of Zariski sheaves (on the small sites X_{Zar} or $\mathfrak{X}_{\mathrm{Zar}}$), and one can define

$$\mathrm{CH}^n_{\mathrm{M}}(X) = \mathbb{H}^{2n}_{\mathrm{Zar}}(X, \mathbb{Z}(n)) \text{ and } \mathrm{CH}^n_{\mathrm{M}}(\mathfrak{X}) = \mathbb{H}^{2n}_{\mathrm{Zar}}(\mathfrak{X}, \mathbb{Z}(n)).$$

It follows from Zariski descent that the group $\mathrm{CH}^n_{\mathrm{M}}(X)$ defined in this way coincides with the classical Chow group $\mathrm{CH}^n(X)$, and an easy argument shows that in our case of interest we also have $\mathrm{CH}^2(\mathfrak{X}) \cong \mathrm{CH}^2_{\mathrm{M}}(\mathfrak{X})$. We may also view $\mathbb{Z}(n)$ as a complex of sheaves in the étale topology (on the small sites $X_{\text{ét}}$ and $\mathfrak{X}_{\text{ét}}$), and write $\mathbb{Z}(n)_{\text{ét}}$ for this complex. The étale motivic or Lichtenbaum Chow groups we consider here are defined as the étale hypercohomology groups

$$\mathrm{CH}^n_{\mathrm{L}}(X) = \mathbb{H}^{2n}_{\text{ét}}(X, \mathbb{Z}(n)) \text{ and } \mathrm{CH}^n_{\mathrm{L}}(\mathfrak{X}) = \mathbb{H}^{2n}_{\text{ét}}(\mathfrak{X}, \mathbb{Z}(n));$$

for details, see section 2. Since $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ is a quotient and $\mathrm{CH}^2(\mathfrak{X})\{p\}$ is a subquotient of $\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, the p-primary torsion in both Chow groups is of finite cotype, and a priori there could be more torsion in the Lichtenbaum Chow group. We show:

Theorem 1.1. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication and conductor N. Let p be a rational prime such that $p \nmid 6N$ and such that E*

has ordinary reduction at p, and let S be the set or primes dividing $6Np$. Set $X = E \times_{\mathbb{Q}} E$, and let $\mathfrak{X} \to \operatorname{Spec}(\mathbb{Z}_S)$ be a smooth proper model of X. Then

(a) $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ *has positive corank,*
(b) $\mathrm{CH}^2_{\mathrm{L}}(X)\{p\}$ *contains a copy of $\mathbb{Q}_p/\mathbb{Z}_p$.*

To prove this, we first establish a sufficient criterion in terms of étale cohomology for $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ to have positive corank, and then show that this condition is satisfied. The second step involves local computations using p-adic Hodge theory and is already implicit in the proof of Langer-Raskind. For the convenience of the reader we give the full argument.

2. Preliminaries

We summarize the definitions and properties of motivic and Lichtenbaum cohomology needed in the sequel.

Let X be a smooth quasi-projective variety over a field k. The motivic cohomology groups of X with coefficients in an abelian group A are defined as

$$\mathrm{H}^m_{\mathrm{M}}(X, A(n)) = \mathbb{H}^m_{\mathrm{Zar}}(X, (z^n(-,\bullet) \otimes A)[-2n]),$$

where $z^n(-,\bullet)$ is the complex of Zariski sheaves given by Bloch's cycle complex [2]. In particular, $\mathrm{H}^{2n}_{\mathrm{M}}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ is the Chow group of codimension n cycles. The complex $z^n(-,\bullet)$ is also a complex of étale sheaves [2, §11] whose hypercohomology groups define the étale motivic or Lichtenbaum cohomology

$$\mathrm{H}^m_{\mathrm{L}}(X, A(n)) = \mathbb{H}^m_{\text{ét}}(X, (z^n(-,\bullet)_{\text{ét}} \otimes A)[-2n]);$$

in particular, $\mathrm{CH}^n_{\mathrm{L}}(X) = \mathrm{H}^{2n}_{\mathrm{L}}(X, \mathbb{Z}(n))$. It is known that the comparison map

$$\mathrm{H}^m_{\mathrm{M}}(X, A(n)) \to \mathrm{H}^m_{\mathrm{L}}(X, A(n))$$

is an isomorphism with rational coefficients. Furthermore, if $\ell \nmid \operatorname{char}(k)$ is a prime, there is a quasi-isomorphism $(\mathbb{Z}/\ell^r\mathbb{Z})_X(n)_{\text{ét}} \xrightarrow{\sim} \mu_{\ell^r}^{\otimes n}$ [8, Theorem 1.5], thus with finite coefficients Lichtenbaum and étale cohomology groups coincide

$$\mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}/\ell^r(n)) \cong \mathrm{H}^m_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n}).$$

Let R be a Dedekind domain and let $\mathfrak{X} \to B = \operatorname{Spec}(R)$ be an essentially smooth B-scheme. The complex $z^n(-,\bullet)$ of presheaves on $\mathfrak{X}$ defines a complex of sheaves for the Zariski and for the étale cohomology. Following Geisser [9], we define the motivic and étale motivic or Lichtenbaum cohomology of $\mathfrak{X}$ as

$$\mathrm{H}^m_{\mathrm{M}}(\mathfrak{X}, A(n)) = \mathbb{H}^m_{\mathrm{Zar}}(\mathfrak{X}, A(n)) \text{ and } \mathrm{H}^m_{\mathrm{L}}(\mathfrak{X}, A(n)) = \mathbb{H}^m_{\text{ét}}(\mathfrak{X}, A(n));$$

we also set $\mathrm{CH}^n_{\mathrm{M}}(\mathfrak{X}) = \mathrm{H}^{2n}_{\mathrm{M}}(\mathfrak{X}, \mathbb{Z}(n))$ and $\mathrm{CH}^n_{\mathrm{L}}(\mathfrak{X}) = \mathrm{H}^{2n}_{\mathrm{L}}(\mathfrak{X}, \mathbb{Z}(n))$. We remark that it not clear that the groups $\mathrm{H}^m_{\mathrm{M}}(\mathfrak{X}, \mathbb{Z}(n))$ coincides with the cohomology of Bloch's cycle complex of abelian groups in general (this is known, for example, in case X is essentially of finite type over $B = \operatorname{Spec}(A)$, where A is a discrete valuation ring [9, Proposition 3.6]). However, in our case of interest we can compare the exact localization sequences along the evident maps

$$\begin{array}{ccccccccc}
\mathrm{CH}^2(X,1) & \to & \bigoplus\limits_{v\notin S}\mathrm{Pic}(Y_v) & \to & \mathrm{CH}^2(\mathfrak{X}) & \to & \mathrm{CH}^2(X) & \to 0 \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \downarrow\cong & \\
\mathrm{H}^3_{\mathrm{M}}(X,\mathbb{Z}(2)) & \to & \bigoplus\limits_{v\notin S}\mathrm{H}^2_{\mathrm{M}}(Y_v,\mathbb{Z}(1)) & \to & \mathrm{H}^4_{\mathrm{M}}(\mathfrak{X},\mathbb{Z}(2)) & \to & \mathrm{H}^4_{\mathrm{M}}(X,\mathbb{Z}(2)) & \to 0
\end{array}$$

where the isomorphisms result from the facts that for X one has Zariski descent, and that for a smooth variety Y one has $\mathbb{Z}(1)_{\mathrm{Zar}} \sim \mathcal{O}_Y^\times[-1]$ [2, Corollary 6.4]. Hence $\mathrm{CH}^2(\mathfrak{X}) \cong \mathrm{CH}^2_{\mathrm{M}}(\mathfrak{X})$, i.e. the motivic Chow group of the model we consider here does indeed coincide with the classical Chow group.

We remark that if $\epsilon : \mathfrak{X}_{\text{ét}} \to \mathfrak{X}_{\mathrm{Zar}}$ is the canonical morphism of sites and $\mathfrak{X}$ is essentially of finite type over B, the map $\mathbb{Q}(n)_{\mathrm{Zar}} \to R\epsilon_*\mathbb{Q}(n)_{\text{ét}}$ is a quasi-isomorphism [9, Proposition 3.6], hence we have with rational coefficients

$$\mathrm{H}^m_{\mathrm{M}}(\mathfrak{X},\mathbb{Q}(n)) \cong \mathrm{H}^m_{\mathrm{L}}(\mathfrak{X},\mathbb{Q}(n)).$$

Let $\mathfrak{X} \to B$ be an essentially smooth scheme over the spectrum of a Dedekind ring. In what follows we will use two properties of the Lichtenbaum cohomology groups of $\mathfrak{X}$ whose proofs require the use of the proof of the Bloch-Kato conjecture, i.e. the assertion that for a field F and a prime number $p \nmid \mathrm{char}(F)$ the norm-residue map between Milnor K-theory and étale cohomology

$$\mathrm{K}^M_m(F)/p^r \to \mathrm{H}^m_{\text{ét}}(F,\mu_{p^r}^{\otimes m})$$

is an isomorphism. This has been shown for the prime $p = 2$ by Voevodsky [20], and in general by Rost-Voevodsky [21], see also [22]. Making use of the norm-residue isomorphism, Geisser has shown the following [9, Theorem 1.2]:

(A) The morphism $\mathbb{Z}(n)_{\mathrm{Zar}} \to \tau_{\leq n+1}R\epsilon_*\mathbb{Z}(n)_{\text{ét}}$ is a quasi-isomorphism, thus

$$\mathrm{H}^m_{\mathrm{M}}(\mathfrak{X},\mathbb{Z}(n)) \cong \mathrm{H}^m_{\mathrm{L}}(\mathfrak{X},\mathbb{Z}(n)) \text{ for } m \leq n+1.$$

(B) If the prime p is invertible in B, there is a quasi-isomorphism of complexes of étale sheaves $\mathbb{Z}/p^r(n)_{\text{ét}} \xrightarrow{\sim} \mu_{p^r}^{\otimes n}$ (where the sheaf $\mu_{p^r}^{\otimes n}$ is in degree 0). Hence

$$\mathrm{H}^m_{\mathrm{L}}(\mathfrak{X},\mathbb{Z}/p^r(n)) \cong \mathrm{H}^m_{\text{ét}}(\mathfrak{X},\mu_{p^r}^{\otimes n}) \quad \text{for } m \in \mathbb{Z}.$$

3. Proof of the Theorem

Proof. Let $p > 3$ be a prime such that E has good ordinary reduction at p, and let $X = E \times_{\mathbb{Q}} E$ and $\mathfrak{X} \to \mathrm{Spec}(\mathbb{Z}_S)$ be as in Theorem 1.1; in particular, the prime p is invertible in $\mathbb{Z}_S$. Note that by standard arguments involving the Leray spectral sequence associated with $\mathfrak{X} \to \mathrm{Spec}(\mathbb{Z}_S)$ the étale cohomology groups $\mathrm{H}^m_{\text{ét}}(\mathfrak{X},\mu_{p^r}^{\otimes n})$ are finite, the groups $\mathrm{H}^m_{\text{ét}}(\mathfrak{X},\mathbb{Q}_p(n))$ are finite-dimensional $\mathbb{Q}_p$-vector spaces, and the groups $\mathrm{H}^m_{\text{ét}}(\mathfrak{X},\mathbb{Q}_p/\mathbb{Z}_p(n))$ are of finite cotype.

In both topologies $\tau = \mathrm{Zar}$ and $\tau = \text{ét}$, the map 'multiplication by p^r' induces short exact sequence $0 \to \mathbb{Z}(2)_\tau \to \mathbb{Z}(2)_\tau \to \mathbb{Z}/p^r(2)_\tau \to 0$ of complexes of sheaves, from which we obtain short exact universal coefficient sequences in motivic and Lichtenbaum cohomology. Comparing these sequences along the morphism $\mathfrak{X}_{\text{ét}} \to \mathfrak{X}_{\text{Zar}}$ we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^3_{\mathrm{M}}(\mathfrak{X}, \mathbb{Z}(2))/p^r & \to & \mathrm{H}^3_{\mathrm{M}}(\mathfrak{X}, \mathbb{Z}/p^r(2)) & \to & \mathrm{CH}^2(\mathfrak{X})[p^r] & \to & 0 \\ & & \cong\big\downarrow & & \kappa_{p^r}\big\downarrow & & \big\downarrow & & \\ 0 & \to & \mathrm{H}^3_{\mathrm{L}}(\mathfrak{X}, \mathbb{Z}(2))/p^r & \to & \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) & \to & \mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})[p^r] & \to & 0 \end{array} \tag{1}$$

Here the isomorphism on the left results from the quasi-isomorphism $\mathbb{Z}(n) \xrightarrow{\sim} \tau_{\leq n+1} R\epsilon_* \mathbb{Z}(n)_{\text{ét}}$ from (A), and in the bottom row we have used the quasi-isomorphism $\mathbb{Z}/p^r(2)_{\text{ét}} \xrightarrow{\sim} \mu_{p^r}^{\otimes 2}$ from (B), which allows us to identify the groups

$$\mathrm{H}^3_{\mathrm{L}}(\mathfrak{X}, \mathbb{Z}/p^r(2)) \cong \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2}).$$

Note that given our assumptions $\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2})$ and $\mathrm{CH}^2(\mathfrak{X})[p^r]$ are finite, hence all groups in the above diagram are finite as well. Taking the direct limit over all r of the bottom row shows that $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ is a quotient of $\mathrm{H}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, thus is itself of finite cotype. Hence we have an isomorphism

$$\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus k} \oplus F, \quad F \text{ a finite group.}$$

If A is an abelian group, let $\mathrm{T}_p(A) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A)$ be the p-adic Tate module, and let $V_p(A) = \mathrm{T}_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the corresponding Tate $\mathbb{Q}_p$-vector space. From the above description of $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ we see that $\mathrm{T}_p(\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}) \cong \mathbb{Z}_p^{\oplus k}$, thus $\mathrm{V}_p(\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}) \cong \mathbb{Q}_p^{\oplus k}$, and we have the following equivalence

$$\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\} \text{ has positive corank } \Leftrightarrow \ V_p(\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}) > 0.$$

Consider the $\mathbb{Q}_p$-vector spaces

$$\begin{aligned} \mathrm{H}_{\mathrm{M}}(\mathfrak{X}) &= [\varprojlim_r \mathrm{H}^3_{\mathrm{M}}(\mathfrak{X}, \mathbb{Z}(2))/p^r] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \\ \mathrm{H}_{\mathrm{L}}(\mathfrak{X}) &= [\varprojlim_r \mathrm{H}^3_{\mathrm{L}}(\mathfrak{X}, \mathbb{Z}(2))/p^r] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \end{aligned}$$

Since $\mathrm{CH}^2(\mathfrak{X})\{p\}$ is finite, $\mathrm{T}_p(\mathrm{CH}^2(\mathfrak{X})\{p\}) = 0$ and $\mathrm{V}_p(\mathrm{CH}^2(\mathfrak{X})\{p\}) = 0$. Taking in the diagram (1) the inverse limit over all r (which is exact in this case) and tensoring with $\mathbb{Q}_p$, we obtain the following commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{H}_{\mathrm{M}}(\mathfrak{X}) & \xrightarrow{\cong} & [\varprojlim_r \mathrm{H}^3_{\mathrm{M}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2})] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & & \\ & & \cong\big\downarrow & & \kappa\big\downarrow & & \\ 0 & \to & \mathrm{H}_{\mathrm{L}}(\mathfrak{X}) & \to & \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) & \to \mathrm{V}_p(\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})) & \to 0 \end{array} \tag{2}$$

whose bottom row is exact. If $\mathrm{K} = \mathrm{im}(\kappa)$, this immediately implies

$$\mathrm{V}_p(\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}) \cong \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))/\,\mathrm{K}. \tag{3}$$

Let $\mathbb{Q}(X)$ be the function field of X, and define

$$\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) = \ker\{\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) \to \mathrm{H}^3(\mathbb{Q}(X), \mu_{p^r}^{\otimes 2})\},$$

and

$$\mathrm{N}\,\mathrm{H}^3(\mathfrak{X}, \mathbb{Q}_p(2)) = [\lim_{\substack{\leftarrow \\ r}} \mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2})] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Note that the $\mathbb{Q}_p$-vector space $\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$ defined above does not necessarily coincide with the first level of the usual coniveau filtration, i.e. if

$$\mathrm{N}'\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) = \lim_{\rightarrow} \ker\{\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) \to \mathrm{H}^3_{\text{ét}}(\mathfrak{X} \setminus \mathfrak{Z}, \mathbb{Q}_p(2))\},$$

where the limit is taken over all closed subschemes $\mathfrak{Z} \subseteq \mathfrak{X}$ of codimension one, there is an injective map $\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) \to \mathrm{N}'\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$, which is not an isomorphism in general, see [12, Remark 2.1].

We claim $\mathrm{K} = \mathrm{im}(\kappa) \subseteq \mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$. To see this, note first that

$$\mathrm{K} = [\lim_{\substack{\leftarrow \\ r}} \mathrm{im}(\kappa_{p^r})] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{4}$$

The composition $\mathrm{H}^3_{\mathrm{M}}(\mathfrak{X}, \mathbb{Z}/p^r(2)) \xrightarrow{\kappa_{p^r}} \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) \to \mathrm{H}^3_{\text{ét}}(\mathbb{Q}(X), \mu_{p^r}^{\otimes 2})$ coincides with $\mathrm{H}^3_{\mathrm{M}}(\mathfrak{X}, \mathbb{Z}/p^r(2)) \to \mathrm{H}^3_{\mathrm{M}}(\mathbb{Q}(X), \mathbb{Z}/p^r(2)) \to \mathrm{H}^3_{\text{ét}}(\mathbb{Q}(X), \mathbb{Z}/p^r(2))$, where the first map is trivial, because the motivic Zariski sheaf $\mathcal{H}^3_M(\mathbb{Z}/p^r(2))$ on X is trivial. This shows $\mathrm{im}(\kappa_{p^r}) \subseteq \mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mu_{p^r}^{\otimes 2})$, and our claim follows from (4).

Given $\mathrm{K} \subseteq \mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$, the composition of (3) with the quotient map $\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))/\mathrm{K} \to \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))/\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$ yields a surjective map

$$\mathrm{V}_p(\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}) \to \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))/\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)). \tag{5}$$

In particular, to show $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ has positive corank it suffices to show

$$\dim_{\mathbb{Q}_p} \mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) \leq \dim_{\mathbb{Q}_p} \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) - 1. \tag{6}$$

Let G_S be the Galois group of the maximal extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ which is unramified outside of S. For $v \in S$ let $X_v = \mathfrak{X} \times_B \mathbb{Q}_v$ and write $G_v = \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ for the absolute Galois group, and $\mathrm{H}^n(G_v, M) = \mathrm{H}^n(\mathbb{Q}_v, M)$ for the continuous Galois cohomology groups with values in a G_v-module M.

Let $V = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2))$ and consider the localization map

$$\mathrm{H}^1(G_S, V) \to \bigoplus_{v \in S} \mathrm{H}^1(G_v, V);$$

we recall that by results of Flach [5] this map becomes surjective after passing to a suitable quotient on the right hand side.

In general, given any $\mathbb{Q}_p$-linear representation V of $G_{\mathbb{Q}}$, we have for every non-archimedian completion $\mathbb{Q}_v$ of $\mathbb{Q}$ the following $\mathbb{Q}_p$-subspaces of $\mathrm{H}^1(\mathbb{Q}_v, V)$

$$\mathrm{H}^1_f(G_v, V) \subseteq \mathrm{H}^1_g(G_v, V) \subseteq \mathrm{H}^1(G_v, V),$$

defined by Bloch-Kato in [3, §3] as follows: If $v \neq p$, let $\mathbb{Q}_v^{nr}$ be the maximal unramified extension of $\mathbb{Q}_v$, set $\mathrm{H}^1_g(\mathbb{Q}_v, V) = \mathrm{H}^1(\mathbb{Q}_v, V)$, and

$$\mathrm{H}^1_f(\mathbb{Q}_v, V) = \ker\{\mathrm{H}^1(\mathbb{Q}_v, V) \to \mathrm{H}^1(\mathbb{Q}_v^{nr}, V)\}.$$

If $v = p$, let B_{cris} and B_{DR} be the rings defined by Fontaine in [6], and set

$$\begin{aligned} \mathrm{H}^1_f(\mathbb{Q}_v, V) &= \ker\{\mathrm{H}^1(\mathbb{Q}_v, V) \to \mathrm{H}^1(\mathbb{Q}_v, V \otimes_{\mathbb{Q}_p} B_{cris})\}, \\ \mathrm{H}^1_g(\mathbb{Q}_v, V) &= \ker\{\mathrm{H}^1(\mathbb{Q}_v, V) \to \mathrm{H}^1(\mathbb{Q}_v, V \otimes_{\mathbb{Q}_p} B_{DR})\}. \end{aligned}$$

If T is a G_S-stable $\mathbb{Z}_p$-lattice, define A by the exactness of the sequence

$$0 \to T \xrightarrow{\iota} V \xrightarrow{\mathrm{pr}} A \to 0.$$

Let $W_v = \mathrm{H}^1_f(\mathbb{Q}_v, V) \subseteq \mathrm{H}^1(\mathbb{Q}_v, V)$ (where we set $\mathrm{H}^1_f(\mathbb{Q}_v, V) = 0$ for the archimedean place $v = \infty$). Let $\mathrm{M}_{\mathbb{Q}}$ be the set of all valuations of $\mathbb{Q}$ and define

$$\mathrm{H}^1_{f,\mathbb{Z}}(\mathbb{Q}, V) = \{x \in \mathrm{H}^1(\mathbb{Q}, V) \mid x_v \in \mathrm{H}^1_f(\mathbb{Q}_v, V) \text{ for all } v \in \mathrm{M}_{\mathbb{Q}}\}.$$

Using Tate's global duality theorem, Flach has shown [5, Proposition 1.4] (cp. also [3, Lemma 5.16]) that the localization maps induce an exact sequence

$$\mathrm{H}^1(G_S, A) \to \bigoplus_{v \in S} \frac{\mathrm{H}^1(\mathbb{Q}_v, A)}{\mathrm{pr}_*(W_v)} \to \iota_*^{-1}(W')^*,$$

where

$$W' = \mathrm{H}^1_{f,\mathbb{Z}}(\mathbb{Q}, V^*(1)) \text{ and } V^*(1) \cong \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(1)).$$

Moreover, given our assumptions on E, Flach also showed [5] that the Selmer groups associated with V and $V^*(1)$ are finite over $\mathbb{Q}$. This implies that the group W' appearing in the above exact sequence is finite, and further that the above localization maps induce a surjective map (cp. [12, Corollary 5.2])

$$\mathrm{H}^1(G_S, V) \to \bigoplus_{v \in S} \frac{\mathrm{H}^1(\mathbb{Q}_v, V)}{\mathrm{H}^1_f(\mathbb{Q}_v, \mathrm{V})}. \tag{7}$$

Consider now a smooth proper model $\mathfrak{X} \to \mathrm{Spec}(\mathbb{Z}_S)$ of X as in Theorem 1.1. The surjective localization map from (7) fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) & \to & \bigoplus_{v \in S} \mathrm{H}^3_{\text{ét}}(X_v, \mathbb{Q}_p(2)) \\ {\scriptstyle \text{onto}}\downarrow & & \downarrow \\ \mathrm{H}^1(G_S, V) & \xrightarrow{\text{onto}} & \bigoplus_{v \in S} \mathrm{H}^1(\mathbb{Q}_v, V) / \mathrm{H}^1_f(\mathbb{Q}_v, V) \end{array}$$

whose vertical maps are induced by the Hochschild-Serre spectral sequences. Since $p \in S$, we may pass first to the quotient $\mathrm{H}^1(\mathbb{Q}_p, V) / \mathrm{H}^1_f(\mathbb{Q}_p, V)$, and then further to $\mathrm{H}^1(\mathbb{Q}_p, V) / \mathrm{H}^1_g(\mathbb{Q}_p, V)$ to obtain a surjective map of $\mathbb{Q}_p$-vector spaces

$$\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) \to \mathrm{H}^1(\mathbb{Q}_p, V) / \mathrm{H}^1_g(\mathbb{Q}_p, V). \tag{8}$$

We show next that the surjective map (8) factors through the quotient $\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) / \mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$. We have in the Hochschild-Serre spectral sequence

$$\mathrm{E}_2^{r,s} = \mathrm{H}^r(\mathbb{Q}_p, \mathrm{H}^r_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2))) \Rightarrow \mathrm{H}^{r+s}_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p(2))$$

$\mathrm{E}_2^{0,3} = 0$ by a weight argument, and $\mathrm{E}_2^{2,1} = 0$ by [1, Proposition 2.4]. Since furthermore $\mathrm{E}_2^{p,q} = 0$ for $p > 2$ by cohomological dimension, this implies

$$\mathrm{H}^3_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p(2)) \cong \mathrm{H}^1(\mathbb{Q}_p, V). \tag{9}$$

Let $\mathfrak{X}_{\mathbb{Z}_p} = \mathfrak{X} \times_B \mathbb{Z}_p$, and let $j : X_{\mathbb{Q}_p} \to \mathfrak{X}_{\mathbb{Z}_p}$ be the inclusion. It is immediate that the image of $\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$ in $\mathrm{H}^1(\mathbb{Q}_p, V)$ is contained in the image of $\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}_{\mathbb{Z}_p}, \mathbb{Q}_p(2))$ in $\mathrm{H}^1(\mathbb{Q}_p, V)$. In particular, to show the map (8) factors through $\mathrm{H}^3_{\text{ét}}(\mathfrak{X},\mathbb{Q}_p(2))/\,\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X},\mathbb{Q}_p(2))$ it suffices to show that the image of $\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}_{\mathbb{Z}_p},\mathbb{Q}_p(2))$ in $\mathrm{H}^1(\mathbb{Q}_p,V)$ is contained in the subspace $\mathrm{H}^1_g(\mathbb{Q}_p,V)$.

If $C^\bullet$ is a complex of sheaves on $\mathfrak{X}_{\mathbb{Z}_p}$, we write $\tau_{\leq n}C^\bullet$ for its truncation consisting of terms in degree $\leq n$. In particular, in case $C^\bullet = Rj_*\mu_{p^r}^{\otimes 2}$ we have

$$\mathrm{H}^3(\mathfrak{X}_{\mathbb{Z}_p},\tau_{\leq 2}Rj_*\mathbb{Q}_p(2)) = [\varprojlim_r \mathrm{H}^3(\mathfrak{X}_{\mathbb{Z}_p},\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2})]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p,$$

and it follows from [13, Lemma 5-4] that there is an inclusion

$$\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X}_{\mathbb{Z}_p},\mathbb{Q}_p(2)) \subseteq \mathrm{H}^3(\mathfrak{X}_{\mathbb{Z}_p},\tau_{\leq 2}Rj_*\mathbb{Q}_p(2)). \tag{10}$$

We need a result from p-adic Hodge theory, for an overview in a more general setting, see [17, §2], for instance. There is a natural (injective) pullback map

$$\alpha:\mathrm{H}^3(\mathfrak{X}_{\mathbb{Z}_p},\tau_{\leq 2}Rj_*\mathbb{Q}_p(2)) \to \mathrm{H}^3_{\text{ét}}(X_{\mathbb{Q}_p},\mathbb{Q}_p(2)) \cong \mathrm{H}^1(\mathbb{Q}_p,V). \tag{11}$$

Let $\iota: Y_p \to \mathfrak{X}_{\mathbb{Z}_p}$ be the inclusion of the closed fiber, and let $s_r^{\log}(n)$ be the log syntomic complex defined by Kato in [11]. By a result of Tsuji [19, Theorem 5.1], we have in $D^b_{\text{ét}}(\mathfrak{X},\mathbb{Z}/p^r\mathbb{Z})$ for $0\leq n\leq p-2$ a quasi-isomorphism

$$\eta: s_r^{\log}(n) \xrightarrow{\sim} \iota_*\iota^*(\tau_{\leq n}Rj_*\mu_{p^r}^{\otimes n});$$

in particular, this holds for $n=2$, provided $p>3$. Consider now the group

$$\mathrm{H}^n(\mathfrak{X}_{\mathbb{Z}_p},s_{\mathbb{Q}_p}^{\log}(n)) = [\varprojlim_r \mathrm{H}^n(\mathfrak{X}_{\mathbb{Z}_p},s_r^{\log}(n))]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

Since we assume $p>3$, the quasi-isomorphism η induces an isomorphism

$$\mathrm{H}^3(\mathfrak{X}_{\mathbb{Z}_p},s_{\mathbb{Q}_p}^{\log}(2)) \xrightarrow{\cong} \mathrm{H}^3(\mathfrak{X}_{\mathbb{Z}_p},\tau_{\leq 2}Rj_*\mathbb{Q}_p(2)). \tag{12}$$

Let η' be the composition of this isomorphism with the map α from (11), i.e.

$$\eta':\mathrm{H}^3(\mathfrak{X}_{\mathbb{Z}_p},s_{\mathbb{Q}_p}^{\log}(2)) \to \mathrm{H}^1(\mathbb{Q}_p,V).$$

The result we need is the following, which was proved by Langer [14] in a special case, and by Nekovár [16, Theorem 3.1] in general

$$\mathrm{im}(\eta') = \mathrm{H}^1_g(\mathbb{Q}_p,V). \tag{13}$$

In particular, by (10), (12) and (13) the image of $\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X},\mathbb{Q}_p(2))$ in $\mathrm{H}^1(\mathbb{Q}_p,V)$ lies in the subspace $\mathrm{H}^1_g(\mathbb{Q}_p,V)$, so that (8) induces a surjective map

$$\mathrm{H}^3_{\text{ét}}(\mathfrak{X},\mathbb{Q}_p(2))/\,\mathrm{N}\,\mathrm{H}^3_{\text{ét}}(\mathfrak{X},\mathbb{Q}_p(2)) \to \mathrm{H}^1(\mathbb{Q}_p,V)/\,\mathrm{H}^1_g(\mathbb{Q}_p,V) =: \Theta_p. \tag{14}$$

To complete the proof it remains to show $\Theta_p\neq 0$. The dimension of this space can be computed as follows [12, pg. 16] (see also [13, proof of Lemma 4-5]): Let $W = V(1)^* \cong \mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Q}_p(1))$, viewed as a $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-module. Then

$$\dim_{\mathbb{Q}_p}\Theta_p = \dim_{\mathbb{Q}_p} D_{dR}(W)/D^0_{dR}(W) - \dim_{\mathbb{Q}_p} D_{cris}(W)^{f=1} + \dim_{\mathbb{Q}_p} W^{G_p}.$$

For the first term we have isomorphisms

$$\mathrm{D}_{dR}(W) = \mathrm{H}^2_{dR}(X/\mathbb{Q})\otimes\mathbb{Q}_p \text{ and } D^0_{dR}(W) = \mathrm{F}^1_{\mathrm{Hdg}}\,\mathrm{H}^2_{dR}(X/\mathbb{Q})\otimes\mathbb{Q}_p,$$

hence the dimension of this term is equal to $\dim \mathrm{H}^2(X,\mathcal{O}_X) = 1$.

For the second term, let $E_{\mathbb{F}_p}$ be the reduction of E at the prime p, and let $X_{\mathbb{F}_p} = E_{\mathbb{F}_p} \times E_{\mathbb{F}_p}$. By the crystalline conjecture (cp. [7] and [4]) and the crystalline Tate conjecture (proved for abelian varieties in [18] and [10])

$$D_{cris}(W)^{f=1} \cong (\mathrm{H}^2_{cris}(X_{\mathbb{F}_p}/W(\mathbb{F}_p)) \otimes \mathbb{Q}_p)^{\phi_p = p} \cong \mathrm{Pic}(X_{\mathbb{F}_p}) \otimes \mathbb{Q}_p,$$

and therefore

$$\dim_{\mathbb{Q}_p} \mathrm{D}_{cris}(W)^{f=1} = 2 + \dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{F}_p}(E_{\mathbb{F}_p}) \otimes \mathbb{Q}) = 4.$$

For the remaining term, let $\mathrm{V}_p(E) = \mathrm{H}^1(\overline{E}, \mathbb{Q}_p(1))$, so that

$$\dim_{\mathbb{Q}_p} W^{G_p} = 2 + \dim_{\mathbb{Q}_p} \mathrm{End}_{G_p}(\mathrm{V}_p(E)).$$

Here we use that we have ordinary reduction at p. It follows that the prime p splits in the CM field K, $\mathbb{Q}_p$ contains K, and the complex multiplication is defined over $\mathbb{Q}_p$. Thus $\dim_{\mathbb{Q}_p} \mathrm{End}_{G_p}(V_p(E)) = 2$, and therefore

$$\dim_{\mathbb{Q}_p} \Theta_p = 1 - 4 + 4 = 1. \tag{15}$$

Now the combination of (6), (14) and (15) implies that $\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$ has positive corank, which proves part (a) of Theorem 1.1.

To show $\mathrm{CH}^2_{\mathrm{L}}(X)\{p\}$ contains a copy of $\mathbb{Q}_p/\mathbb{Z}_p$ we show that the kernel of

$$\mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\} \to \mathrm{CH}^2_{\mathrm{L}}(X)\{p\}$$

is finite. Consider the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & \bigoplus_{v\notin S} \mathrm{H}^1_{\text{ét}}(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \arrow[d] & & \\
0 \arrow[r] & \mathrm{H}^3_{\mathrm{L}}(\mathfrak{X}, \mathbb{Z}(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "\alpha"'] & \mathrm{H}^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r] \arrow[d, "\beta"'] & \mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathrm{H}^3_{\mathrm{L}}(X, \mathbb{Z}(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & \mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r] \arrow[d] & \mathrm{CH}^2_{\mathrm{L}}(X)\{p\} \arrow[r] & 0 \\
 & & \bigoplus_{v\notin S} \mathrm{H}^2_{\text{ét}}(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) & &
\end{tikzcd}$$

whose middle column is the localization sequence in étale cohomology. We will make use of Mildenhall's result [15, Theorem 5.8] that $\Sigma = \ker\{\mathrm{CH}^2(\mathfrak{X}) \to \mathrm{CH}^2(X)\}$ is a finite group. Since in bidegree $(3, 2)$ we can identify motivic and Lichtenbaum cohomology for both $\mathfrak{X}$ and X, we may rewrite the localization sequence in motivic cohomology with integral coefficients in the form

$$\mathrm{H}^3_{\mathrm{L}}(\mathfrak{X}, \mathbb{Z}(2)) \to \mathrm{H}^3_{\mathrm{L}}(X, \mathbb{Z}(2)) \xrightarrow{\partial} \bigoplus_{v\notin S} \mathrm{CH}^1(Y_v) \to \Sigma \to 0.$$

From the map 'multiplication by p^r' we obtain a long exact sequence

$$\Sigma[p^r] \to \mathrm{im}(\partial)/p^r \to \bigoplus_{v\notin S} \mathrm{CH}^1(Y_v)/p^r \to \Sigma/p^r \to 0,$$

and taking the direct limit over all r it is immediate that the map

$$\mathrm{coker}(\alpha) = \mathrm{im}(\partial) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \bigoplus_{v\notin S} \mathrm{CH}^1(Y_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

has finite kernel. If Y_v is a smooth closed fiber, we have from the Kummer sequence an injective map $\mathrm{CH}^1(Y_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{H}^2_{\text{ét}}(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1))$ such that

$$\begin{tikzcd} \mathrm{coker}(\alpha) \arrow[r] \arrow[d] & \mathrm{coker}(\beta) \arrow[d, "\text{into}"] \\ \bigoplus_{v \notin S} \mathrm{CH}^1(Y_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r, "\text{into}"] & \bigoplus_{v \notin S} \mathrm{H}^2_{\text{ét}}(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \end{tikzcd}$$

commutes. Hence the kernel of the top horizontal map is finite. Given this, it suffices to show the image of $\ker(\beta) \to \ker(\gamma)$, or equivalently, the image of

$$\bigoplus_{v \notin S} \mathrm{CH}^1_{\mathrm{L}}(Y_v)\{p\} \to \mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\}$$

is finite, which also follows easily from Mildenhall's Theorem. The diagram

$$\begin{tikzcd} \bigoplus_{v \notin S} \mathrm{CH}^1(Y_v)\{p\} \arrow[r] \arrow[d, "\cong"'] & \mathrm{CH}^2(\mathfrak{X})\{p\} \arrow[d] \\ \bigoplus_{v \notin S} \mathrm{CH}^1_{\mathrm{L}}(Y_v)\{p\} \arrow[r] & \mathrm{CH}^2_{\mathrm{L}}(\mathfrak{X})\{p\} \end{tikzcd}$$

commutes, and the top horizontal map factors through the finite group $\Sigma\{p\}$. This proves our second claim (b) and completes the proof of Theorem 1.1. □

Remark 3.1. If $E/\mathbb{Q}$ does not have complex multiplication and $p > 3$ is a prime such that E has good ordinary reduction at p, it follows from the proof of [13, Lemma 4-5] that $\dim_{\mathbb{Q}_p} \Theta_p = 0$.

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