CHAPTER 8

Toolbox

9. Adjoint Functors and the Yoneda Lemma

Theorem 8.9.1. *(Yoneda Lemma) Let $\mathcal{C}$ be a category. Given a covariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ and an object $A \in \mathcal{C}$. Then the map*

$$\pi : \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}) \ni \phi \mapsto \phi(A)(1_A) \in \mathcal{F}(A)$$

is bijective with the inverse map

$$\pi^{-1} : \mathcal{F}(A) \ni a \mapsto h^a \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}),$$

where $h^a(B)(f) = \mathcal{F}(f)(a)$.

Proof. For $\phi \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F})$ we have a map $\phi(A) : \mathrm{Mor}_{\mathcal{C}}(A, A) \longrightarrow \mathcal{F}(A)$, hence π with $\pi(\phi) := \phi(A)(1_A)$ is a well defined map. For π^{-1} we have to check that h^a is a natural transformation. Given $f : B \longrightarrow C$ in $\mathcal{C}$. Then the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\mathrm{Mor}(A,f)"] \arrow[d, "h^a(B)"'] & \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[d, "h^a(C)"] \\
\mathcal{F}(B) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(C)
\end{tikzcd}$$

is commutative. In fact if $g \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ then $h^a(C)\mathrm{Mor}_{\mathcal{C}}(A, f)(g) = h^a(C)(fg) = \mathcal{F}(fg)(a) = \mathcal{F}(f)\mathcal{F}(g)(a) = \mathcal{F}(f)h^a(B)(a)$. Thus π^{-1} is well defined.

Let $\pi^{-1}(a) = h^a$. Then $\pi\pi^{-1}(a) = h^a(A)(1_A) = \mathcal{F}(1_A)(a) = a$. Let $\pi(\phi) = \phi(A)(1_A) = a$. Then $\pi^{-1}\pi(\phi) = h^a$ and $h^a(B)(f) = \mathcal{F}(f)(a) = \mathcal{F}(f)(\phi(A)(1_A)) = \phi(B)\mathrm{Mor}_{\mathcal{C}}(A, f)(1_A) = \phi(B)(f)$, also $h^a = \phi$. $\square$

Corollary 8.9.2. *Given $A, B \in \mathcal{C}$. Then the following hold*

1. $\mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathrm{Mor}_{\mathcal{C}}(f, \text{-}) \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B, \text{-}), \mathrm{Mor}_{\mathcal{C}}(A, \text{-}))$ *is a bijective map.*
2. *With the bijective map from 1. the isomorphisms from* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *correspond to natural isomorphisms from* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B, \text{-}), \mathrm{Mor}_{\mathcal{C}}(A, \text{-}))$.
3. *For contravariant functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ we have* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathcal{F}) \cong \mathcal{F}(A)$.
4. $\mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathrm{Mor}_{\mathcal{C}}(\text{-}, f) \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathrm{Mor}_{\mathcal{C}}(\text{-}, B))$ *is a bijective map that defines a one-to-one correspondence between the isomorphisms from* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *and the natural isomorphisms from* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathrm{Mor}_{\mathcal{C}}(\text{-}, B))$.

Proof. 1. follows from the Yoneda Lemma with $\mathcal{F} = \mathrm{Mor}_{\mathcal{C}}(A, \text{-})$.

2. Observe that $h^f(C)(g) = \mathrm{Mor}_{\mathcal{C}}(A, g)(f) = gf = \mathrm{Mor}_{\mathcal{C}}(f, C)(g)$ hence $h^f = \mathrm{Mor}_{\mathcal{C}}(f, \text{-})$. Since we have $\mathrm{Mor}_{\mathcal{C}}(f, \text{-})\mathrm{Mor}_{\mathcal{C}}(g, \text{-}) = \mathrm{Mor}_{\mathcal{C}}(gf, \text{-})$ and $\mathrm{Mor}_{\mathcal{C}}(f, \text{-}) = \mathrm{id}_{\mathrm{Mor}_{\mathcal{C}}(A,\text{-})}$ if and only if $f = 1_A$ we get the one-to-one correspondence between the isomorphisms from 1.

3. and 4. follow by dualizing. $\square$

Remark 8.9.3. The map π is a natural transformation in the arguments A and $\mathcal{F}$. More precisely: if $f : A \longrightarrow B$ and $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ are given then the following diagrams commute

$$\begin{tikzcd} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\mathcal{F}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Nat}(\mathrm{Mor}(A,-),\phi)"'] & \mathcal{F}(A) \arrow[d, "\phi(A)"] \\ \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\mathcal{G}) \arrow[r, "\pi"'] & \mathcal{G}(A) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\mathcal{F}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Nat}(\mathrm{Mor}(f,-),\mathcal{F})"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"] \\ \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B,\text{-}),\mathcal{F}) \arrow[r, "\pi"'] & \mathcal{F}(B). \end{tikzcd}$$

This can be easily checked. Furthermore we have for $\psi : \mathrm{Mor}_{\mathcal{C}}(A,\text{-}) \longrightarrow \mathcal{F}$

$$\pi\, \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A,\text{-}),\phi)(\psi) = \pi(\phi\psi) = (\phi\psi)(A)(1_A) = \phi(A)\psi(A)(1_A) = \phi(A)\pi(\psi)$$

and

$$\begin{aligned} &\pi\, \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(f,\text{-}),\mathcal{F})(\psi) = \pi(\psi \mathrm{Mor}_{\mathcal{C}}(f,\text{-})) = (\psi \mathrm{Mor}_{\mathcal{C}}(f,\text{-}))(B)(1_B) = \psi(B)(f) \\ &= \psi(B)\mathrm{Mor}_{\mathcal{C}}(A,f)(1_A) = \mathcal{F}(f)\psi(A)(1_A) = \mathcal{F}(f)\pi(\psi). \end{aligned}$$

Remark 8.9.4. By the previous corollary the representing object A is uniquely determined up to isomorphism by the isomorphism class of the functor $\mathrm{Mor}_{\mathcal{C}}(A,\text{-})$.

Problem 8.9.1. 1. Determine explicitly all natural endomorphisms from $\mathbb{G}_a$ to $\mathbb{G}_a$ (as defined in Lemma 2.3.5).

2. Determine all additive natural endomorphisms of $\mathbb{G}_a$.
3. Determine all natural transformations from $\mathbb{G}_a$ to $\mathbb{G}_m$ (see Lemma 2.3.7).
4. Determine all natural automorphisms of $\mathbb{G}_m$.

Proposition 8.9.5. *Let $\mathcal{G} : \mathcal{C} \times \mathcal{D} \longrightarrow$ **Set** be a covariant bifunctor such that the functor $\mathcal{G}(C,\text{-}) : \mathcal{D} \longrightarrow$ **Set** is representable for all $C \in \mathcal{C}$. Then there exists a contravariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathcal{G} \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-})$ holds. Furthermore $\mathcal{F}$ is uniquely determined by $\mathcal{G}$ up to isomorphism.*

Proof. For each $C \in \mathcal{C}$ choose an object $\mathcal{F}(C) \in \mathcal{D}$ and an isomorphism $\xi_C : \mathcal{G}(C,\text{-}) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),\text{-})$. Given $f : C \longrightarrow C'$ in $\mathcal{C}$ then let $\mathcal{F}(f) : \mathcal{F}(C') \longrightarrow \mathcal{F}(C)$ be the uniquely determined morphism (by the Yoneda Lemma) in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{G}(C,\text{-}) \arrow[r, "\xi_C"] \arrow[d, "\mathcal{G}(f,-)"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),\text{-}) \arrow[d, "\mathrm{Mor}(\mathcal{F}(f),-)"] \\ \mathcal{G}(C',\text{-}) \arrow[r, "\xi_{C'}"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C'),\text{-}) \end{tikzcd}$$

commutes. Because of the uniqueness $\mathcal{F}(f)$ and because of the functoriality of $\mathcal{G}$ it is easy to see that $\mathcal{F}(fg) = \mathcal{F}(g)\mathcal{F}(f)$ and $\mathcal{F}(1_C) = 1_{\mathcal{F}(C)}$ hold and that $\mathcal{F}$ is a contravariant functor.

If $\mathcal{F}' : \mathcal{C} \longrightarrow \mathcal{D}$ is given with $\mathcal{G} \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-},\text{-})$ then $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-}) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-},\text{-})$. Hence by the Yoneda Lemma $\psi(C) : \mathcal{F}(C) \cong \mathcal{F}'(C)$ is an isomorphism for all $C \in \mathcal{C}$. With these isomorphisms induced by ϕ the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'(C),\text{-}) \arrow[r, "\mathrm{Mor}(\psi(C),\text{-})"] \arrow[d, "\mathrm{Mor}(\mathcal{F}'(f),\text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),\text{-}) \arrow[d, "\mathrm{Mor}(\mathcal{F}(f),\text{-})"] \\
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'(C'),\text{-}) \arrow[r, "\mathrm{Mor}(\psi(C'),\text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C'),\text{-})
\end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd}
\mathcal{F}(C') \arrow[r, "\psi(C')"] \arrow[d, "\mathcal{F}'(f)"'] & \mathcal{F}'(C') \arrow[d, "\mathcal{F}(f)"] \\
\mathcal{F}(C) \arrow[r, "\psi(C)"'] & \mathcal{F}'(C)
\end{tikzcd}$$

commutes. Thus $\psi : \mathcal{F} \longrightarrow \mathcal{F}'$ is a natural isomorphism. $\square$

Definition 8.9.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be covariant functors. $\mathcal{F}$ is called *leftadjoint* to $\mathcal{G}$ and $\mathcal{G}$ *rightadjoint* to $\mathcal{F}$ if there is a natural isomorphism of bifunctors $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-},\mathcal{G}\text{-})$ from $\mathcal{C}^{op} \times \mathcal{D}$ to **Set**.

Lemma 8.9.7. *If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is leftadjoint to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ then $\mathcal{F}$ is uniquely determined by $\mathcal{G}$ up to isomorphism. Similarly $\mathcal{G}$ is uniquely determined by $\mathcal{F}$ up to isomorphism.*

Proof. Now we prove the first claim. Assume that also $\mathcal{F}'$ is leftadjoint to $\mathcal{G}$ with $\phi' : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-},\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-},\mathcal{G}\text{-})$. Then we have a natural isomorphism ${\phi'}^{-1}\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-},\text{-})$. By Proposition 8.9.5 we get $\mathcal{F} \cong \mathcal{F}'$. $\square$

Lemma 8.9.8. *A functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ has a leftadjoint functor iff all functors $\mathrm{Mor}_{\mathcal{C}}(C,\mathcal{G}\text{-})$ are representable.*

Proof. follows from 8.9.5. $\square$

Lemma 8.9.9. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be covariant functors. Then*

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}},\mathcal{G}\mathcal{F}) \ni \Phi \mapsto \mathcal{G}\text{-}\Phi\text{-} \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-}),\mathrm{Mor}_{\mathcal{C}}(\text{-},\mathcal{G}\text{-}))$$

is a bijective map with inverse map

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-},\text{-}),\mathrm{Mor}_{\mathcal{C}}(\text{-},\mathcal{G}\text{-})) \ni \phi \mapsto \phi(\text{-},\mathcal{F}\text{-})(1_{\mathcal{F}\text{-}}) \in \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}},\mathcal{G}\mathcal{F}).$$

Furthermore

$$\mathrm{Nat}(\mathcal{F}\mathcal{G}, \mathrm{Id}_{\mathcal{C}}) \ni \Psi \mapsto \Psi\text{-}\mathcal{F}\text{-} \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}), \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}))$$

is a bijective map with inverse map

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}), \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})) \ni \psi \mapsto \psi(\mathcal{G}\text{-}, \text{-})(1_{\mathcal{G}\text{-}}) \in \mathrm{Nat}(\mathcal{F}\mathcal{G}, \mathrm{Id}_{\mathcal{C}}).$$

Proof. The natural transformation $\mathcal{G}$-Φ- is defined as follows. Given $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D)$ then let $(\mathcal{G}\text{-}\Phi\text{-})(C, D)(f) := \mathcal{G}(f)\Phi(C) : C \longrightarrow \mathcal{G}\mathcal{F}(C) \longrightarrow \mathcal{G}(D)$. It is easy to check the properties of a natural transformation.

Given Φ then one obtains by composition of the two maps $\mathcal{G}(1_{\mathcal{F}(C)})\Phi(C) = \mathcal{G}\mathcal{F}(1_C)\Phi(C) = \Phi(C)$. Given ϕ one obtains

$$\begin{aligned}&\mathcal{G}(f)(\phi(C, \mathcal{F}(C))(1_{\mathcal{F}(C)}) = \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(f))\phi(C, \mathcal{F}(C))(1_{\mathcal{F}(C)})\\ &= \phi(C, D)\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), f)(1_{\mathcal{F}(C)}) = \phi(C, D)(f).\end{aligned}$$

The second part of the lemma is proved similarly. □

Proposition 8.9.10. *Let*

$$\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \quad \text{and} \quad \psi : \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})$$

be natural transformations with associated natural transformations (by Lemma 8.9.9) $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{G}\mathcal{F}$ *resp.* $\Psi : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{Id}_{\mathcal{D}}$.

1) Then we have $\phi\psi = \mathrm{id}_{\mathrm{Mor}(\text{-}, \mathcal{G}\text{-})}$ *if and only if* $(\mathcal{G} \xrightarrow{\Phi\mathcal{G}} \mathcal{G}\mathcal{F}\mathcal{G} \xrightarrow{\mathcal{G}\Psi} \mathcal{G}) = \mathrm{id}_{\mathcal{G}}$.

2) We also have $\psi\phi = \mathrm{id}_{\mathrm{Mor}(\mathcal{F}\text{-}, \text{-})}$ *if and only if* $(\mathcal{F} \xrightarrow{\mathcal{F}\Phi} \mathcal{F}\mathcal{G}\mathcal{F} \xrightarrow{\Psi\mathcal{F}} \mathcal{F}) = \mathrm{id}_{\mathcal{F}}$.

Proof. We get

$$\begin{aligned}&\mathcal{G}\Psi(D)\Phi\mathcal{G}(D) = \mathcal{G}\Psi(D)\phi(\mathcal{G}(D), \mathcal{F}\mathcal{G}(D))(1_{\mathcal{F}\mathcal{G}(D)})\\ &= \mathrm{Mor}_{\mathcal{C}}(\mathcal{G}(D), \mathcal{G}\Psi(D))\phi(\mathcal{G}(D), \mathcal{F}\mathcal{G}(D))(1_{\mathcal{F}\mathcal{G}(D)})\\ &= \phi(\mathcal{G}(D), D)\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\mathcal{G}(D), \Psi(D))(1_{\mathcal{F}\mathcal{G}(D)})\\ &= \phi(\mathcal{G}(D), D)(\Psi(D))\\ &= \phi(\mathcal{G}(D), D)\psi(\mathcal{G}(D), D)(1_{\mathcal{G}(D)})\\ &= \phi\psi(\mathcal{G}(D), D)(1_{\mathcal{G}(D)}).\end{aligned}$$

Similarly we get

$$\begin{aligned}&\phi\psi(C, D)(f) = \phi(C, D)\psi(C, D)(f) = \mathcal{G}(\Psi(D)\mathcal{F}(f))\Phi(C)\\ &= \mathcal{G}\Psi(D)\mathcal{G}\mathcal{F}(f)\Phi(C) = \mathcal{G}\Psi(D)\Phi\mathcal{G}(D)f.\end{aligned}$$ □

Corollary 8.9.11. *Let* $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *and* $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ *be functors.* $\mathcal{F}$ *is leftadjoint to* $\mathcal{G}$ *if and only if there are natural transformations* $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{G}\mathcal{F}$ *and* $\Psi : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{Id}_{\mathcal{D}}$ *such that* $(\mathcal{G}\Psi)(\Phi\mathcal{G}) = \mathrm{id}_{\mathcal{G}}$ *and* $(\Psi\mathcal{F})(\mathcal{F}\Phi) = \mathrm{id}_{\mathcal{F}}$.

Definition 8.9.12. The natural transformations $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{G}\mathcal{F}$ and $\Psi : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{Id}_{\mathcal{D}}$ given in 8.9.11 are called *unit* and *counit* resp. for the adjoint functors $\mathcal{F}$ and $\mathcal{G}$.

Problem 8.9.2. 1. Let ${}_RM_S$ be a bimodule. Show that the functor $M \otimes_S$ - : ${}_S\mathcal{M} \longrightarrow {}_R\mathcal{M}$ is leftadjoint to $\mathrm{Hom}_R(M,\text{-}) : {}_R\mathcal{M} \longrightarrow {}_S\mathcal{M}$. Determine the associated unit and counit.

b) Show that there is a natural isomorphism $\mathrm{Map}(A \times B, C) \cong \mathrm{Map}(B, \mathrm{Map}(A, C))$. Determine the associated unit and counit.

c) Show that there is a natural isomorphism $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}G, A) \cong \mathbf{Gr}(G, U(A))$. Determine the associated unit and counit.

d) Show that there is a natural isomorphism $\mathbb{K}\text{-}\mathbf{Alg}(U(\mathfrak{g}), A) \cong \text{Lie-Alg}(\mathfrak{g}, A^L)$. Determine the corresponding leftadjoint functor and the associated unit and counit.

Definition 8.9.13. Let $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be a covariant functor. $\mathcal{G}$ generates a *(co-)universal problem* a follows:

Given $C \in \mathcal{C}$. Find an object $\mathcal{F}(C) \in \mathcal{D}$ and a morphism $\iota : C \longrightarrow \mathcal{G}(\mathcal{F}(C))$ in $\mathcal{C}$ such that there is a unique morphism $g : \mathcal{F}(C) \longrightarrow D$ in $\mathcal{D}$ for each object $D \in \mathcal{D}$ and for each morphism $f : C \longrightarrow \mathcal{G}(D)$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathcal{G}(\mathcal{F}(C)) \arrow[d, "\mathcal{G}(g)"] \\ & \mathcal{G}(D) \end{tikzcd}$$

commutes.

A pair $(\mathcal{F}(C), \iota)$ that satisfies the above conditions is called a *universal solution* of the (co-)universal problem defined by $\mathcal{G}$ and C.

Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor. $\mathcal{F}$ generates a *universal problem* a follows:

Given $D \in \mathcal{D}$. Find an object $\mathcal{G}(D) \in \mathcal{C}$ and a morphism $\nu : \mathcal{F}(\mathcal{G}(D)) \longrightarrow D$ in $\mathcal{D}$ such that there is a unique morphism $g : C \longrightarrow \mathcal{G}(D)$ in $\mathcal{C}$ for each object $C \in \mathcal{C}$ and for each morphism $f : \mathcal{F}(C) \longrightarrow D$ in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{F}(C) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "f"] & \\ \mathcal{F}\mathcal{G}(D) \arrow[r, "\nu"'] & D \end{tikzcd}$$

commutes.

A pair $(\mathcal{G}(D), \nu)$ that satisfies the above conditions is called a *universal solution* of the (co-)universal problem defined by $\mathcal{F}$ and D.

Proposition 8.9.14. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be leftadjoint to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$. Then $\mathcal{F}(C)$ and the unit $\iota = \Phi(C) : C \longrightarrow \mathcal{G}\mathcal{F}(C)$ form a (co-)universal solution for the (co-)universal problem defined by $\mathcal{G}$ and C.*

Furthermore $\mathcal{G}(D)$ and the counit $\nu = \Psi(D) : \mathcal{F}\mathcal{G}(D) \longrightarrow D$ form a universal solution for the universal problem defined by $\mathcal{F}$ and D.

PROOF. By Theorem 8.9.10 the morphisms $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})$ and $\psi : \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})$ are inverses of each other. They are defined with unit and counit as $\phi(C, D)(g) = \mathcal{G}(g)\Phi(C)$ resp. $\psi(C, D)(f) = \Psi(D)\mathcal{F}(f)$. Hence for each $f : C \longrightarrow \mathcal{G}(D)$ there is a unique $g : \mathcal{F}(C) \longrightarrow D$ such that $f = \phi(C, D)(g) = \mathcal{G}(g)\Phi(C) = \mathcal{G}(g)\iota$.

The second statement follows analogously. $\square$

Remark 8.9.15. If $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and $C \in \mathcal{C}$ are given then the (co-)universal solution $(\mathcal{F}(C), \iota : C \longrightarrow \mathcal{G}(D))$ can be considered as the best (co-)approximation of the object C in $\mathcal{C}$ by an object D in $\mathcal{D}$ with the help of a functor $\mathcal{G}$. The object $D \in \mathcal{D}$ turns out to be $\mathcal{F}(C)$.

If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $D \in \mathcal{D}$ are given then the universal solution $(\mathcal{G}(D), \nu : \mathcal{F}\mathcal{G}(D) \longrightarrow D)$ can be considered as the best approximation of the object D in $\mathcal{D}$ by an object C in $\mathcal{C}$ with the help of a functor $\mathcal{F}$. The object $C \in \mathcal{C}$ turns out to be $\mathcal{G}(D)$.

Proposition 8.9.16. *Given $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$. Assume that for each $C \in \mathcal{C}$ the universal problem defined by $\mathcal{G}$ and C is solvable. Then there is a leftadjoint functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ to $\mathcal{G}$.*

Given $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. Assume that for each $D \in \mathcal{D}$ the universal problem defined by $\mathcal{F}$ and D is solvable. Then there is a leftadjoint functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ to $\mathcal{F}$.

PROOF. Assume that the (co-)universal problem defined by $\mathcal{G}$ and C is solved by $\iota : C \longrightarrow \mathcal{F}(C)$. Then the map $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D)) \ni f \mapsto g \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D)$ with $\mathcal{G}(g)\iota = f$ is bijective. The inverse map is given by $g \mapsto \mathcal{G}(g)\iota$. This is a natural transformation since the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D) \arrow[r, "\mathcal{G}(\text{-})\iota"] \arrow[d, "\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),h)"'] & \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D)) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(C,\mathcal{G}(h))"] \\
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D') \arrow[r, "\mathcal{G}(\text{-})\iota"'] & \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D'))
\end{tikzcd}$$

commutes for each $h \in \mathrm{Mor}_D(D, D')$. In fact we have

$$\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(h))(\mathcal{G}(g)\iota) = \mathcal{G}(h)\mathcal{G}(g)\iota = \mathcal{G}(hg)\iota = \mathcal{G}(\mathrm{Mor}_{\mathcal{C}}(\mathcal{F}(C), h)(g))\iota.$$

Hence for all $C \in \mathcal{C}$ the functor $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(\text{-})) : \mathcal{D} \longrightarrow \mathbf{Set}$ induced by the bifunctor $\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\text{-})) : \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathbf{Set}$ is representable. By Theorem 8.9.5 there is a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\text{-})) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(\text{-}), \text{-})$.

The second statement follows analogously. $\square$

Remark 8.9.17. One can characterize the properties that $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ (resp. $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$) must have in order to possess a left-(right-)adjoint functor. One of the essential properties for this is that $\mathcal{G}$ preserves limits (hence direct products and difference kernels).