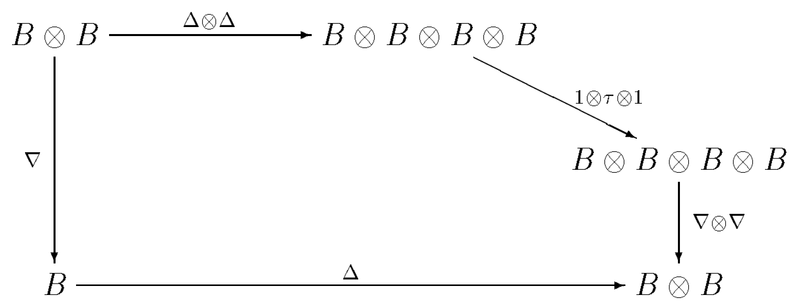
CHAPTER 8

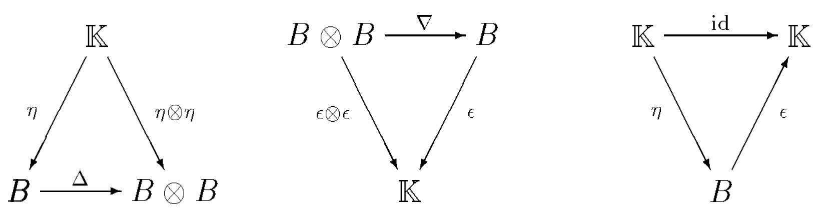
Toolbox

7. Bialgebras

Definition 8.7.1. 1. A *bialgebra* $(B, \nabla, \eta, \Delta, \epsilon)$ consists of an algebra (B, ∇, η) and a coalgebra (B, Δ, ϵ) such that the diagrams



and



commute, i.e. Δ and ϵ are homomorphisms of algebras resp. ∇ and η are homomorphisms of coalgebras.

2. Given bialgebras A and B. A map $f : A \longrightarrow B$ is called a *homomorphism of bialgebras* if it is a homomorphism of algebras and a homomorphism of coalgebras.

3. The category of bialgebras is denoted by $\mathbb{K}$-**Bialg**.

Problem 8.7.1. 1. Let (B, ∇, η) be an algebra and (B, Δ, ε) be a coalgebra. The following are equivalent:

a) $(B, \nabla, \eta, \Delta, \varepsilon)$ is a bialgebra.
b) $\Delta : B \longrightarrow B \otimes B$ and $\varepsilon : B \longrightarrow \mathbb{K}$ are homomorphisms of $\mathbb{K}$-algebras.
c) $\nabla : B \otimes B \longrightarrow B$ and $\eta : \mathbb{K} \longrightarrow B$ are homomorphisms of $\mathbb{K}$-coalgebras.

2. Let B be a finite dimensional bialgebra over field $\mathbb{K}$. Show that the dual space B^* is a bialgebra.

One of the most important properties of bialgebras B is that the tensor product over $\mathbb{K}$ of two B-modules or two B-comodules is again a B-module.

Proposition 8.7.2. 1. *Let B be a bialgebra. Let M and N be left B-modules. Then $M \otimes_{\mathbb{K}} N$ is a B-module by the map*

$$B \otimes M \otimes N \xrightarrow{\Delta\otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{1\otimes\tau\otimes 1} B \otimes M \otimes B \otimes N \xrightarrow{\mu\otimes\mu} M \otimes N.$$

2. *Let B be a bialgebra. Let M and N be left B-comodules. Then $M \otimes_{\mathbb{K}} N$ is a B-comodule by the map*

$$M \otimes N \xrightarrow{\delta\otimes\delta} B \otimes M \otimes B \otimes N \xrightarrow{1\otimes\tau\otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{\nabla\otimes 1} B \otimes M \otimes N.$$

3. $\mathbb{K}$ *is a* B*-module by the map* $B \otimes \mathbb{K} \cong B \xrightarrow{\varepsilon} \mathbb{K}$.
4. $\mathbb{K}$ *is a* B*-comodule by the map* $\mathbb{K} \xrightarrow{\eta} B \cong B \otimes \mathbb{K}$.

PROOF. We give a diagrammatic proof for 1. The associativity law is given by

$$\begin{array}{ccccccc}
B\otimes B\otimes M\otimes N & \xrightarrow{1\otimes\Delta\otimes 1\otimes 1} & B\otimes B\otimes B\otimes M\otimes N & \xrightarrow{1\otimes 1\otimes\tau\otimes 1} & B\otimes B\otimes M\otimes B\otimes N & \xrightarrow{1\otimes\mu\otimes\mu} & B\otimes M\otimes N \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \Delta\otimes 1\otimes 1\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle \Delta\otimes 1\otimes 1\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle \Delta\otimes 1\otimes 1} \\
\Big| & & B\otimes B\otimes B\otimes B\otimes M\otimes N & \xrightarrow{1\otimes 1\otimes 1\otimes\tau\otimes 1} & B\otimes B\otimes B\otimes M\otimes B\otimes N & \xrightarrow{1\otimes 1\otimes\mu\otimes\mu} & B\otimes B\otimes M\otimes N \\
\Big|{\scriptstyle \nabla\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle 1\otimes\tau\otimes 1\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle 1\otimes\tau(B,B\otimes M)\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle 1\otimes\tau\otimes 1} \\
\Big| & & B\otimes B\otimes B\otimes B\otimes M\otimes N & \xrightarrow{1\otimes 1\otimes\tau(B\otimes B,M)\otimes 1} & B\otimes B\otimes M\otimes B\otimes B\otimes N & \xrightarrow{1\otimes\mu\otimes 1\otimes\mu} & B\otimes M\otimes B\otimes N \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \nabla\otimes\nabla\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle \nabla\otimes 1\otimes\nabla\otimes 1} & & \Big\downarrow{\scriptstyle \mu\otimes\mu} \\
B\otimes M\otimes N & \xrightarrow{\Delta\otimes 1\otimes 1} & B\otimes B\otimes M\otimes N & \xrightarrow{1\otimes\tau\otimes 1} & B\otimes M\otimes B\otimes N & \xrightarrow{\mu\otimes\mu} & M\otimes N
\end{array}$$

The unit law is the commutativity of

$$\begin{array}{ccccc}
M\otimes N & \cong & \mathbb{K}\otimes M\otimes N & \xrightarrow{\eta\otimes 1\otimes 1} & B\otimes M\otimes N \\
\Big| & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle \Delta\otimes 1\otimes 1} \\
= & & \mathbb{K}\otimes\mathbb{K}\otimes M\otimes N & \xrightarrow{\eta\otimes\eta\otimes 1\otimes 1} & B\otimes B\otimes M\otimes N \\
\Big\downarrow & & \Big\downarrow{\scriptstyle 1\otimes\tau\otimes 1} & & \Big\downarrow{\scriptstyle 1\otimes\tau\otimes 1} \\
M\otimes N & \cong & \mathbb{K}\otimes M\otimes\mathbb{K}\otimes N & \xrightarrow{\eta\otimes 1\otimes\eta\otimes 1} & B\otimes M\otimes B\otimes N \\
 & & & \searrow{\scriptstyle 1} & \Big\downarrow{\scriptstyle \mu\otimes\mu} \\
 & & & & M\otimes N
\end{array}$$

The corresponding properties for comodules follows from the dualized diagrams. The module and comodule properties of $\mathbb{K}$ are easily checked. □

Definition 8.7.3. 1. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let A be a left B-module with structure map $\mu : B \otimes A \longrightarrow A$. Let furthermore (A, ∇_A, η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-modules. Then $(A, \nabla_A, \eta_A, \mu)$ is called a *B-module algebra.*

2. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let C be a left B-module with structure map $\mu : B \otimes C \longrightarrow C$. Let furthermore $(C, \Delta_C, \varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B- modules. Then $(C, \Delta_C, \varepsilon_C, \mu)$ is called a *B-module coalgebra.*

3. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let A be a left B-comodule with structure map $\delta : A \longrightarrow B \otimes A$. Let furthermore (A, ∇_A, η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-comodules. Then $(A, \nabla_A, \eta_A, \delta)$ is called a *B-comodule algebra.*

4. Let $(B, \nabla, \eta, \Delta, \epsilon)$ be a bialgebra. Let C be a left B-comodule with structure map $\delta : C \longrightarrow B \otimes C$. Let furthermore $(C, \Delta_C, \varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B-comodules. Then $(C, \Delta_C, \varepsilon_C, \delta)$ is called a *B-comodule coalgebra.*

Remark 8.7.4. If $(C, \Delta_C, \varepsilon_C)$ is a $\mathbb{K}$-coalgebra and (C, μ) is a B-module, then $(C, \Delta_C, \varepsilon_C, \mu)$ is a B-module coalgebra iff μ is a homomorphism of $\mathbb{K}$-coalgebras.

If (A, ∇_A, η_A) is a $\mathbb{K}$-algebra and (A, δ) is a B-comodule, then $(A, \nabla_A, \eta_A, \delta)$ is a B-comodule algebra iff δ is a homomorphism of $\mathbb{K}$-algebras.

Similar statement for module algebras or comodule coalgebras do *not* hold.