CHAPTER 8

Toolbox

4. Tensor Products

Definition and Remark 8.4.1. Let M_R and ${}_RN$ be R-modules, and let A be an abelian group. A map $f : M \times N \longrightarrow A$ is called *R-bilinear* if

1. $f(m + m', n) = f(m, n) + f(m', n)$,
2. $f(m, n + n') = f(m, n) + f(m, n')$,
3. $f(mr, n) = f(m, rn)$

for all $r \in R,\ m, m' \in M,\ n, n' \in N$.

Let $\mathrm{Bil}_R(M, N; A)$ denote the set of all R-bilinear maps $f : M \times N \longrightarrow A$.

$\mathrm{Bil}_R(M, N; A)$ is an abelian group with $(f + g)(m, n) := f(m, n) + g(m, n)$.

Definition 8.4.2. Let M_R and ${}_RN$ be R-modules. An abelian group $M \otimes_R N$ together with an R-bilinear map

$$\otimes : M \times N \ni (m, n) \mapsto m \otimes n \in M \otimes_R N$$

is called a *tensor product of M and N over R* if for each abelian group A and for each R-bilinear map $f : M \times N \longrightarrow A$ there exists a unique group homomorphism $g : M \otimes_R N \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "g"] \\
& A
\end{tikzcd}$$

commutes. The elements of $M \otimes_R N$ are called *tensors*, the elements of the form $m \otimes n$ are called *decomposable tensors*.

Warning: If you want to define a homomorphism $f : M \otimes_R N \longrightarrow A$ with a tensor product as domain you *must* define it by giving an R-bilinear map defined on $M \times N$.

Lemma 8.4.3. *A tensor product $(M \otimes_R N, \otimes)$ defined by M_R and ${}_RN$ is unique up to a unique isomorphism.*

PROOF. Let $(M \otimes_R N, \otimes)$ and $(M \boxtimes_R N, \boxtimes)$ be tensor products. Then

$$\begin{tikzcd}
& & M \times N \arrow[dll, "\otimes"'] \arrow[dl, "\boxtimes"] \arrow[dr, "\otimes"'] \arrow[drr, "\boxtimes"] & & \\
M \otimes_R N \arrow[r, "h"] & M \boxtimes_R N \arrow[rr, "k"] & & M \otimes_R N \arrow[r, "h"] & M \boxtimes_R N
\end{tikzcd}$$

implies $k = h^{-1}$. $\square$

Because of this fact we will henceforth talk about *the* tensor product of M and N over R.

Proposition 8.4.4. (Rules of computation in a tensor product) *Let $(M \otimes_R N, \otimes)$ be the tensor product. Then we have for all $r \in R$, $m, m' \in M$, $n, n' \in N$*

1. $M \otimes_R N = \{\sum_i m_i \otimes n_i \mid m_i \in M, n_i \in N\}$,
2. $(m + m') \otimes n = m \otimes n + m' \otimes n$,
3. $m \otimes (n + n') = m \otimes n + m \otimes n'$,
4. $mr \otimes n = m \otimes rn$ *(observe in particular, that* $\otimes : M \times N \longrightarrow M \otimes N$ *is not injective in general),*
5. *if* $f : M \times N \longrightarrow A$ *is an* R*-bilinear map and* $g : M \otimes_R N \longrightarrow A$ *is the induced homomorphism, then*
$$g(m \otimes n) = f(m, n).$$

PROOF. 1. Let $B := \langle m \otimes n \rangle \subseteq M \otimes_R N$ denote the subgroup of $M \otimes_R N$ generated by the decomposable tensors $m \otimes n$. Let $j : B \longrightarrow M \otimes_R N$ be the embedding homomorphism. We get an induced map $\otimes' : M \times N \longrightarrow B$. In the following diagram

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes'"] \arrow[dr, "\otimes'"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"] & M \otimes_R N \arrow[dl, "p"'] \arrow[d, "jp"] \\
 & B \arrow[r, "j"] & M \otimes_R N
\end{tikzcd}$$

we have $\mathrm{id}_B \circ \otimes' = \otimes'$, p with $p \circ j \circ \otimes' = p \circ \otimes = \otimes'$ exists since $\otimes'$ is R-bilinear. Because of $jp \circ \otimes = j \circ \otimes' = \otimes = \mathrm{id}_{M \otimes_R N} \circ \otimes$ we get $jp = \mathrm{id}_{M \otimes_R N}$, hence the embedding j is surjective and thus the identity.

2. $(m + m') \otimes n = \otimes(m + m', n) = \otimes(m, n) + \otimes(m', n) = m \otimes n + m' \otimes n$.

3. and 4. analogously.

5. is precisely the definition of the induced homomorphism. $\square$

Remark 8.4.5. To construct tensor products, we use the notion of a free module.

Let X be a set and R be a ring. An R-module RX together with a map $\iota : X \longrightarrow RX$ is called a *free* R*-module generated by* X, if for every R-module M and for every map $f : X \longrightarrow M$ there exists a unique homomorphism of R-modules $g : RX \longrightarrow M$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "f"'] & RX \arrow[d, "g"] \\
 & M
\end{tikzcd}$$

commutes.

Free R-modules exist and can be constructed as $RX := \{\alpha : X \longrightarrow R \mid$ for almost all $x \in X : \alpha(x) = 0\}$.

Proposition 8.4.6. *Given* R*-modules* M_R *and* ${}_RN$*. Then there exists a tensor product* $(M \otimes_R N, \otimes)$.

PROOF. Define $M \otimes_R N := \mathbb{Z}\{M \times N\}/U$ where $\mathbb{Z}\{M \times N\}$ is a free $\mathbb{Z}$-module over $M \times N$ (the free abelian group) and U is generated by

$$\iota(m+m',n)-\iota(m,n)-\iota(m',n)$$
$$\iota(m,m+n')-\iota(m,n)-\iota(m,n')$$
$$\iota(mr,n)-\iota(m,rn)$$

for all $r\in R$, $m,m'\in M$, $n,n'\in N$. Consider

$$\begin{tikzcd} M\times N \arrow[r, "\iota"] \arrow[drr, "\psi"'] & \mathbb{Z}\{M\times N\} \arrow[r, "\nu"] \arrow[dr, "\rho"] & M\otimes_R N = \mathbb{Z}\{M\times N\}/U \arrow[d, "g"] \\ & & A \end{tikzcd}$$

Let ψ be given. Then there is a unique $\rho\in\mathrm{Hom}(\mathbb{Z}\{M\times N\},A)$ such that $\rho\iota=\psi$. Since ψ is R-bilinear we get $\rho(\iota(m+m',n)-\iota(m,n)-\iota(m'n))=\psi(m+m',n)-\psi(m,n)-\psi(m',n)=0$ and similarly $\rho(\iota(m,n+n')-\iota(m,n)-\iota(m,n'))=0$ and $\rho(\iota(mr,n)-\iota(m,rn))=0$. So we get $\rho(U)=0$. This implies that there is a unique $g\in\mathrm{Hom}(M\otimes_R N,A)$ such that $g\nu=\rho$ (homomorphism theorem). Let $\otimes:=\nu\circ\iota$. Then $\otimes$ is bilinear since $(m+m')\otimes n=\nu\circ\iota(m+m',n)=\nu(\iota(m+m',n))=\nu(\iota(m+m',n)-\iota(m,n)-\iota(m',n)+\iota(m,n)+\iota(m',n))=\nu(\iota(m,n)+\iota(m',n))=\nu\circ\iota(m,n)+\nu\circ\iota(m',n)=m\otimes n+m'\otimes n$. The other two properties are obtained in an analogous way.

We have to show that $(M\otimes_R N,\otimes)$ is a tensor product. The above diagram shows that for each abelian group A and for each R-bilinear map $\psi: M\times N\longrightarrow A$ there is a $g\in\mathrm{Hom}(M\otimes_R N,A)$ such that $g\circ\otimes=\psi$. Given $h\in\mathrm{Hom}(M\otimes_R N,A)$ with $h\circ\otimes=\psi$. Then $h\circ\nu\circ\iota=\psi$. This implies $h\circ\nu=\rho=g\circ\nu$ hence $g=h$. $\square$

Proposition and Definition 8.4.7. *Given two homomorphisms*

$$f\in\mathrm{Hom}_R(M.,M'.)\ \textit{and}\ g\in\mathrm{Hom}_R(.N,.N').$$

Then there is a unique homomorphism

$$f\otimes_R g\in\mathrm{Hom}(M\otimes_R N,M'\otimes_R N')$$

such that $f\otimes_R g(m\otimes n)=f(m)\otimes g(n)$, i.e. the following diagram commutes

$$\begin{tikzcd} M\times N \arrow[r, "\otimes"] \arrow[d, "f\times g"'] & M\otimes_R N \arrow[d, "f\otimes_R g"] \\ M'\times N' \arrow[r, "\otimes"'] & M'\otimes_R N' \end{tikzcd}$$

PROOF. $\otimes\circ(f\times g)$ is bilinear. $\square$

Notation 8.4.8. We often write $f\otimes_R N:=f\otimes_R 1_N$ and $M\otimes_R g:=1_M\otimes_R g$. We have the following rule of computation:

$$f\otimes_R g=(f\otimes_R N')\circ(M\otimes_R g)=(M'\otimes_R g)\circ(f\otimes_R N)$$

since $f\times g=(f\times N')\circ(M\times g)=(M'\times g)\circ(f\times N)$.

Proposition 8.4.9. *The following define covariant functors*

1. $- \otimes N : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Ab}$;
2. $M \otimes - : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$;
3. $- \otimes - : \mathbf{Mod}\text{-}R \times R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$.

PROOF. $(f \times g) \circ (f' \times g') = ff' \times gg'$ implies $(f \otimes_R g) \circ (f' \otimes_R g') = ff' \times gg'$. Furthermore $1_M \times 1_N = 1_{M \times N}$ implies $1_M \otimes_R 1_N = 1_{M \otimes_R N}$. $\square$

Definition 8.4.10. Let R, S be rings and let M be a left R-module and a right S-module. M is called an *R-S-bimodule* if $(rm)s = r(ms)$. We define $\mathrm{Hom}_{R\text{-}S}(.M.,.N.) := \mathrm{Hom}_R(.M,.N) \cap \mathrm{Hom}_S(M.,N.)$.

Remark 8.4.11. Let M_S be a right S-module and let $R \times M \longrightarrow M$ a map. M is an R-S-bimodule if and only if

1. $\forall r \in R : (M \ni m \mapsto rm \in M) \in \mathrm{Hom}_S(M.,M.)$,
2. $\forall r, r' \in R, m \in M : (r + r')m = rm + r'm$,
3. $\forall r, r' \in R, m \in M : (rr')m = r(r'm)$,
4. $\forall m \in M : 1m = m$.

Lemma 8.4.12. *Let ${}_RM_S$ and ${}_SN_T$ be bimodules. Then ${}_R(M \otimes_S N)_T$ is a bimodule by $r(m \otimes n) := rm \otimes n$ and $(m \otimes n)t := m \otimes nt$.*

PROOF. Obviously we have 2.-4. Furthermore $(r \otimes_S \mathrm{id})(m \otimes n) = rm \otimes n = r(m \otimes n)$ is a homomorphism. $\square$

Corollary 8.4.13. *Given bimodules ${}_RM_S$, ${}_SN_T$, ${}_RM'_S$, ${}_SN'_T$ and homomorphisms $f \in \mathrm{Hom}_{R\text{-}S}(.M.,.M'.)$ and $g \in \mathrm{Hom}_{S\text{-}T}(.N.,.N'.)$. Then we have $f \otimes_S g \in \mathrm{Hom}_{R\text{-}T}(.M \otimes_S N.,.M' \otimes_S N'.)$.*

PROOF. $f \otimes_S g(rm \otimes nt) = f(rm) \otimes g(nt) = r(f \otimes_S g)(m \otimes n)t$. $\square$

Remark 8.4.14. Every module M over a commutative ring $\mathbb{K}$ and in particular every vector space over a field $\mathbb{K}$ is a $\mathbb{K}$-$\mathbb{K}$-bimodule by $\lambda m = m\lambda$. So there is an embedding functor $\iota : \mathbb{K}\text{-}\mathbf{Mod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K}$. Observe that there are $\mathbb{K}$-$\mathbb{K}$-bimodules that do not satisfy $\lambda m = m\lambda$. Take for example an automorphism $\alpha : \mathbb{K} \longrightarrow \mathbb{K}$ and a left $\mathbb{K}$-module M and define $m\lambda := \alpha(\lambda)m$. Then M is such a $\mathbb{K}$-$\mathbb{K}$-bimodule.

The tensor product $M \otimes_{\mathbb{K}} N$ of two $\mathbb{K}$-$\mathbb{K}$-bimodules M and N is again a $\mathbb{K}$-$\mathbb{K}$-bimodule. If we have, however, $\mathbb{K}$-$\mathbb{K}$-bimodules M and N arising from $\mathbb{K}$-modules as above, i.e. satisfying $\lambda m = m\lambda$, then their tensor product $M \otimes_{\mathbb{K}} N$ also satisfies this equation, so $M \otimes_{\mathbb{K}} N$ comes from a module in $\mathbb{K}$-$\mathbf{Mod}$. Indeed we have $\lambda m \otimes n = m\lambda \otimes n = m \otimes \lambda n = m \otimes n\lambda$. Thus the following diagram of functors commutes:

$$\begin{tikzcd}
\mathbb{K}\text{-}\mathbf{Mod} \times \mathbb{K}\text{-}\mathbf{Mod} \arrow[r, "\iota \times \iota"] \arrow[d, "\otimes_{\mathbb{K}}"'] & \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K} \times \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K} \arrow[d, "\otimes_{\mathbb{K}}"] \\
\mathbb{K}\text{-}\mathbf{Mod} \arrow[r, "\iota"'] & \mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{K}.
\end{tikzcd}$$

So we can consider $\mathbb{K}$-**Mod** as a (proper) subcategory of $\mathbb{K}$-**Mod**-$\mathbb{K}$. The tensor product over $\mathbb{K}$ can be restricted to $\mathbb{K}$-**Mod**.

We write the tensor product of two vector spaces M and N as $M \otimes N$.

Theorem 8.4.15. *In the category* $\mathbb{K}$-**Mod** *there are natural isomorphisms*

1. Associativity Law: $\alpha : (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$.
2. Law of the Left Unit: $\lambda : \mathbb{K} \otimes M \cong M$.
3. Law of the Right Unit: $\rho : M \otimes \mathbb{K} \cong M$.
4. Symmetry Law: $\tau : M \otimes N \cong N \otimes M$.
5. Existence of Inner Hom-Functors: $\mathrm{Hom}(P \otimes M, N) \cong \mathrm{Hom}(P, \mathrm{Hom}(M, N))$.

Proof. We only describe the corresponding homomorphisms.

1. Use (8.4.45.) to define $\alpha((m \otimes n) \otimes p) := m \otimes (n \otimes p)$.
2. Define $\lambda : \mathbb{K} \otimes M \longrightarrow M$ by $\lambda(r \otimes m) := rm$.
3. Define $\rho : M \otimes \mathbb{K} \longrightarrow M$ by $\rho(m \otimes r) := mr$.
4. Define $\tau(m \otimes n) := n \otimes m$.
5. For $f : P \otimes M \longrightarrow N$ define $\phi(f) : P \longrightarrow \mathrm{Hom}(M, N)$ by $\phi(f)(p)(m) := f(p \otimes m)$. $\square$

Usually one identifies threefold tensor products along the map α so that we use $M \otimes N \otimes P = (M \otimes N) \otimes P = M \otimes (N \otimes P)$. For the notion of a monoidal or tensor category, however, this natural transformation is of central importance.

Problem 8.4.1. 1. Give an explicit proof of $M \otimes (X \oplus Y) \cong M \otimes X \oplus M \otimes Y$.

2. Show that for every finite dimensional vector space V there is a unique element $\sum_{i=1}^{n} v_i \otimes v_i^* \in V \otimes V^*$ such that the following holds

$$\forall v \in V : \quad \sum_i v_i^*(v) v_i = v.$$

(Hint: Use an isomorphism $\mathrm{End}(V) \cong V \otimes V^*$ and dual bases $\{v_i\}$ of V and $\{v_i^*\}$ of V^*.)

3. Show that the following diagrams (coherence diagrams or constraints) commute in $\mathbb{K}$-**Mod**:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\alpha(A\otimes B,C,D)"] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha(B,C,D)"'] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \mathbb{K}) \otimes B \arrow[rr, "\alpha(A,\mathbb{K},B)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (\mathbb{K} \otimes B) \arrow[dl, "1\otimes\lambda(B)"] \\
& A \otimes B &
\end{tikzcd}$$

4. Write $\tau(A,B) : A \otimes B \longrightarrow B \otimes A$ for $\tau(A,B) : a \otimes b \mapsto b \otimes a$. Show that τ is a natural transformation (between which functors?). Show that

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\tau(X,B)\otimes 1"] \arrow[d, "\alpha"] & (B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[d, "1\otimes\tau(A,C)"] \\
A \otimes (B \otimes C) \arrow[r, "\tau(A,B\otimes C)"] & (B \otimes C) \otimes A \arrow[r, "\alpha"] & B \otimes (C \otimes A)
\end{tikzcd}$$

commutes for all $A, B, C \in \mathbb{K}$-$\mathbf{Mod}$ and that

$$\tau(B,A)\tau(A,B) = \mathrm{id}_{A\otimes B}$$

for all A, B in $\mathbb{K}$-$\mathbf{Mod}$.

5. Find an example of M, $N \in \mathbb{K}$-$\mathbf{Mod}$-$\mathbb{K}$ such that $M \otimes_{\mathbb{K}} N \not\cong N \otimes_{\mathbb{K}} M$.