CHAPTER 8

Toolbox

2. Functors

Definition 8.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $\mathcal{F}$ consist of

1. a map $\mathrm{Ob}\,\mathcal{C} \ni A \mapsto \mathcal{F}(A) \in \mathrm{Ob}\,\mathcal{D}$,
2. a family of maps

$$\{\mathcal{F}_{A,B} : \mathrm{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \mathcal{F}_{A,B}(f) \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(A),\mathcal{F}(B)) | A, B \in \mathcal{C}\}$$

$$[\text{or } \{\mathcal{F}_{A,B} : \mathrm{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \mathcal{F}_{A,B}(f) \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(B),\mathcal{F}(A)) | A, B \in \mathcal{C}\}]$$

$\mathcal{F}$ is called a *covariant* [*contravariant*] *functor* if

1. $\mathcal{F}_{A,A}(1_A) = 1_{\mathcal{F}(A)}$ for all $A \in \mathrm{Ob}\,\mathcal{C}$,
2. $\mathcal{F}_{A,C}(gf) = \mathcal{F}_{B,C}(g)\mathcal{F}_{A,B}(f)$ for all $A, B, C \in \mathrm{Ob}\,\mathcal{C}$.
   $[\mathcal{F}_{A,C}(gf) = \mathcal{F}_{A,B}(f)\mathcal{F}_{B,C}(g)$ for all $A, B, C \in \mathrm{Ob}\,\mathcal{C}]$.

Notation: We write

$A \in \mathcal{C}$	instead of	$A \in \mathrm{Ob}\,\mathcal{C}$
$f \in \mathcal{C}$	instead of	$f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$
$\mathcal{F}(f)$	instead of	$\mathcal{F}_{A,B}(f)$.

Examples 8.2.2.
1. $\mathrm{Id} : \mathbf{Set} \longrightarrow \mathbf{Set}$
2. Forget : R-$\mathbf{Mod} \longrightarrow \mathbf{Set}$
3. Forget : $\mathbf{Ri} \longrightarrow \mathbf{Ab}$
4. Forget : $\mathbf{Ab} \longrightarrow \mathbf{Gr}$
5. $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}, \mathcal{P}(M) :=$ power set of M. $\mathcal{P}(f)(X) := f^{-1}(X)$ for $f : M \longrightarrow N, X \subseteq N$ is a contravariant functor.
6. $\mathcal{Q} : \mathbf{Set} \longrightarrow \mathbf{Set}, \mathcal{Q}(M) :=$ power set of M. $\mathcal{Q}(f)(X) := f(X)$ for $f : M \longrightarrow N, X \subseteq M$ is a covariant functor.

Lemma 8.2.3.
1. *Let $X \in \mathcal{C}$. Then*

$$\mathrm{Ob}\,\mathcal{C} \ni A \mapsto \mathrm{Mor}_{\mathcal{C}}(X,A) \in \mathrm{Ob}\,\mathbf{Set}$$

$$\mathrm{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto Mor_{\mathcal{C}}(X,f) \in \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathcal{C}}(X,A),\mathrm{Mor}_{C}(X,B)),$$

   with $\mathrm{Mor}_C(X,f) : \mathrm{Mor}_C(X,A) \ni g \mapsto fg \in \mathrm{Mor}_C(X,B)$ *or* $\mathrm{Mor}_C(X,f)(g) = fg$ *is a covariant functor* $\mathrm{Mor}_C(X,\text{-})$.
2. *Let $X \in C$. Then*

$$\mathrm{Ob}\,\mathcal{C} \ni A \mapsto \mathrm{Mor}_{\mathcal{C}}(A,X) \in \mathrm{Ob}\,\mathbf{Set}$$

$$\mathrm{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \mathrm{Mor}_{\mathcal{C}}(f,X) \in \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathcal{C}}(B,X),\mathrm{Mor}_{\mathcal{C}}(A,X))$$

   with $\mathrm{Mor}_{\mathcal{C}}(f,X) : \mathrm{Mor}_{\mathcal{C}}(B,X) \ni g \mapsto gf \in \mathrm{Mor}_{\mathcal{C}}(A,X)$ *or* $\mathrm{Mor}_{\mathcal{C}}(f,X)(g) = gf$ *is a contravariant functor* $\mathrm{Mor}_{\mathcal{C}}(\text{-},X)$.

Proof. 1. $\mathrm{Mor}_{\mathcal{C}}(X,1_A)(g) = 1_A g = g = \mathrm{id}(g)$, $\mathrm{Mor}_C(X,f)\mathrm{Mor}_C(X,g)(h) = fgh = \mathrm{Mor}_C(X,fg)(h)$.
2. analogously. □

Remark 8.2.4. The preceding lemma shows that $\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})$ is a functor in both arguments. A functor in two arguments is called a *bifunctor*. We can regards the bifunctor $\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})$ as a covariant functor

$$\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-}) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{Set}.$$

The use of the dual category removes the fact that the bifunctor $\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})$ is contravariant in the first variable.

Obviously the composition of two functors is again a functor and this composition is associative. Furthermore for each category $\mathcal{C}$ there is an identity functor $\mathrm{Id}_{\mathcal{C}}$.

Functors of the form $\mathrm{Mor}_{\mathcal{C}}(X,\text{-})$ resp. $\mathrm{Mor}_{\mathcal{C}}(\text{-},X)$ are called *representable functors* (covariant resp. contravariant) and X is called the *representing object* (see also section 8.8).