CHAPTER 4

The Infinitesimal Theory

3. The Lie Algebra of Primitive Elements

Lemma 4.3.1. *Let H be a Hopf algebra and H° be its Sweedler dual. If $d \in \mathrm{Der}_{\mathbb{K}}(H, {}_\varepsilon\mathbb{K}_\varepsilon) \subseteq \mathrm{Hom}(H, \mathbb{K})$ is a derivation then d is a primitive element of H°. Furthermore every primitive element $d \in H^\circ$ is a derivation in $\mathrm{Der}_{\mathbb{K}}(H, {}_\varepsilon\mathbb{K}_\varepsilon)$.*

Proof. Let $d : H \longrightarrow \mathbb{K}$ be a derivation and let $a, b \in H$. Then $(b \rightharpoonup d)(a) = d(ab) = \varepsilon(a)d(b) + d(a)\varepsilon(b) = (d(b)\varepsilon + \varepsilon(b)d)(a)$ hence $(b \rightharpoonup d) = d(b)\varepsilon + \varepsilon(b)d$. Consequently we have $Hd = (H \rightharpoonup d) \subseteq \mathbb{K}\varepsilon + \mathbb{K}d$ so that $\dim Hd \leq 2 < \infty$. This shows $d \in H^\circ$. Furthermore we have $\langle \Delta d, a \otimes b\rangle = \langle d, ab\rangle = d(ab) = d(a)\varepsilon(b) + \varepsilon(a)d(b) = \langle d \otimes \varepsilon, a \otimes b\rangle + \langle \varepsilon \otimes d, a \otimes b\rangle = \langle 1_{H^\circ} \otimes d + d \otimes 1_{H^\circ}, a \otimes b\rangle$ hence $\Delta(d) = d \otimes 1_{H^\circ} + 1_{H^\circ} \otimes d$ so that d is a primitive element in H°.

Conversely let $d \in H^\circ$ be primitive. then $d(ab) = \langle \Delta(d), a \otimes b\rangle = d(a)\varepsilon(b) + \varepsilon(a)d(b)$. □

Proposition and Definition 4.3.2. *Let H be a Hopf algebra. The set of primitive elements of H will be denoted by* $\mathbf{Lie}(H)$ *and is a Lie algebra. If* $\mathrm{char}(\mathbb{K}) = p > 0$ *then* $\mathbf{Lie}(H)$ *is a restricted Lie algebra or a p-Lie algebra.*

Proof. Let $a, b \in H$ be primitive elements. Then $\Delta([a, b]) = \Delta(ab - ba) = (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) - (b \otimes 1 + 1 \otimes b)(a \otimes 1 + 1 \otimes a) = (ab - ba) \otimes 1 + 1 \otimes (ab - ba)$ hence $\mathbf{Lie}(H) \subseteq H^L$ is a Lie algebra. If the characteristic of $\mathbb{K}$ is $p > 0$ then we have $(a \otimes 1 + 1 \otimes a)^p = a^p \otimes 1 + 1 \otimes a^p$. Thus $\mathbf{Lie}(H)$ is a restricted Lie subalgebra of H^L with the structure maps $[a, b] = ab - ba$ and $a^{[p]} = a^p$. □

Corollary 4.3.3. *Let H be a Hopf algebra. Then the set of left translation invariant derivations $D : H \longrightarrow H$ is a Lie algebra under $[D, D'] = DD' - D'D$. If* $\mathrm{char} = p$ *then these derivations are a restricted Lie algebra with $D^{[p]} = D^p$.*

Proof. The map $\Psi : H^\circ \longrightarrow H^* \xrightarrow{\Phi} \mathrm{End}(H)$ is a homomorphism of algebras by 4.2.6. Hence $\Psi(d * d' - d' * d) = \Phi(d * d' - d' * d) = \Phi(d)\Phi(d') - \Phi(d')\Phi(d)$. If d is a primitive element in H° then by 4.2.7 and 4.3.1 the image $D := \Psi(d)$ in $\mathrm{End}(H)$ is a left translation invariant derivation and all left translation invariant derivations are of this form. Since $[d, d'] = d * d' - d' * d$ is again primitive we get that $[D, D'] = DD' - D'D$ is a left translation invariant derivation so that the set of left translation invariant derivations $\mathrm{Der}^H_{\mathbb{K}}(H, H)$ is a Lie algebra resp. a restricted Lie algebra. □

Definition 4.3.4. Let H be a Hopf algebra. An element $c \in H$ is called *cocommutative* if $\tau\Delta(c) = \Delta(c)$, i. e. if $\sum c_{(1)} \otimes c_{(2)} = \sum c_{(2)} \otimes c_{(1)}$. Let $C(H) := \{c \in H | c \text{ is cocommutative }\}$.

Let $G(H)$ denote the set of group like elements of H.

Lemma 4.3.5. *Let H be a Hopf algebra. Then the set of cocommutative elements $C(H)$ is a subalgebra of H and the group like elements $G(H)$ form a linearly independent subset of $C(H)$. Furthermore $G(H)$ is a multiplicative subgroup of the group of units $U(C(H))$.*

PROOF. It is clear that $C(H)$ is a linear subspace of H. If $a, b \in C(H)$ then $\Delta(ab) = \Delta(a)\Delta(b) = (\tau\Delta)(a)(\tau\Delta)(b) = \tau(\Delta(a)\Delta(b)) = \tau\Delta(ab)$ and $\Delta(1) = 1 \otimes 1 = \tau\Delta(1)$. Thus $C(H)$ is a subalgebra of H.

The group like elements obviously are cocommutative and form a multiplicative group, hence a subgroup of $U(C(H))$. They are linearly independent by Lemma 2.1.14. □

Proposition 4.3.6. *Let H be a Hopf algebra with $S^2 = \mathrm{id}_H$. Then there is a left module structure*

$$C(H) \otimes \mathbf{Lie}(H) \ni c \otimes a \mapsto c \cdot a \in \mathbf{Lie}(H)$$

with $c \cdot a := \nabla_H(\nabla_H \otimes 1)(1 \otimes \tau)(1 \otimes S \otimes 1)(\Delta \otimes 1)(c \otimes a) = \sum c_{(1)} a S(c_{(2)})$ *such that*

$$c \cdot [a, b] = \sum [c_{(1)} \cdot a, c_{(2)} \cdot b].$$

In particular $G(H)$ acts by Lie automorphisms on $\mathbf{Lie}(H)$.

PROOF. The given action is actually the action $H \otimes H \longrightarrow H$ with $h \cdot a = \sum h_{(1)} a S(h_{(2)})$, the so-called *adjoint action.*

We first show that the given map has image in $\mathbf{Lie}(H)$. For $c \in C(H)$ and $a \in \mathbf{Lie}(H)$ we have $\Delta(c \cdot a) = \Delta(\sum c_{(1)} a S(c_{(2)})) = \sum \Delta(c_{(1)})(a \otimes 1 + 1 \otimes a)\Delta(S(c_{(2)})) = \sum \Delta(c_{(1)})(a \otimes 1)\Delta(S(c_{(2)})) + \sum \Delta(c_{(2)})(1 \otimes a)\Delta(S(c_{(1)})) = \sum c_{(1)} a S(c_{(4)}) \otimes c_{(2)} S(c_{(3)}) + \sum c_{(3)} S(c_{(2)}) \otimes c_{(4)} a S(c_{(1)}) = c \cdot a \otimes 1 + 1 \otimes c \cdot a$ since c is cocommutative, $S^2 = \mathrm{id}_H$ and a is primitive.

We show now that $\mathbf{Lie}(H)$ is a $C(H)$-module. $(cd) \cdot a = \sum c_{(1)} d_{(1)} a S(c_{(2)} d_{(2)}) = \sum c_{(1)} d_{(1)} a S(d_{(2)}) S(c_{(2)}) = c \cdot (d \cdot a)$. Furthermore we have $1 \cdot a = 1 a S(1) = a$.

To show the given formula let $a, b \in \mathbf{Lie}(H)$ and $c \in C(H)$. Then $c \cdot [a, b] = \sum c_{(1)}(ab - ba) S(c_{(2)}) = \sum c_{(1)} a S(c_{(2)}) c_{(3)} b S(c_{(4)}) - \sum c_{(1)} b S(c_{(2)}) c_{(3)} a S(c_{(4)}) = \sum (c_{(1)} \cdot a)(c_{(2)} \cdot b) - \sum (c_{(1)} \cdot b)(c_{(2)} \cdot a) = \sum [c_{(1)} \cdot a, c_{(2)} \cdot b]$ again since $c \in C(H)$ is cocommutative.

Now let $g \in G(H)$. Then $g \cdot a = gaS(g) = gag^{-1}$ since $S(g) = g^{-1}$ for any group like element. Furthermore $g \cdot [a, b] = [g \cdot a, g \cdot b]$ hence g defines a Lie algebra automorphism of $\mathbf{Lie}(H)$. □

Problem 4.3.1. Show that the *adjoint action* $H \otimes H \ni h \otimes a \mapsto \sum h_{(1)} a S(h_{(2)}) \in H$ makes H an H-module algebra.

Definition and Remark 4.3.7. The algebra $\mathbb{K}(\delta) = \mathbb{K}[\delta]/(\delta^2)$ is called the algebra of *dual numbers.* Observe that $\mathbb{K}(\delta) = \mathbb{K} \oplus \mathbb{K}\delta$ as a $\mathbb{K}$-module.

We consider δ as a "small quantity" whose square vanishes.

The maps $p : \mathbb{K}(\delta) \longrightarrow K$ with $p(\delta) = 0$ and $j : \mathbb{K} \longrightarrow \mathbb{K}(\delta)$ are algebra homomorphism satisfying $pj = \mathrm{id}$.

Let $\mathbb{K}(\delta, \delta') := \mathbb{K}[\delta, \delta']/(\delta^2, \delta'^2)$. Then $\mathbb{K}(\delta, \delta') = \mathbb{K} \oplus \mathbb{K}\delta \oplus \mathbb{K}\delta' \oplus \mathbb{K}\delta\delta'$. The map $\mathbb{K}(\delta) \ni \delta \mapsto \delta\delta' \in \mathbb{K}(\delta, \delta')$ is an injective algebra homomorphism. Furthermore for every $\alpha \in \mathbb{K}$ we have an algebra homomorphism $\varphi_\alpha : \mathbb{K}(\delta) \ni \delta \mapsto \alpha\delta \in \mathbb{K}(\delta)$.

These algebra homomorphisms induce algebra homomorphisms $H \otimes \mathbb{K}(\delta) \longrightarrow H \otimes \mathbb{K}(\delta)$ resp. $H \otimes \mathbb{K}(\delta) \longrightarrow H \otimes \mathbb{K}(\delta, \delta')$ for every Hopf algebra H.

Proposition 4.3.8. *The map*

$$e^{\delta -} : \mathbf{Lie}(H) \longrightarrow H \otimes \mathbb{K}(\delta) \subseteq H \otimes \mathbb{K}(\delta, \delta')$$

with $e^{\delta a} := 1 + a \otimes \delta = 1 + \delta a$ *is called the* exponential map *and satisfies*

$$\begin{aligned} e^{\delta(a+b)} &= e^{\delta a} e^{\delta b}, \\ e^{\delta \alpha a} &= \varphi_\alpha(e^{\delta a}), \\ e^{\delta\delta'[a,b]} &= e^{\delta a} e^{\delta' b} (e^{\delta a})^{-1} (e^{\delta' b})^{-1}. \end{aligned}$$

Furthermore all elements $e^{\delta a} \in H \otimes \mathbb{K}(\delta)$ *are group like in the* $\mathbb{K}(\delta)$*-Hopf algebra* $H \otimes \mathbb{K}(\delta)$.

PROOF. 1. $e^{\delta(a+b)} = (1 + \delta(a+b)) = (1 + \delta a)(1 + \delta b) = e^{\delta a} e^{\delta b}$.

2. $e^{\delta \alpha a} = 1 + \delta \alpha a = \varphi_\alpha(1 + \delta a) = \varphi_\alpha(e^{\delta a})$.

3. Since $(1+\delta a)(1-\delta a) = 1$ we have $(e^{\delta a}) = 1 - \delta a$. So we get $e^{\delta\delta'[a,b]} = 1 + \delta[a,b] = 1 + \delta(a - a) + \delta'(b - b) + \delta\delta'(ab - ab - ba + ab) = (1 + \delta a)(1 + \delta' b)(1 - \delta a)(1 - \delta' b) = e^{\delta a} e^{\delta' b} (e^{\delta a})^{-1} (e^{\delta' b})^{-1}$.

4. $\Delta_{\mathbb{K}(\delta)}(e^{\delta a}) = \Delta(1 + a \otimes \delta) = 1 \otimes_{\mathbb{K}(\delta)} 1 + (a \otimes 1 + 1 \otimes a) \otimes \delta = 1 \otimes_{\mathbb{K}(\delta)} 1 + \delta a \otimes_{\mathbb{K}(\delta)} 1 + 1 \otimes_{\mathbb{K}(\delta)} \delta a + \delta a \otimes_{\mathbb{K}(\delta)} \delta a = (1 + \delta a) \otimes_{\mathbb{K}(\delta)} (1 + \delta a) = e^{\delta a} \otimes_{\mathbb{K}(\delta)} e^{\delta a}$ and $\varepsilon_{\mathbb{K}(\delta)}(e^{\delta a}) = \varepsilon_{\mathbb{K}(\delta)}(1 + \delta a) = 1 + \delta \varepsilon(a) = 1$. ☐

Corollary 4.3.9. $(\mathbf{Lie}(H), e^{\delta -})$ *is the kernel of the group homomorphism* $p : G_{\mathbb{K}(\delta)}(H \otimes \mathbb{K}(\delta)) \longrightarrow G(H)$.

PROOF. $p = 1 \otimes p : H \otimes \mathbb{K}(\delta) \longrightarrow H \otimes \mathbb{K} = H$ is a homomorphism of $\mathbb{K}$-algebras. We show that it preserves group like elements. Observe that group like elements in $H \otimes \mathbb{K}(\delta)$ are defined by the Hopf algebra structure over $\mathbb{K}(\delta)$. Let $g \in G_{\mathbb{K}(\delta)}(H \otimes \mathbb{K}(\delta))$. Then $(\Delta_H \otimes 1)(g) = g \otimes_{\mathbb{K}(\delta)} g$ and $(\varepsilon_H \otimes 1)(g) = 1 \in \mathbb{K}(\delta)$.

Since $p : \mathbb{K}(\delta) \longrightarrow \mathbb{K}$ is an algebra homomorphism the following diagram commutes

$$\begin{array}{ccc} (H \otimes \mathbb{K}(\delta)) \otimes_{\mathbb{K}(\delta)} (H \otimes \mathbb{K}(\delta)) & \xrightarrow{\cong} & H \otimes H \otimes \mathbb{K}(\delta) \\ \Big\downarrow {\scriptstyle (1\otimes p)\otimes(1\otimes p)} & & \Big\downarrow {\scriptstyle 1\otimes p} \\ (H \otimes \mathbb{K}) \otimes (H \otimes \mathbb{K}) & \xrightarrow[\cong]{} & H \otimes H \otimes \mathbb{K}. \end{array}$$

We identify elements along the isomorphisms. Thus we get $(\Delta_H \otimes 1_{\mathbb{K}})(1_H \otimes p)(g) = (1_{H \otimes H} \otimes p)(\Delta_H \otimes 1_{\mathbb{K}(\delta)})(g) = ((1_H \otimes p) \otimes_{\mathbb{K}(\delta)} (1_H \otimes p))(g \otimes_{\mathbb{K}(\delta)} g) = (1_H \otimes p)(g) \otimes (1_H \otimes p)(g)$, so that $1_H \otimes p : G_{\mathbb{K}(\delta)}(H \otimes \mathbb{K}(\delta)) \longrightarrow G(H)$. Now we have $(1_H \otimes p)(gg') = (1_H \otimes p)(g)(1_H \otimes p)(g')$ so that $1_H \otimes p$ is a group homomorphism.

Now let $g = g_0 \otimes 1 + g_1 \otimes \delta \in G_{\mathbb{K}(\delta)}(H \otimes \mathbb{K}(\delta)) \subseteq H \otimes \mathbb{K} \oplus H \otimes \mathbb{K}\delta$. Then we have $(1_H \otimes p)(g) = 1$ iff $g_0 = 1$ iff $g = 1_H \otimes 1_{\mathbb{K}(\delta)} + g_1 \otimes \delta$. Furthermore we have

$$\begin{aligned} &\Delta_{H \otimes \mathbb{K}(\delta)}(g) = g \otimes_{\mathbb{K}(\delta)} g \Longleftrightarrow \\ &1_H \otimes 1_H \otimes 1_{\mathbb{K}(\delta)} + \Delta_H(g_1) \otimes \delta = (1_H \otimes 1_{\mathbb{K}(\delta)} + g_1 \otimes \delta) \otimes_{\mathbb{K}(\delta)} (1_H \otimes 1_{\mathbb{K}(\delta)} + g_1 \otimes \delta) \\ &\qquad = 1_H \otimes 1_H \otimes 1_{\mathbb{K}(\delta)} + (g_1 \otimes 1_H + 1_H \otimes g_1) \otimes \delta \Longleftrightarrow \\ &\Delta_H(g_1) = g_1 \otimes 1_H + 1_H \otimes g_1. \end{aligned}$$

Similarly we have $\varepsilon_{\mathbb{K}(\delta)})(g) = 1$ iff $1 \otimes 1 + \varepsilon(g_1) \otimes \delta = 1$ iff $\varepsilon(g_1) = 0$. $\square$