CHAPTER 4

The Infinitesimal Theory

1. Integrals and Fourier Transforms

Assume for this chapter that $\mathbb{K}$ is a field.

Lemma 4.1.1. *Let C be a finite dimensional coalgebra. Every right C-comodule M is a left C^*-module by $c^*m = \sum m_{(M)}\langle c^*, m_{(1)}\rangle$ and conversely by $\delta(m) = \sum_i c_i^* m \otimes c_i$ where $\sum c_i^* \otimes c_i$ is the dual basis.*

PROOF. We check that M becomes a left C^*-module

$$\begin{aligned}(c^*c'^*)m &= \sum m_{(M)}\langle c^*c'^*, m_{(1)}\rangle = \sum m_{(M)}\langle c^*, m_{(1)}\rangle\langle c'^*, m_{(2)}\rangle \\ &= c^* \sum m_{(M)}\langle c'^*, m_{(1)}\rangle = c^*(c'^*m).\end{aligned}$$

It is easy to check that the two constructions are inverses of each other. In particular assume that M is a right C-comodule. Choose m_i such that $\delta(m) = \sum m_i \otimes c_i$. Then $c_j^* m = \sum m_i \langle c_j^*, c_i\rangle = m_j$ and $\sum c_i^* m \otimes c_i = \sum m_i \otimes c_i = \delta(m)$. □

Definition 4.1.2. 1. Let A be an algebra with augmentation $\varepsilon : A \longrightarrow \mathbb{K}$, an algebra homomorphism. Let M be a left A-module. Then ${}^AM = \{m \in M | am = \varepsilon(a)m\}$ is called the *space of left invariants* of M.

This defines a functor ^{A}- : $A - \mathbf{Mod} \longrightarrow \mathbf{Vec}$.

2. Let C be a coalgebra with a group-like element $1 \in C$. Let M be a right C-comodule. Then $M^{coC} := \{m \in M | \delta(m) = m \otimes 1\}$ is called the *space of right coinvariants* of M.

This defines a functor -^{coC} : $\mathbf{Comod}$-$C \longrightarrow \mathbf{Vec}$.

Lemma 4.1.3. *Let C be a finite dimensional coalgebra with a group like element $1 \in C$. Then $A := C^*$ is an augmented algebra with augmentation $\varepsilon : C^* \ni a \mapsto \langle a, 1\rangle \in \mathbb{K}$. Let M be a right C-comodule. Then M is a left C^*-module and we have*

$${}^{C^*}M = M^{coC}.$$

PROOF. Since $1 \in C$ is group-like we have $\varepsilon_A(ab) = \langle ab, 1\rangle = \langle a, 1\rangle\langle b, 1\rangle = \varepsilon_A(a)\varepsilon_A(b)$ and $\varepsilon_A(1_A) = \langle 1_A, 1_C\rangle = \varepsilon_C(1_C) = 1$.

We have $m \in M^{coH}$ iff $\delta(m) = \sum m_{(M)} \otimes m_{(1)} = m \otimes 1$ iff $\sum m_{(M)}\langle a, m_{(1)}\rangle = m\langle a, 1\rangle$ for all $a \in A = C^*$ and by identifying $C^* \otimes C = \mathrm{Hom}(C^*, C^*)$ iff $am = \varepsilon_A(a)m$ iff $m \in {}^AM$. □

Remark 4.1.4. The theory of Fourier transforms contains the following statements. Let H be the (Schwartz) space of infinitely differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{C}$, such that f and all derivatives rapidly decrease at infinity. (f decreases rapidly at infinity if $|x|^m f(x)$ is bounded for all m.) This space is an algebra (without unit) under the multiplication of values. There is a second multiplication on H, the convolution

$$(f * g)(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)g(x-t)dt.$$

The Fourier transform is a homomorphism $\hat{} : H \longrightarrow H$ defined by

$$\hat{f}(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(t)e^{-itx}dt.$$

It satisfies the identity $(f * g)\hat{} = \hat{f}\hat{g}$ hence it is an algebra homomorphism. We want to find an analogue of this theory for finite quantum groups.

A similar example is the following. Let G be a locally compact topological group. Let μ be the (left) *Haar measure* on G and $\int f := \int_G f(x)d\mu(x)$ be the *Haar integral*.

The Haar measure is left invariant in the sense that $\mu(E) = \mu(gE)$ for all $g \in G$ and all compact subsets E of G. The Haar measure exists and is unique up to a positive factor. The Haar integral is translation invariant i.e. for all $y \in G$ we have $\int f(yx)d\mu(x) = \int f(x)d\mu(x)$.

If μ is a left-invariant Haar measure then there is a continuous homomorphism $\mod : G \longrightarrow (\mathbb{R}^+, \cdot)$ such that $\int f(xy^{-1})d\mu(x) = \mod(y)\int(f(x)d\mu(x)$. The homomorphism μ does not depend on f and is called the *modulus* of G. The group G is called *unimodular* if the homomorphism $\mod$ is the identity.

If G is a compact, or discrete, or Abelian group, or a connected semisimple or nilpotent Lie group, then G is unimodular.

Let G be a quantum group (or a quantum monoid) with function algebra H an arbitrary Hopf algebra. We also use the algebra of linear functionals $H^* = \mathrm{Hom}(H, \mathbb{K})$ (called the bialgebra of G in the French literature). The operation $H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in \mathbb{K}$ is nondegenerate on both sides. We denote the elements of H by $f, g, h \in H$, the elements of H^* by $a, b, c \in H^*$, the (non existing) elements of the quantum group G by $x, y, z \in G$.

Remark 4.1.5. In 2.4.8 we have seen that the dual vector space H^* of a finite dimensional Hopf algebra H is again a Hopf algebra. The Hopf algebra structures are connected by the evaluation bilinear form

$$\mathrm{ev} : H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in \mathbb{K}$$

as follows:

$$\begin{gathered} \langle a \otimes b, \sum f_{(1)} \otimes f_{(2)} \rangle = \langle ab, f \rangle, \quad \langle \sum a_{(1)} \otimes a_{(2)}, f \otimes g \rangle = \langle a, fg \rangle, \\ \langle a, 1 \rangle = \varepsilon(a), \qquad \langle 1, f \rangle = \varepsilon(f), \\ \langle a, S(f) \rangle = \langle S(a), f \rangle. \end{gathered}$$

Definition 4.1.6. 1. The linear functionals $a \in H^*$ are called *generalized integrals on H* ([Riesz-Nagy] S.123).

2. An element $\int \in H^*$ is called a *left (invariant) integral on* H if

$$a\int = \langle a, 1_H \rangle \int$$

or $a\int = \varepsilon_{H^*}(a)\int$ for all $a \in H^*$.

3. An element $\delta \in H$ is called a *left integral in* H if

$$f\delta = \varepsilon(f)\delta$$

for all $f \in H$.

4. The set of left integrals in H is denoted by $\mathrm{Int}_l(H)$, the set of right integrals by $\mathrm{Int}_r(H)$. The set of left (right) integrals on H is $\mathrm{Int}_l(H^*)$ ($\mathrm{Int}_r(H^*)$).

5. A Hopf algebra H is called *unimodular* if $\mathrm{Int}_l(H) = \mathrm{Int}_r(H)$.

Lemma 4.1.7. *The left integrals* $\mathrm{Int}_l(H^*)$ *form a two sided ideal of* H^*. *If the antipode* S *is bijective then* S *induces an isomorphism* $S : \mathrm{Int}_l(H^*) \longrightarrow \mathrm{Int}_r(H^*)$.

PROOF. For $\int$ in $\mathrm{Int}_l(H^*)$ we have $a\int = \varepsilon(a)\int \in \mathrm{Int}_l(H^*)$ and $a\int b = \varepsilon(a)\int b$ hence $\int b \in \mathrm{Int}_l(H^*)$. If S is bijective then the induced map $S : H^* \longrightarrow H^*$ is also bijective and satisfies $S(\int)b = S(\int)S(S^{-1}(b)) = S(S^{-1}(b)\int) = S(\int)\varepsilon(b)$ hence $S(\int) \in \mathrm{Int}_r(H^*)$. □

Remark 4.1.8. Maschke's Theorem has an extension to finite dimensional Hopf algebras: $\varepsilon(\int) \neq 0$ iff H^* is semisimple.

Corollary 4.1.9. *Let* H *be a finite dimensional Hopf algebra. Then* H^* *is a left* H^**-module by the usual multiplication, hence a right* H*-comodule. We have*

$$(H^*)^{coH} = \mathrm{Int}_l(H^*).$$

PROOF. By definition we have $\mathrm{Int}_l(H^*) = {}^{H^*}H^*$. □

Example 4.1.10. Let G be a finite group. Let $H := \mathrm{Map}(G, \mathbb{K})$ be the Hopf algebra defined by the following isomorphism

$$\mathbb{K}^G = \mathrm{Map}(G, \mathbb{K}) \cong \mathrm{Hom}(\mathbb{K}G, \mathbb{K}) = (\mathbb{K}G)^*.$$

This isomorphism between the vector space $\mathbb{K}^G$ of all set maps from the group G to the base ring $\mathbb{K}$ and the dual vector space $(\mathbb{K}G)^*$ of the group algebra $\mathbb{K}G$ defines the structure of a Hopf algebra on $\mathbb{K}^G$.

We regard $H := \mathbb{K}^G$ as the function algebra on the set G. In the sense of algebraic geometry this is not quite true. The algebra $\mathbb{K}^G$ represents a functor from $\mathbb{K}$-**cAlg** to **Set** that has G as value for all connected algebras A in particular for all field extensions of $\mathbb{K}$.

As before we use the map $\mathrm{ev} : \mathbb{K}G \otimes \mathbb{K}^G \longrightarrow \mathbb{K}$. The multiplication of $\mathbb{K}^G$ is given by pointwise multiplication of maps since $\langle x, ff' \rangle = \langle \sum x_{(1)} \otimes x_{(2)}, f \otimes f' \rangle = \langle x \otimes x, f \otimes f' \rangle = \langle x, f \rangle \langle x, f' \rangle$ for all $f, f' \in \mathbb{K}^G$ and all $x \in G$. The unit element $1_{\mathbb{K}^G}$ of $\mathbb{K}^G$ is the map $\varepsilon : \mathbb{K}G \longrightarrow \mathbb{K}$ restricted to G, hence $\varepsilon(x) = 1 = \langle x, 1_{\mathbb{K}^G} \rangle$ for all $x \in G$. The antipode of $f \in \mathbb{K}^G$ is given by $S(f)(x) = \langle x, S(f) \rangle = f(x^{-1})$.

The elements of the dual basis $(x^*|x \in G)$ with $\langle x, y^* \rangle = \delta_{x,y}$ considered as maps from G to $\mathbb{K}$ form a basis of $\mathbb{K}^G$. They satisfy the conditions

$$x^* y^* = \delta_{x,y} x^* \text{ and } \sum_{x \in G} x^* = 1_{\mathbb{K}^G}$$

since $\langle z, x^* y^* \rangle = \langle z, x^* \rangle \langle z, y^* \rangle = \delta_{z,x}\delta_{z,y} = \delta_{x,y}\langle z, x^* \rangle$ and $\langle z, \sum_{x \in G} x^* \rangle = 1 = \langle z, 1_{\mathbb{K}^G} \rangle$.

Hence the dual basis $(x^*|x \in G)$ is a decomposition of the unit into a set of minimal orthogonal idempotents and the algebra of $\mathbb{K}^G$ has the structure

$$\mathbb{K}^G = \oplus_{x \in G} \mathbb{K} x^* \cong \mathbb{K} \times \ldots \times \mathbb{K}.$$

In particular $\mathbb{K}^G$ is commutative and semisimple.

The diagonal of $\mathbb{K}^G$ is

$$\Delta(x^*) = \sum_{y \in G} y^* \otimes (y^{-1}x)^* = \sum_{y,z \in G, yz = x} y^* \otimes z^*$$

since

$$\begin{aligned} \langle z \otimes u, \Delta(x^*) \rangle &= \langle zu, x^* \rangle = \delta_{x,zu} = \delta_{z^{-1}x,u} = \textstyle\sum_{y \in G} \delta_{y,z}\delta_{y^{-1}x,u} \\ &= \textstyle\sum_{y \in G} \langle z, y^* \rangle \langle u, (y^{-1}x)^* \rangle = \langle z \otimes u, \sum_{y \in G} y^* \otimes (y^{-1}x)^* \rangle. \end{aligned}$$

Let $a \in \mathbb{K}G$. Then a defines a map $\widetilde{a} : G \longrightarrow \mathbb{K} \in \mathbb{K}^G$ by $a = \sum_{x \in G} \widetilde{a}(x)x$. For arbitrary $f \in \mathbb{K}^G$ and $a \in \mathbb{K}G$ this gives

$$\langle a, f \rangle = f(\sum_{x \in G} \widetilde{a}(x)x) = \sum_{x \in G} \widetilde{a}(x) f(x).$$

The counit of $\mathbb{K}^G$ is given by $\varepsilon(x^*) = \delta_{x,e}$ where $e \in G$ is the unit element.

The antipode is, as above, $S(x^*) = (x^{-1})^*$.

We consider $H = \mathbb{K}^G$ as the function algebra on the finite group G and $\mathbb{K}G$ as the dual space of $H = \mathbb{K}^G$ hence as the set of distributions on H.

Then

$$\textstyle\int := \sum_{x \in G} x \in H^* = \mathbb{K}G \tag{1}$$

is a (two sided) integral on H since $\sum_{x \in G} yx = \sum_{x \in G} x = \varepsilon(y) \sum_{x \in G} x = \sum_{x \in G} yx$.

We write

$$\int f(x)dx := \langle \textstyle\int, f \rangle = \sum_{x \in G} f(x).$$

We have seen that there is a decomposition of the unit $1 \in \mathbb{K}^G$ into a set of primitive orthogonal idempotents $\{x^*|x \in G\}$ such that every element $f \in \mathbb{K}^G$ has a unique representation $f = \sum f(x)x^*$. Since $\int y^* = \sum_{x \in G} \langle x, y^* \rangle$ we get $\int f y^* = \sum_{x \in G} \langle x, f y^* \rangle = \sum f(x) y^*(x) = f(y)$ hence

$$f = \sum (\int f(x) y^*(x) dx) y^*.$$

Problem 4.1.1. Describe the group valued functor $\mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}^G, -)$ in terms of sets and their group structure.

Definition and Remark 4.1.11. Let $\mathbb{K}$ be an algebraicly closed field and let G be a finite abelian group (replacing $\mathbb{R}$ above). Assume that the characteristic of $\mathbb{K}$ does not divide the order of G. Let $H = \mathbb{K}^G$. We identify $\mathbb{K}^G = \mathrm{Hom}(\mathbb{K}G, \mathbb{K})$ along the linear expansion of maps as in Example 2.1.10.

Let us consider the set $\hat{G} := \{\chi : G \longrightarrow \mathbb{K}^* | \chi \text{ group homomorphism}\}$. Since $\mathbb{K}^*$ is an abelian group, the set $\hat{G}$ is an abelian group by pointwise multiplication.

The group $\hat{G}$ is called the *character group* of G.

Obviously the character group is a multiplicative subset of $\mathbb{K}^G = \mathrm{Hom}(\mathbb{K}G, \mathbb{K})$. Actually it is a subgroup of $\mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}G, \mathbb{K}) \subseteq \mathrm{Hom}(\mathbb{K}G, \mathbb{K})$ since the elements $\chi \in \hat{G}$ expand to algebra homomorphisms: $\chi(ab) = \chi(\sum \alpha_x x \sum \beta_y y) = \sum \alpha_x \beta_y \chi(xy) = \chi(a)\chi(b)$ and $\chi(1) = \chi(e) = 1$. Conversely an algebra homomorphism $f \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}G, \mathbb{K})$ restricts to a character $f : G \longrightarrow \mathbb{K}^*$. Thus $\hat{G} = \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}G, \mathbb{K})$, the set of rational points of the affine algebraic group represented by $\mathbb{K}G$.

There is a more general observation behind this remark.

Lemma 4.1.12. *Let H be a finite dimensional Hopf algebra. Then the set $Gr(H^*)$ of group like elements of H^* is equal to $\mathbb{K}$-$\mathbf{Alg}(H, \mathbb{K})$.*

Proof. In fact $f : H \longrightarrow \mathbb{K}$ is an algebra homomorphism iff $\langle f \otimes f, a \otimes b \rangle = \langle f, a \rangle \langle f, b \rangle = \langle f, ab \rangle = \langle \Delta(f), a \otimes b \rangle$ and $1 = \langle f, 1 \rangle = \varepsilon(f)$. □

Hence there is a Hopf algebra homomorphism $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ by 2.1.5.

Proposition 4.1.13. *The Hopf algebra homomorphism $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ is bijective.*

Proof. We give the proof by several lemmas.

Lemma 4.1.14. *Any set of group like elements in a Hopf algebra H is linearly independent.*

Proof. Assume there is a linearly dependent set $\{x_0, x_1, \ldots, x_n\}$ of group like elements in H. Choose such a set with n minimal. Obviously $n \geq 1$ since all elements are non zero. Thus $x_0 = \sum_{i=1}^n \alpha_i x_i$ and $\{x_1, \ldots, x_n\}$ linearly independent. We get

$$\sum_{i,j} \alpha_i \alpha_j x_i \otimes x_j = x_0 \otimes x_0 = \Delta(x_0) = \sum_i \alpha_i x_i \otimes x_i.$$

Since all $\alpha_i \neq 0$ and the $x_i \otimes x_j$ are linearly independent we get $n = 1$ and $\alpha_1 = 1$ so that $x_0 = x_1$, a contradiction. □

Corollary 4.1.15. *(Dedekind's Lemma) Any set of characters in $\mathbb{K}^G$ is linearly independent.*

Thus $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ is injective. Now we prove that the map $\varphi : \mathbb{K}\hat{G} \longrightarrow \mathbb{K}^G$ is surjective.

Lemma 4.1.16. *(Pontryagin duality) The evaluation $\hat{G} \times G \longrightarrow \mathbb{K}^*$ is a non-degenerate bilinear map of abelian groups.*

PROOF. First we observe that $\operatorname{Hom}(C_n, \mathbb{K}^*) \cong C_n$ for a cyclic group of order n since $\mathbb{K}$ has a primitive n-th root of unity ($\operatorname{char}(\mathbb{K}) \not| \ |G|$).

Since the direct product and the direct sum coincide in **Ab** we can use the fundamental theorem for finite abelian groups $G \cong C_{n_1} \times \ldots \times C_{n_t}$ to get $\operatorname{Hom}(G, \mathbb{K}^*) \cong G$ for any abelian group G with $\operatorname{char}(\mathbb{K}) \not| \ |G|$. Thus $\hat{G} \cong G$ and $\hat{\hat{G}} = G$. In particular $\chi(x) = 1$ for all $x \in G$ iff $\chi = 1$. By the symmetry of the situation we get that the bilinear form $\langle .,. \rangle : \hat{G} \times G \longrightarrow \mathbb{K}^*$ is non-degenerate. ☐

Thus $|\hat{G}| = |G|$ hence $\dim(\mathbb{K}\hat{G}) = \dim(\mathbb{K}^G)$. This proves Proposition 2.1.13. ☐

Definition 4.1.17. Let H be a Hopf algebra. A $\mathbb{K}$-module M that is a right H-module by $\rho : M \otimes H \longrightarrow M$ and a right H-comodule by $\delta : M \longrightarrow M \otimes H$ is called a *Hopf module* if the diagram

$$\begin{array}{ccccc} M \otimes H & \xrightarrow{\rho} & H & \xrightarrow{\delta} & M \otimes H \\ \downarrow{\scriptstyle \delta\otimes\Delta} & & & & \uparrow{\scriptstyle \rho\otimes\nabla} \\ M \otimes H \otimes H \otimes H & & \xrightarrow{1\otimes\tau\otimes 1} & & M \otimes H \otimes H \otimes H \end{array}$$

commutes, i.e. if $\delta(mh) = \sum m_{(M)}h_{(1)} \otimes m_{(1)}h_{(2)}$ holds for all $h \in H$ and all $m \in M$.

Observe that H is an Hopf module over itself. Furthermore each module of the form $V \otimes H$ is a Hopf module by the induced structure. More generally there is a functor $\mathbf{Vec} \ni V \mapsto V \otimes H \in \mathbf{Hopf\text{-}Mod}\text{-}H$.

Proposition 4.1.18. *The two functors $\text{-}^{coH} : \mathbf{Hopf\text{-}Mod}\text{-}H \longrightarrow \mathbf{Vec}$ and $\text{-}\otimes H : \mathbf{Vec} \ni V \mapsto V \otimes H \in \mathbf{Hopf\text{-}Mod}\text{-}H$ are inverse equivalences of each other.*

PROOF. Define natural isomorphisms

$$\alpha : M^{coH} \otimes H \ni m \otimes h \mapsto mh \in M$$

with inverse map

$$\alpha^{-1} : M \ni m \mapsto \sum m_{(M)}S(m_{(1)}) \otimes m_{(2)} \in M^{coH} \otimes H$$

and

$$\beta : V \ni v \mapsto v \otimes 1 \in (V \otimes H)^{coH}$$

with inverse map

$$(V \otimes H)^{coH} \ni v \otimes h \mapsto v\varepsilon(h) \in V.$$

Obviously these homomorphisms are natural transformations in M and V. Furthermore α is a homomorphism of H-modules. α^{-1} is well-defined since

$$\begin{aligned}\delta(\sum m_{(M)}S(m(1))) &= \sum m_{(M)}S(m_{(3)}) \otimes m_{(1)}S(m_{(2)}) \\ &\quad \text{(since } M \text{ is a Hopf module)} \\ &= \sum m_{(M)}S(m_{(2)}) \otimes \eta\varepsilon(m_{(1)}) \\ &= \sum m_{(M)}S(m_{(1)}) \otimes 1\end{aligned}$$

hence $\sum m_{(M)}S(m_{(1)}) \in M^{coH}$. Furthermore α^{-1} is a homomorphism of comodules since

$$\begin{aligned}\delta\alpha^{-1}(m) &= \delta(\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)}) \otimes m_{(2)} \otimes m_{(3)} \\ &= \sum \alpha^{-1}(m_{(M)}) \otimes m_{(1)} = (\alpha^{-1} \otimes 1)\delta(m).\end{aligned}$$

Finally α and α^{-1} are inverse to each other by

$$\alpha\alpha^{-1}(m) = \alpha(\sum m_{(M)}S(m_{(1)}) \otimes m_{(2)}) = \sum m_{(M)}S(m_{(1)})m_{(2)} = m$$

and

$$\begin{aligned}\alpha^{-1}\alpha(m \otimes h) &= \alpha^{-1}(mh) = \sum m_{(M)}h_{(1)}S(m_{(1)}h_{(2)}) \otimes m_{(2)}h_{(3)} \\ &= \sum mh_{(1)}S(h_{(2)}) \otimes h_{(3)} \text{ (by } \delta(m) = m \otimes 1 \text{)} = m \otimes h.\end{aligned}$$

Thus α and α^{-1} are mutually inverse homomorphisms of Hopf modules.

The image of β is in $(V \otimes H)^{coH}$ by $\delta(v \otimes 1) = v \otimes \Delta(1) = (v \otimes 1) \otimes 1$. Both β and β^{-1} are $\mathbb{K}$-linear maps. Furthermore we have

$$\beta^{-1}\beta(v) = \beta^{-1}(v \otimes 1) = v\varepsilon(1) = v$$

and

$$\begin{aligned}\beta\beta^{-1}(\sum v_i \otimes h_i) &= \beta(\sum v_i\varepsilon(h_i)) = \sum v_i\varepsilon(h_i) \otimes 1 = \sum v_i \otimes \varepsilon(h_i)1 \\ &= \sum v_i \otimes \varepsilon(h_{i(1)})h_{i(2)} \text{ (since } \textstyle\sum v_i \otimes h_i \in (V \otimes H)^{coH} \text{)} = \sum v_i \otimes h_i.\end{aligned}$$

Thus β and β^{-1} are mutually inverse homomorphisms. □

Since $H^* = \mathrm{Hom}(H, \mathbb{K})$ and $S : H \to H$ is an algebra antihomomorphism, the dual H^* is an H-module in four different ways:

$$\begin{aligned}&\langle (f \rightharpoonup a), g\rangle := \langle a, gf\rangle, \qquad \langle (a \leftharpoonup f), g\rangle := \langle a, fg\rangle, \\ &\langle (f \rightharpoondown a), g\rangle := \langle a, S(f)g\rangle, \quad \langle (a \leftharpoondown f), g\rangle := \langle a, gS(f)\rangle.\end{aligned} \tag{2}$$

If H is finite dimensional then H^* is a Hopf algebra. The equality $\langle (f \rightharpoonup a), g\rangle = \langle a, gf\rangle = \sum\langle a_{(1)}, g\rangle\langle a_{(2)}, f\rangle$ implies

$$(f \rightharpoonup a) = \sum a_{(1)}\langle a_{(2)}, f\rangle. \tag{3}$$

Analogously we have

$$(a \leftharpoonup f) = \sum \langle a_{(1)}, f\rangle a_{(2)}. \tag{4}$$

Proposition 4.1.19. *Let H be a finite dimensional Hopf algebra. Then H^* is a right Hopf module over H.*

PROOF. H^* is a left H^*-module by left multiplication hence by 2.1.1 a right H-comodule by $\delta(a) = \sum_i b_i^* a \otimes b_i$. Let $f, g \in H$ and $a, b \in H^*$. The (left) multiplication of H^* satisfies

$$ab = \sum b_{(H^*)}\langle a, b_{(1)}\rangle.$$

We use the right H-module structure

$$(a \leftharpoondown f) = \sum a_{(1)}\langle S(f), a_{(2)}\rangle.$$

on $H^* = \mathrm{Hom}(H, \mathbb{K})$.

Now we check the Hopf module property. Let $a, b \in H^*$ and $f, g \in H$. We apply $H^* \otimes H$ to its dual $H \otimes H^*$ and get

$$\begin{aligned}\delta(a \leftharpoondown f)(g \otimes b) &= \sum\langle(a \leftharpoondown f)_{(H^*)}, g\rangle\langle b, (a \leftharpoondown f)_{(1)}\rangle = \langle b(a \leftharpoondown f), g\rangle \\ &= \sum\langle b, g_{(1)}\rangle\langle(a \leftharpoondown f), g_{(2)}\rangle = \sum\langle b, g_{(1)}\rangle\langle a, g_{(2)}S(f)\rangle \\ &= \sum\langle b, g_{(1)}\varepsilon(f_{(2)})\rangle\langle a, g_{(2)}S(f_{(1)})\rangle = \sum\langle(f_{(3)} \rightharpoonup b), g_{(1)}S(f_{(2)})\rangle\langle a, g_{(2)}S(f_{(1)})\rangle \\ &= \sum\langle(f_{(2)} \rightharpoonup b)a, gS(f_{(1)})\rangle = \sum\langle((f_{(2)} \rightharpoonup b)a) \leftharpoondown f_{(1)}, g\rangle \\ &= \sum\langle(a_{(H^*)}\langle(f_{(2)} \rightharpoonup b), a_{(1)}\rangle) \leftharpoondown f_{(1)}, g\rangle \\ &= \sum\langle(a_{(H^*)} \leftharpoondown f_{(1)})\langle(f_{(2)} \rightharpoonup b), a_{(1)}\rangle, g\rangle = \sum\langle(a_{(H^*)} \leftharpoondown f_{(1)})\langle b, a_{(1)}f_{(2)}\rangle, g\rangle\end{aligned}$$

hence $\delta(a \leftharpoondown f) = \sum(a_{(H^*)} \leftharpoondown f_{(1)}) \otimes a_{(1)}f_{(2)}$. $\square$

Theorem 4.1.20. *Let H be a finite dimensional Hopf algebra. Then the antipode S is bijective, the space of left integrals $\mathrm{Int}_l(H^*)$ has dimension 1, and the homomorphism*

$$H \ni f \mapsto (f \rightharpoonup \textstyle\int) = \sum \int_{(1)}\langle \int_{(2)}, f\rangle \ni H^*$$

is bijective for any $0 \neq \int \in \mathrm{Int}_l(H^*)$.

PROOF. By Proposition 2.1.19 H^* is a right Hopf module over H. By Proposition 2.1.18 there is an isomorphism $\alpha : (H^*)^{coH} \otimes H \ni a \otimes f \mapsto (a \leftharpoondown f) = (S(f) \rightharpoonup a) \in H^*$. Since $(H^*)^{coH} \cong \mathrm{Int}_l(H^*)$ by 2.1.9 we get

$$\mathrm{Int}_l(H^*) \otimes H \cong H^*$$

as right H-Hopf modules by the given map. This shows $\dim(\mathrm{Int}_l(H^*)) = 1$. So we get an isomorphism $H \ni f \mapsto (\int \leftharpoondown f) \in H^*$ that is a composition of S and $f \mapsto (f \rightharpoonup \int)$. Since H is finite dimensional both of these maps are bijective. $\square$

If G is a finite group then every generalized integral $a \in \mathbb{K}G$ can be written with a uniquely determined $g \in H = \mathbb{K}^G$ as

$$\langle a, f\rangle = \int f(x)S(g)(x)dx = \sum_{x \in G} f(x)g(x^{-1}) \tag{5}$$

for all $f \in H$.

If G is a finite Abelian group then each group element (rational integral) $y \in G \subseteq \mathbb{K}G$ can be written as

$$y = \sum_{x\in G}\sum_{\chi\in\hat{G}} \beta_\chi \langle x^{-1}, \chi\rangle x$$

since

$$\begin{aligned}\langle y, f\rangle &= \langle (\textstyle\int \leftharpoondown \sum_{\chi\in\hat{G}} \beta_\chi \chi), f\rangle = \langle \textstyle\int, fS(\sum_{\chi\in\hat{G}} \beta_\chi\chi)\rangle \\ &= \textstyle\sum_{x\in G}\langle x, f\rangle \sum_{\chi\in\hat{G}} \beta_\chi \langle x, S(\chi)\rangle = \langle \sum_{x\in G}\sum_{\chi\in\hat{G}} \beta_\chi\langle x^{-1},\chi\rangle x, f\rangle.\end{aligned}$$

In particular the matrix $(\langle x^{-1}, \chi\rangle)$ is invertible.

Let H be finite dimensional. Since $\langle \int, fg\rangle = \langle (\int \leftharpoondown f), g\rangle$ as a functional on g is a generalized integral, there is a unique $\nu(f) \in H$ such that

$$\langle \textstyle\int, fg\rangle = \langle \textstyle\int, g\nu(f)\rangle \tag{6}$$

or

$$\int f(x)g(x)dx = \int g(x)\nu(f)(x)dx. \tag{7}$$

Although the functions $f, g \in H$ of the quantum group do not commute under multiplication, there is a simple commutation rule if the product is integrated.

Proposition and Definition 4.1.21. *The map $\nu : H \longrightarrow H$ is an algebra automorphism, called the* Nakayama automorphism.

PROOF. It is clear that ν is a linear map. We have $\int f\nu(gh) = \int ghf = \int hf\nu(g) = \int f\nu(g)\nu(h)$ hence $\nu(gh) = \nu(g)\nu(h)$ and $\int f\nu(1) = \int f$ hence $\nu(1) = 1$. Furthermore if $\nu(g) = 0$ then $0 = \langle \int, f\nu(g)\rangle = \langle \int, gf\rangle = \langle (f \rightharpoonup \int), g\rangle$ for all $f \in H$ hence $\langle a, g\rangle = 0$ for all $a \in H^*$ hence $g = 0$. So ν is injective hence bijective. $\square$

Corollary 4.1.22. *The map $H \ni f \mapsto (\int \leftharpoondown f) \in H^*$ is an isomorphism.*

PROOF. We have

$$(\textstyle\int \leftharpoondown f) = (\nu(f) \rightharpoonup \textstyle\int)$$

since $\langle (\int \leftharpoondown f), g\rangle = \langle \int, fg\rangle = \langle \int, g\nu(f)\rangle = \langle (\nu(f) \rightharpoonup \int), g\rangle$. This implies the corollary. $\square$

If G is a finite group and $H = \mathbb{K}^G$ then H is commutative hence $\nu = \mathrm{id}$.

Definition 4.1.23. An element $\delta \in H$ is called a *Dirac δ-function* if δ is a left invariant integral in H with $\langle \int, \delta\rangle = 1$, i.e. if δ satisfies

$$f\delta = \varepsilon(f)\delta \qquad \text{and} \qquad \int \delta(x)dx = 1$$

for all $f \in H$. If H has a Dirac δ-function then we write

$$\int^* a(x)dx = \textstyle\int^* a := \langle a, \delta\rangle. \tag{8}$$

Proposition 4.1.24.

1. *If H is finite dimensional then there exists a unique Dirac δ-function δ.*
2. *If H is infinite dimensional then there exists no Dirac δ-function.*

PROOF. 1. Since $H \ni f \mapsto (f \rightharpoonup \int) \in H^*$ is an isomorphism there is a $\delta \in H$ such that $(\delta \rightharpoonup \int) = \varepsilon$. Then $(f\delta \rightharpoonup \int) = (f \rightharpoonup (\delta \rightharpoonup \int)) = (f \rightharpoonup \varepsilon) = \varepsilon(f)\varepsilon = \varepsilon(f)(\delta \rightharpoonup \int)$ which implies $f\delta = \varepsilon(f)\delta$. Furthermore we have $\langle \int, \delta \rangle = \langle \int, 1_H \delta \rangle = \langle (\delta \rightharpoonup \int), 1_H \rangle = \varepsilon(1_H) = 1_{\mathbb{K}}$.

2. is [Sweedler] exercise V.4. □

Lemma 4.1.25. *Let H be a finite dimensional Hopf algebra. Then $\int \in H^*$ is a left integral iff*

$$a(\sum \int_{(1)} \otimes S(\int_{(2)})) = (\sum \int_{(1)} \otimes S(\int_{(2)}))a \tag{9}$$

iff

$$\sum S(a)\int_{(1)} \otimes \int_{(2)} = \sum \int_{(1)} \otimes a\int_{(2)} \tag{10}$$

iff

$$\sum f_{(1)} \langle \int, f_{(2)} \rangle = \langle \int, f \rangle 1_H. \tag{11}$$

PROOF. Let $\int$ be a left integral. Then

$$\sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)}) = \sum (a\int)_{(1)} \otimes S((a\int)_{(2)}) = \varepsilon(a)(\sum \int_{(1)} \otimes S(\int_{(2)}))$$

for all $a \in H$. Hence

$$\begin{aligned}(\sum \int_{(1)} \otimes S(\int_{(2)}))a &= \sum \varepsilon(a_{(1)})(\int_{(1)} \otimes S(\int_{(2)}))a_{(2)} \\ &= \sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)})a_{(3)} \\ &= \sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})\varepsilon(a_{(2)}) = a(\sum \int_{(1)} \otimes S(\int_{(2)})).\end{aligned}$$

Conversely $a(\sum \int_{(1)} \varepsilon(S(\int_{(2)}))) = (\sum \int_{(1)} \varepsilon(S(\int_{(2)})a)) = \varepsilon(a)(\sum \int_{(1)} \varepsilon(S(\int_{(2)})))$, hence $\int = \sum \int_{(1)} \varepsilon(S(\int_{(2)}))$ is a left integral.

Since S is bijective the following holds

$$\begin{aligned}&\sum S(a)\int_{(1)} \otimes \int_{(2)} = \sum S(a)\int_{(1)} \otimes S^{-1}(S(\int_{(2)})) \\ &= \sum \int_{(1)} \otimes S^{-1}(S(\int_{(2)})S(a)) = \sum \int_{(1)} \otimes a\int_{(2)}.\end{aligned}$$

The converse follows easily.

If $\int \in \mathrm{Int}_l(H)$ is a left integral then $\sum \langle a, f_{(1)} \rangle \langle \int, f_{(2)} \rangle = \langle a\int, f \rangle = \langle a, 1_H \rangle \langle \int, f \rangle$.

Conversely if $\lambda \in H^*$ with (11) is given then $\langle a\lambda, f \rangle = \sum \langle a, f_{(1)} \rangle \langle \lambda, f_{(2)} \rangle = \langle a, 1_H \rangle \langle \lambda, f \rangle$ hence $a\lambda = \varepsilon(a)\lambda$. □

If G is a finite group then

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq e; \\ 1 \text{ if } x = e. \end{cases} \tag{12}$$

In fact since δ is left invariant we get $f(x)\delta(x) = f(e)\delta(x)$ for all $x \in G$ and $f \in \mathbb{K}^G$. Since $G \subset H^* = \mathbb{K}G$ is a basis, we get $\delta(x) = 0$ if $x \neq e$. Furthermore $\int \delta(x)dx = \sum_{x\in G} \delta(x) = 1$ implies $\delta(e) = 1$. So we have $\delta = e^*$.

If G is a finite Abelian group we get $\delta = \alpha \sum_{\chi\in\hat{G}} \chi$ for some $\alpha \in \mathbb{K}$. The evaluation gives $1 = \langle \int, \delta\rangle = \alpha \sum_{x\in G,\chi\in\hat{G}} \langle \chi, x\rangle$. Now let $\lambda \in \widehat{G}$. Then $\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = \sum_{\chi\in\hat{G}} \langle \lambda\chi, x\rangle = \langle \lambda, x\rangle \sum_{\chi\in\hat{G}} \langle \chi, x\rangle$. Since for each $x \in G \setminus \{e\}$ there is a λ such that $\langle \lambda, x\rangle \neq 1$ and we get

$$\sum_{\chi\in\hat{G}} \langle \chi, x\rangle = |G|\delta_{e,x}.$$

Hence $\sum_{x\in G,\chi\in\hat{G}} \langle \chi, x\rangle = |G| = \alpha^{-1}$ and

$$\delta = |G|^{-1} \sum_{\chi\in\hat{G}} \chi. \tag{13}$$

Let H be finite dimensional for the rest of this section. In Corollary 1.22 we have seen that the map $H \ni f \mapsto (\int \leftharpoondown f) \in H^*$ is an isomorphism. This map will be called the *Fourier transform*.

Theorem 4.1.26. *The Fourier transform $H \ni f \mapsto \widetilde{f} \in H^*$ is bijective with*

$$\widetilde{f} = (\textstyle\int \leftharpoondown f) = \sum \langle \int_{(1)}, f\rangle \int_{(2)} \tag{14}$$

The inverse Fourier transform is defined by

$$\widetilde{a} = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle. \tag{15}$$

Since these maps are inverses of each other the following formulas hold

$$\begin{array}{ll} \langle \widetilde{f}, g\rangle = \int f(x)g(x)dx & \langle a, \widetilde{b}\rangle = \int^* S^{-1}(a)(x)b(x)dx \\ f = \sum S^{-1}(\delta_{(1)})\langle \widetilde{f}, \delta_{(2)}\rangle & a = \sum \langle \int_{(1)}, \widetilde{a}\rangle \int_{(2)}. \end{array} \tag{16}$$

Proof. We use the isomorphisms $H \longrightarrow H^*$ defined by $\widehat{f} := \widetilde{f} = (\int \leftharpoondown f) = \sum \langle \int_{(1)}, f\rangle \int_{(2)}$ and $H^* \longrightarrow H$ defined by $\widehat{a} := (a \rightharpoonup \delta) = \sum \delta_{(1)}\langle a, \delta_{(2)}\rangle$. Because of

$$\langle a, \widehat{b}\rangle = \langle a, (b \rightharpoonup \delta)\rangle = \langle ab, \delta\rangle \tag{17}$$

and

$$\langle \widetilde{f}, g\rangle = \langle (\textstyle\int \leftharpoondown f), g\rangle = \langle \int, fg\rangle \tag{18}$$

we get for all $a \in H^*$ and $f \in H$

$$\begin{aligned} \langle a, \widehat{\widehat{f}}\rangle &= \langle a\widehat{f}, \delta\rangle = \textstyle\sum \langle a, \delta_{(1)}\rangle\langle \widehat{f}, \delta_{(2)}\rangle = \sum \langle a, \delta_{(1)}\rangle\langle \int, f\delta_{(2)}\rangle \qquad (\text{ by Lemma 1.25 }) \\ &= \textstyle\sum \langle a, S(f)\delta_{(1)}\rangle\langle \int, \delta_{(2)}\rangle = \langle a, S(f)\rangle\langle \int, \delta\rangle = \langle a, S(f)\rangle. \end{aligned}$$

This gives $\widehat{\widehat{f}} = S(f)$. So the inverse map of $H \longrightarrow H^*$ with $\widehat{f} = (\int \leftharpoonup f) = \widetilde{f}$ is $H^* \longrightarrow H$ with $S^{-1}(\widehat{a}) = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle = \widetilde{a}$. Then the given inversion formulas are clear.

We note for later use $\langle a, \widetilde{b}\rangle = \langle a, S^{-1}(\widehat{b})\rangle = \langle S^{-1}(a), \widehat{b}\rangle = \langle S^{-1}(a)b, \delta\rangle$. □

If G is a finite group and $H = \mathbb{K}^G$ then

$$\widetilde{f} = \sum_{x \in G} f(x)x.$$

Since $\Delta(\delta) = \sum_{x\in G} x^{-1^*} \otimes x^*$ where the $x^* \in \mathbb{K}^G$ are the dual basis to the $x \in G$, we get

$$\widetilde{a} = \sum_{x \in G} \langle a, x^*\rangle x^*.$$

If G is a finite Abelian group then the groups G and $\widehat{G}$ are isomorphic so the Fourier transform induces a linear automorphism $\widetilde{-} : \mathbb{K}^G \longrightarrow \mathbb{K}^G$ and we have

$$\widetilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} \langle a, \chi\rangle \chi^{-1}$$

By substituting the formulas for the integral and the Dirac δ-function (1) and (13) we get

$$\begin{array}{ll} \widetilde{f} = \sum_{x\in G} f(x)x, & \widetilde{a} = |G|^{-1}\sum_{\chi\in\hat{G}} a(\chi)\chi^{-1}, \\ f = |G|^{-1}\sum_{\chi\in\hat{G}} \widetilde{f}(\chi)\chi^{-1}, & a = \sum_{x\in G} \widetilde{a}(x)x. \end{array} \tag{19}$$

This implies

$$\widetilde{f}(\chi) = \sum_{x\in G} f(x)\chi(x) = \int f(x)\chi(x)dx \tag{20}$$

with inverse transform

$$\widetilde{a}(x) = |G|^{-1} \sum_{\chi\in\hat{G}} \chi(a)\chi^{-1}(x). \tag{21}$$

Corollary 4.1.27. *The Fourier transforms of the left invariant integrals in H and H^* are*

$$\widetilde{\delta} = \varepsilon\nu^{-1} \in H^* \qquad \text{and} \qquad \widetilde{\int} = 1 \in H. \tag{22}$$

Proof. We have $\langle \widetilde{\delta}, f\rangle = \langle \int, \delta f\rangle = \langle \int, \nu^{-1}(f)\delta\rangle = \varepsilon\nu^{-1}(f)\langle \int, \delta\rangle = \varepsilon\nu^{-1}(f)$ hence $\widetilde{\delta} = \varepsilon\nu^{-1}$. From $\widetilde{1} = (\int \leftharpoonup 1) = \int$ we get $\widetilde{\int} = 1$. □

Proposition 4.1.28. *Define a* convolution multiplication *on H^* by*

$$\langle a * b, f\rangle := \sum \langle a, S^{-1}(\delta_{(1)})f\rangle\langle b, \delta_{(2)}\rangle.$$

Then the following transformation rule holds for $f, g \in H$:

$$\widetilde{fg} = \widetilde{f} * \widetilde{g}. \tag{23}$$

In particular H^* with the convolution multiplication is an associative algebra with unit $\widetilde{1_H} = \int$, i.e.

$$\int * a = a * \int = a. \tag{24}$$

Proof. Given $f, g, h \in H^*$. Then

$$\begin{aligned} \langle \widetilde{fg}, h\rangle &= \langle \textstyle\int, fgh\rangle = \langle \textstyle\int, fS^{-1}(1_H)gh\rangle\langle \textstyle\int, \delta\rangle \\ &= \textstyle\sum \langle \textstyle\int, fS^{-1}(\delta_{(1)})gh\rangle\langle \textstyle\int, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, fS^{-1}(\delta_{(1)})h\rangle\langle \textstyle\int, g\delta_{(2)}\rangle \\ &= \textstyle\sum\langle \widetilde{f}, S^{-1}(\delta_{(1)})h\rangle\langle \widetilde{g}, \delta_{(2)}\rangle = \langle \widetilde{f} * \widetilde{g}, h\rangle. \end{aligned}$$

From (22) we get $\widetilde{1}_H = \int$. So we have $\widetilde{f} = \widetilde{1f} = \widetilde{1} * \widetilde{f} = \int * \widetilde{f}$. □

If G is a finite Abelian group and $a, b \in H^* = \mathbb{K}^{\hat{G}}$. Then

$$(a * b)(\mu) = |G|^{-1} \sum_{\chi,\lambda\in\hat{G},\chi\lambda=\mu} a(\lambda)b(\chi).$$

In fact we have

$$\begin{aligned} (a * b)(\mu) &= \langle a * b, \mu\rangle = \textstyle\sum\langle a, S^{-1}(\delta_{(1)})\mu\rangle\langle b, \delta_{(2)}\rangle \\ &= |G|^{-1}\textstyle\sum_{\chi\in\hat{G}}\langle a, \chi^{-1}\mu\rangle\langle b, \chi\rangle = |G|^{-1}\sum_{\chi,\lambda\in\hat{G},\chi\lambda=\mu} a(\lambda)b(\chi). \end{aligned}$$

One of the most important formulas for Fourier transforms is the Plancherel formula on the invariance of the inner product under Fourier transforms. We have

Theorem 4.1.29. *(The Plancherel formula)*

$$\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle. \tag{25}$$

Proof. First we have from (16)

$$\begin{aligned} \langle a, f\rangle &= \textstyle\sum\langle \textstyle\int_{(1)}, \widetilde{a}\rangle\langle \textstyle\int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\nu(\widetilde{a})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, S^{-1}(S(\nu(\widetilde{a}))\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \textstyle\int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \nu(\widetilde{a})\delta_{(2)}\rangle = \textstyle\sum\langle \textstyle\int, S^{-1}(\delta)_{(2)}\rangle\langle \widetilde{f}, \nu(\widetilde{a})S(S^{-1}(\delta)_{(1)})\rangle \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, \nu(\widetilde{a})\rangle. \end{aligned}$$

Apply this to $\langle \int, \delta\rangle$. Then we get

$$1 = \langle \textstyle\int, \delta\rangle = \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{\delta}, \nu(\widetilde{\textstyle\int})\rangle = \langle \textstyle\int, S^{-1}(\delta)\rangle\varepsilon\nu^{-1}\nu(1) = \langle \textstyle\int, S^{-1}(\delta)\rangle.$$

Hence we get $\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle$. □

Corollary 4.1.30. *If H is unimodular then $\nu = S^2$.*

PROOF. H unimodular means that δ is left and right invariant. Thus we get

$$\begin{aligned}\langle a,f\rangle &= \sum\langle \textstyle\int_{(1)},\widetilde{a}\rangle\langle \textstyle\int_{(2)},S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f},\delta_{(2)}\rangle\\ &= \sum\langle \textstyle\int,\widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f},\delta_{(2)}\rangle = \sum\langle \textstyle\int,S^{-1}(\delta_{(1)}S(\widetilde{a}))\rangle\langle \widetilde{f},\delta_{(2)}\rangle\\ &= \sum\langle \textstyle\int,S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f},\delta_{(2)}S^2(\widetilde{a})\rangle \quad (\text{ since } \delta \text{ is right invariant})\\ &= \langle \textstyle\int,S^{-1}(\delta)\rangle\langle \widetilde{f},S^2(\widetilde{a})\rangle = \langle \widetilde{f},S^2(\widetilde{a})\rangle.\end{aligned}$$

Hence $S^2=\nu$. □

We also get a special representation of the inner product $H^*\otimes H\longrightarrow \mathbb{K}$ by both integrals:

Corollary 4.1.31.

$$\langle a,f\rangle = \int \widetilde{a}(x)f(x)dx = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx. \tag{26}$$

PROOF. We have the rules for the Fourier transform. From (18) we get $\langle a,f\rangle = \langle \int,\widetilde{a}f\rangle = \int\widetilde{a}(x)f(x)dx$ and from (17) $\langle a,f\rangle = \langle S^{-1}(a)\widetilde{f},\delta\rangle = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx$. □

The Fourier transform leads to an interesting integral transform on H by double application.

Proposition 4.1.32. *The double transform* $\breve{f} := (\delta \leftharpoonup (\int \leftharpoonup f))$ *defines an automorphism* $H\longrightarrow H$ *with*

$$\breve{f}(y) = \int f(x)\delta(xy)dx.$$

PROOF. We have

$$\begin{aligned}\langle y,\breve{f}\rangle &= \langle y,(\delta\leftharpoonup(\textstyle\int\leftharpoonup f))\rangle = \langle(\textstyle\int\leftharpoonup f)y,\delta\rangle\\ &= \sum\langle(\textstyle\int\leftharpoonup f),\delta_{(1)}\rangle\langle y,\delta_{(2)}\rangle = \sum\langle \textstyle\int,f\delta_{(1)}\rangle\langle y,\delta_{(2)}\rangle\\ &= \sum\langle \textstyle\int_{(1)},f\rangle\langle \textstyle\int_{(2)},\delta_{(1)}\rangle\langle y,\delta_{(2)}\rangle = \sum\langle \textstyle\int_{(1)},f\rangle\langle \textstyle\int_{(2)}y,\delta\rangle\\ &= \sum\langle \textstyle\int_{(1)},f\rangle\langle \textstyle\int_{(2)},(y\rightharpoonup\delta)\rangle = \langle \textstyle\int,f(y\rightharpoonup\delta)\rangle\\ &= \textstyle\int f(x)\delta(xy)dx\end{aligned}$$

since $\langle x,(y\rightharpoonup\delta)\rangle = \langle xy,\delta\rangle$. □