CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

6. The bialgebra coend

Let $\omega: \mathcal{D} \to \mathcal{C}$ and $\omega': \mathcal{D}' \to \mathcal{C}$ be diagrams in \mathcal{C} . We call the diagram $(\mathcal{D}, \omega) \otimes (\mathcal{D}', \omega') := (\mathcal{D} \times \mathcal{D}', \omega \otimes \omega')$ with $(\omega \otimes \omega')(X, Y) := \omega(X) \otimes \omega'(Y)$ the tensor product of these two diagrams. The new diagram consists of all possible tensor products of all objects and all morphisms of the original diagrams.

From now on we assume that the category \mathcal{C} is the category of vector spaces and we use the symmetry $\tau: V \otimes W \to W \otimes V$ in **Vec**.

Proposition 3.6.1. Let (\mathcal{D}, ω) and (\mathcal{D}', ω') be finite diagrams in **Vec**. Then

$$\operatorname{coend}(\omega \otimes \omega') \cong \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega').$$

PROOF. First observe the following. If two diagrams $\omega: \mathcal{D} \to \mathbf{Vec}$ and $\omega': \mathcal{D}' \to \mathbf{Vec}$ are given then $\varinjlim_{\mathcal{D}} \varrho \varinjlim_{\mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D}} \varrho (\omega) \otimes \varinjlim_{\mathcal{D}'} \varrho (\omega')$ since the tensor product preserves colimits and colimits commute with colimits. For this consider the diagram

$$\omega(X) \otimes \omega'(Y) \longrightarrow \omega(X) \otimes \varinjlim_{\mathcal{D}'}(\omega')$$

$$\varinjlim_{\mathcal{D}}(\omega) \otimes \omega'(Y) \longrightarrow \varinjlim_{\mathcal{D}}(\omega) \otimes \varinjlim_{\mathcal{D}'}(\omega') \qquad \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'}(\omega \otimes \omega').$$

The maps in the diagram are the injections for the corresponding colimits. In particular we have coend $(\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} ((\omega \otimes \omega')^* \otimes (\omega \otimes \omega')) \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega^* \otimes \omega \otimes \omega'^* \otimes \omega') \cong \varinjlim_{\mathcal{D}} (\omega^* \otimes \omega) \otimes \varinjlim_{\mathcal{D}'} (\omega'^* \otimes \omega') \cong \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega').$

The (universal) morphism

$$(\iota(X) \otimes \iota'(Y))(1 \otimes \tau \otimes 1) : \omega(X)^* \otimes \omega'(Y)^* \otimes \omega(X) \otimes \omega'(Y) \to \underline{\lim}(\omega^* \otimes \omega) \otimes \underline{\lim}(\omega'^* \otimes \omega')$$

can be identified with the universal morphism

$$\iota(X,Y):\omega(X)^*\otimes\omega'(Y)^*\otimes\omega(X)\otimes\omega'(Y)\to\underline{\lim}((\omega\otimes\omega')^*\otimes(\omega\otimes\omega')).$$

Hence the induced morphisms

$$(1 \otimes \tau \otimes 1)(\delta \otimes \delta') : \omega(X) \otimes \omega'(Y) \to \omega(X) \otimes \omega'(Y) \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega')$$

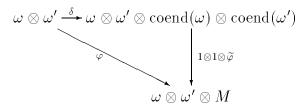
and

$$\delta: \omega(X) \otimes \omega'(Y) \to \omega(X) \otimes \omega'(Y) \otimes \operatorname{coend}(\omega \otimes \omega')$$

can be identified.
$$\Box$$

Corollary 3.6.2. For all finite diagrams (\mathcal{D}, ω) and (\mathcal{D}', ω') in \mathcal{D} there is a universal natural transformation $\delta : \omega \otimes \omega' \to \omega \otimes \omega' \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega')$ so that for each object M and each natural transformation $\varphi : \omega \otimes \omega' \to \omega \otimes \omega' \otimes M$ there exists

a unique morphism $\widetilde{\varphi}$: coend(ω) \otimes coend(ω') $\to M$ such that



commutes.

Definition 3.6.3. Let (\mathcal{D}, ω) be a diagram in $\mathcal{C} = \mathbf{Vec}$. Then ω is called *reconstructive*

- if there is an object coend(ω) in \mathcal{C} and a universal natural transformation $\delta: \omega \to \omega \otimes \operatorname{coend}(\omega)$
- and if $(1 \otimes \tau \otimes 1)(\delta \otimes \delta) : \omega \otimes \omega \to \omega \otimes \omega \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega)$ is a univesarl natural transformation of bifunctors.

Definition 3.6.4. Let (\mathcal{D}, ω) be a diagram in **Vec**. Let \mathcal{D} be a monoidal category and ω be a monoidal functor. Then (\mathcal{D}, ω) is called a *monoidal diagram*.

Let (\mathcal{D}, ω) be a monoidal diagram **Vec**. Let $A \in \mathbf{Vec}$ be an algebra. A natural transformation $\varphi : \omega \to \omega \otimes B$ is called monoidal monoidal if the diagrams

$$\omega(X) \otimes \omega(Y) \xrightarrow{\varphi(X) \otimes \varphi(Y)} \omega(X) \otimes \omega(Y) \otimes B \otimes B$$

$$\downarrow^{\rho} \qquad \qquad \downarrow^{\rho \otimes m} \qquad \downarrow^{\rho \otimes m} \qquad \qquad$$

and

$$\mathbb{K} \xrightarrow{\cong} \mathbb{K} \otimes \mathbb{K}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\omega(I) \xrightarrow{\varphi(I)} \omega(I) \otimes B$$

commute.

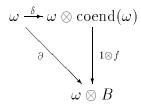
We denote the set of monoidal natural transformations by $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$.

Problem 3.6.1. Show that $Nat^{\otimes}(\omega, \omega \otimes B)$ is a functor in B.

Theorem 3.6.5. Let (\mathcal{D}, ω) be a reconstructive, monoidal diagram in **Vec**. Then coend (ω) is a bialgebra and $\delta : \omega \to \omega \otimes \operatorname{coend}(\omega)$ is a monoidal natural transformation

If B is a bialgebra and $\partial: \omega \to \omega \otimes B$ is a monoidal natural transformation, then there is a unique homomorphism of bialgebras $f: coend(\omega) \to B$ such that the

diagram



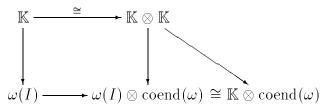
commutes.

PROOF. The multiplication of coend(ω) arises from the following diagram

$$\omega(X) \otimes \omega(Y) \xrightarrow{\delta \otimes \delta} \omega(X) \otimes \omega(Y) \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega)$$

$$\omega(X \otimes Y) \xrightarrow{\delta} \omega(X \otimes Y) \otimes \operatorname{coend}(\omega) \cong \omega(X) \otimes \omega(Y) \otimes \operatorname{coend}(\omega)$$

For the construction of the unit we consider the diagram $\mathcal{D}_0 = (\{I\}, \{id\})$ together with $\omega_0 : \mathcal{D}_0 \to \mathbf{Vec}$, $\omega_0(I) = \mathbb{K}$, the monoidal unit object in the monoidal category of diagrams in \mathbf{Vec} . Then $(\mathbb{K} \to \mathbb{K} \otimes \mathbb{K}) = (\omega_0 \to \omega_0 \otimes \operatorname{coend}(\omega_0))$ is the universal map. The following diagram then induced the unit for $\operatorname{coend}(\omega)$



By using the universal property one checks the laws for bialgebras.

The above diagrams show in particular that the natural transformation $\delta:\omega\to\omega\otimes\mathrm{coend}(\omega)$ is monoidal. \square