CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

5. The coalgebra coend

Proposition 3.5.1. *Let $\mathcal{C}$ be a monoidal category and $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$. Assume that there is a universal object* $\operatorname{coend}(\omega)$ *and natural transformation $\delta : \omega \longrightarrow \omega \otimes \operatorname{coend}(\omega)$.*

Then there is exactly one coalgebra structure on $\operatorname{coend}(\omega)$ *such that the diagrams*

$$\begin{array}{ccc} \omega & \xrightarrow{\delta} & \omega \otimes \operatorname{coend}(\omega) \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ \omega \otimes \operatorname{coend}(\omega) & \xrightarrow[\delta\otimes 1]{} & \omega \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega) \end{array}$$

and

$$\begin{array}{ccc} \omega & \xrightarrow{\delta} & \omega \otimes \operatorname{coend}(\omega) \\ & {\scriptstyle \mathrm{id}_\omega}\searrow & \downarrow{\scriptstyle 1\otimes\epsilon} \\ & & \omega \otimes I \end{array}$$

commute.

Proof. Because of the universal property of $\operatorname{coend}(\omega)$ there are structure morphisms $\Delta : \operatorname{coend}(\omega) \longrightarrow \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega)$ and $\epsilon : \operatorname{coend}(\omega) \longrightarrow I$. This implies the coalgebra property similar to the proof of Corollary 3.3.8. $\square$

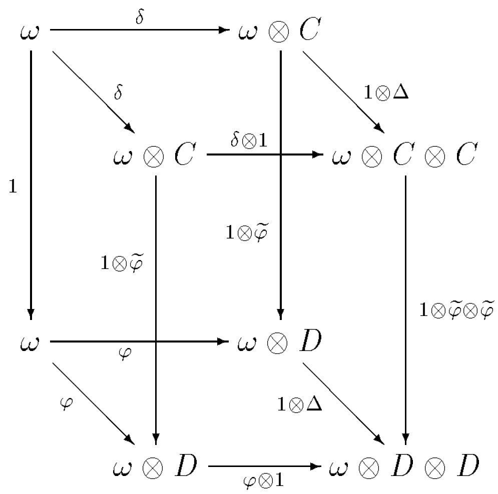
Observe that by this construction all objects and all morphisms of the diagram $\omega : \mathcal{D} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ are comodules or morphisms of comodules over the coalgebra $\operatorname{coend}(\omega)$. In fact $C := \operatorname{coend}(\omega)$ is the universal coalgebra over which the given diagram becomes a diagram of comodules.

Corollary 3.5.2. *Let $(\mathcal{D}, \omega)$ be a diagram $\mathcal{C}$ with objects in $\mathcal{C}_0$. Then all objects of the diagram are comodules over the coalgebra $C :=$* $\operatorname{coend}(\omega)$ *and all morphisms are morphisms of comodules. If D is another coalgebra and all objects of the diagram are D-comodules by $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ and all morphisms of the diagram are morphisms of D-comodules then there exists a unique morphism of coalgebras $\widetilde{\varphi} : \operatorname{coend}(\omega) \longrightarrow D$ such that the diagram*

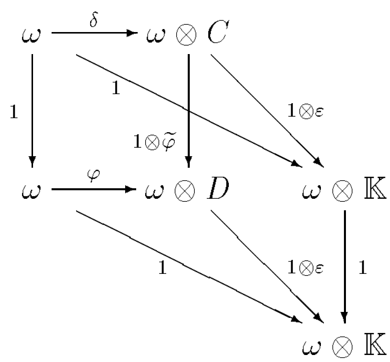
$$\begin{array}{ccc} \omega & \xrightarrow{\delta} & \omega \otimes \operatorname{coend}(\omega) \\ & {\scriptstyle \varphi}\searrow & \downarrow{\scriptstyle 1\otimes\widetilde{\varphi}} \\ & & \omega \otimes D \end{array}$$

commutes.

Proof. The morphisms $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ define a natural transformation since all morphisms of the diagram are morphisms of comodules. So the existence and the uniqueness of a morphism $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow D$ is clear. The only thing to show is that this is a morphism of coalgebras. This follows from the universal property of $C = \mathrm{coend}(\omega)$ and the diagram



where the right side of the cube commutes by the universal property. Similarly we get that $\widetilde{\varphi}$ preserves the counit since the following diagram commutes



$\square$