CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups

4. Finite reconstruction

The endomorphism ring of a vector space enjoys the following universal property. It is a vector space itself and allows a homomorphism $\rho : \mathrm{End}(V) \otimes V \longrightarrow V$. It is universal with respect to this property, i. e. if Z is a vector space and $f : Z \otimes V \longrightarrow V$ is a homomorphism, then there is a unique homomorphism $g : Z \longrightarrow \mathrm{End}(V)$ such that

$$\begin{tikzcd} Z \otimes V \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ \mathrm{End}(V) \otimes V \arrow[r, "\rho"'] & V \end{tikzcd}$$

commutes.

The algebra structure of $\mathrm{End}(V)$ comes for free from this universal property.

If we replace the vector space V by a diagram of vector spaces $\omega : \mathcal{D} \longrightarrow \mathbf{Vec}$ we get a similar universal object $\mathrm{End}(\omega)$. Again the universal property induces a unique algebra structure on $\mathrm{End}(\omega)$.

Problem 3.4.1. 1. Let V be a vector space. Show that there is a universal vector space E and homomorphism $\rho : E \otimes V \longrightarrow V$ (such that for each vector space Z and each homomorphism $f : Z \otimes V \longrightarrow V$ there is a unique homomorphism $g : Z \longrightarrow E$ such that

$$\begin{tikzcd} Z \otimes V \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ E \otimes V \arrow[r, "\rho"'] & V \end{tikzcd}$$

commutes). We call E and $\rho : E \otimes V \longrightarrow V$ a *vector space acting universally on* V.

2. Let E and $\rho : E \otimes V \longrightarrow V$ be a vector space acting universally on V. Show that E uniquely has the structure of an algebra such that V becomes a left E-module.

3. Let $\omega : \mathcal{D} \longrightarrow \mathbf{Vec}$ be a diagram of vector spaces. Show that there is a universal vector space E and natural transformation $\rho : E \otimes \omega \longrightarrow \omega$ (such that for each vector space Z and each natural transformation $f : Z \otimes \omega \longrightarrow \omega$ there is a unique homomorphism $g : Z \longrightarrow E$ such that

$$\begin{tikzcd} Z \otimes \omega \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ E \otimes \omega \arrow[r, "\rho"'] & \omega \end{tikzcd}$$

commutes). We call E and $\rho : E \otimes \omega \longrightarrow \omega$ a *vector space acting universally on* ω.

4. Let E and $\rho : E \otimes \omega \longrightarrow \omega$ be a vector space acting universally on ω. Show that E uniquely has the structure of an algebra such that ω becomes a diagram of left E-modules.

Similar considerations can be carried out for coactions $V \longrightarrow V \otimes C$ or $\omega \longrightarrow \omega \otimes C$ and a coalgebra structure on C. There is one restriction, however. We can only use finite dimensional vector spaces V or diagrams of finite dimensional vector spaces. This will be done further down.

As we have seen, ???

We want to find a universal natural transformation $\delta : \omega \longrightarrow \omega \otimes \operatorname{coend}(\omega)$. For this purpose we consider the isomorphisms

$$\operatorname{Mor}_{\mathcal{C}}(\omega(X), \omega(X) \otimes M) \cong \operatorname{Mor}_{\mathcal{C}}(\omega(X)^* \otimes \omega(X), M)$$

that are given by $f \mapsto (\operatorname{ev} \otimes 1)(1 \otimes f)$ and as inverse $g \mapsto (1 \otimes g)(\operatorname{db} \otimes 1)$. We first develop techniques to describe the properties of a natural transformation $\phi : \omega \longrightarrow \omega \otimes M$ as properties of the associated family $g(X) : \omega(X)^* \otimes \omega(X) \longrightarrow M$. We will see that $g : \omega^* \otimes \omega \longrightarrow M$ will be a *cone*. Then we will show that ϕ is a universal natural transformation if and only if its associated cone is universal. In the literature this is called a coend.

Throughout this section assume the following. Let $\mathcal{D}$ be an arbitrary diagram scheme. Let $\mathcal{C}$ be a cocomplete monoidal category such that the tensor product preserves colimits in both arguments. Let $\mathcal{C}_0$ be the full subcategory of those objects in $\mathcal{C}$ that have a left dual. Let $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$ such that $\omega(X) \in \mathcal{C}_0$ for all $X \in \mathcal{D}$, i. e. ω is given by a functor $\omega_0 : \mathcal{D} \longrightarrow \mathcal{C}_0$. We call such a diagram a *finite diagram* in $\mathcal{C}$. Finally for an object $M \in \mathcal{C}$ let $\omega \otimes M : \mathcal{D} \longrightarrow \mathcal{C}$ be the functor with $(\omega \otimes M)(X) = \omega(X) \otimes M$.

Remark 3.4.1. Consider the following category $\widetilde{\mathcal{D}}$. For each morphism $f : X \longrightarrow Y$ there is an object $\widetilde{f} \in \widetilde{D}$. The object corresponding to the identity $1_X : X \longrightarrow X$ is denoted by $\widetilde{X} \in \widetilde{\mathcal{D}}$. For each morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ there are two morphisms $f_1 : \widetilde{f} \longrightarrow \widetilde{X}$ and $f_2 : \widetilde{f} \longrightarrow \widetilde{Y}$ in $\widetilde{\mathcal{D}}$. Furthermore there are the identities $1_f : \widetilde{f} \longrightarrow \widetilde{f}$ in $\widetilde{\mathcal{D}}$.

Since there are no morphisms with $\widetilde{X}$ as domain other than $(1_X)_i : \widetilde{X} \longrightarrow \widetilde{X}$ and $1_f : \widetilde{f} \longrightarrow \widetilde{f}$ we only have to define the following compositions $(1_X)_i \circ f_j := f_j$. Then $\widetilde{\mathcal{D}}$ becomes a category and we have $1_{\widetilde{X}} = (1_X)_1 = (1_X)_2$.

We define a diagram $\omega^* \otimes \omega : \widetilde{\mathcal{D}} \longrightarrow \mathcal{C}$ as follows. If $f : X \longrightarrow Y$ is given then

$$(\omega^* \otimes \omega)(\widetilde{f}) := \omega(Y)^* \otimes \omega(X)$$

and

$$\begin{aligned} \omega(f_1) &:= \omega(f)^* \otimes \omega(1_X), \\ \omega(f_2) &:= \omega(1_Y)^* \otimes \omega(f). \end{aligned}$$

The colimit of $\omega^* \otimes \omega$ consists of an object $\operatorname{coend}(\omega) \in \mathcal{C}$ together with a family of morphisms $\iota(X,X) : \omega(X)^* \otimes \omega(X) \longrightarrow \operatorname{coend}(\omega)$ such that the diagrams

$$\begin{tikzcd}
 & \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"] & \\
(\omega^* \otimes \omega)(\widetilde{f}) = \omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes\omega(f)"'] & & \operatorname{coend}(\omega) \\
 & \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"'] &
\end{tikzcd}$$

commute for all $f : X \longrightarrow Y$ in $\mathcal{D}$. Indeed, such a family $\iota(\widetilde{X}) := \iota(X,X)$ can be uniquely extended to a natural transformation by defining $\iota(\widetilde{f}) := \iota(X,X)(\omega(f)^* \otimes \omega(X)) = \iota(Y,Y)(\omega(Y)^* \otimes \omega(f))$. In addition the pair $(\operatorname{coend}(\omega), \iota)$ is universal with respect to this property.

In the literature such a universal object is called a *coend* of the bifunctor $\omega^* \otimes \omega : \mathcal{D}^{op} \times \mathcal{D} \longrightarrow \mathcal{C}$.

Corollary 3.4.2. *The following is a coequalizer*

$$\coprod_{f \in \operatorname{Mor}\mathcal{D}} \omega(\operatorname{Zi}(f))^* \otimes \omega(\operatorname{Qu}(f)) \;\underset{q}{\overset{p}{\rightrightarrows}}\; \coprod_{X \in \operatorname{Ob}\mathcal{D}} \omega(X)^* \otimes \omega(X) \longrightarrow \operatorname{coend}(\omega)$$

Proof. This is just a reformulation of Remark A.10.11, since the colimit may also be built from the commutative squares given above.

Observe that for the construction of the colimit not all objects of the diagram have to be used but only those of the form $\omega(X)^* \otimes \omega(X)$. $\square$

Theorem 3.4.3. (**Tannaka-Krein**)

Let $\omega : \mathcal{D} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ be a finite diagram. Then there exists an object $\operatorname{coend}(\omega) \in \mathcal{C}$ *and a natural transformation $\delta : \omega \longrightarrow \omega \otimes \operatorname{coend}(\omega)$ such that for each object $M \in \mathcal{C}$ and each natural transformation $\varphi : \omega \longrightarrow \omega \otimes M$ there exists a unique morphism $\widetilde{\varphi} : \operatorname{coend}(\omega) \longrightarrow M$ such that the diagram*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \operatorname{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\
 & \omega \otimes M
\end{tikzcd}$$

commutes.

Proof. Let $\operatorname{coend}(\omega) \in \mathcal{C}$ together with morphisms $\iota(\widetilde{f}) : \omega(Y)^* \otimes \omega(X) \longrightarrow \operatorname{coend}(\omega)$ be the colimit of the diagram $\omega^* \otimes \omega : \widetilde{\mathcal{D}} \longrightarrow \mathcal{C}$. So we get commutative diagrams

$$\begin{tikzcd}
& \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"] & \\
\omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[rr, "\iota(\widetilde{f})"] \arrow[dr, "1\otimes\omega(f)"'] & & \operatorname{coend}(\omega) \\
& \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"'] &
\end{tikzcd}$$

for each $f : X \longrightarrow Y$ in $\mathcal{C}$.

For $X \in \mathcal{C}$ we define a morphism $\delta(X) : \omega(X) \longrightarrow \omega(X) \otimes \operatorname{coend}(\omega)$ by $(1 \otimes \iota(X,X))(\operatorname{db} \otimes 1) : \omega(X) \longrightarrow \omega(X) \otimes \omega(X)^* \otimes \omega(X) \longrightarrow \omega(X) \otimes \operatorname{coend}(\omega)$. Then we get as in Corollary 3.3.5 $\quad \iota(X,X) = (1 \otimes \operatorname{ev})(1 \otimes \delta(X))$.

We show that δ is a natural transformation. For each $f : X \longrightarrow Y$ the square

$$\begin{tikzcd}
I \arrow[r, "\operatorname{db}_X"] \arrow[d, "\operatorname{db}_Y"'] & \omega(X) \otimes \omega(X)^* \arrow[d, "\omega(f)\otimes 1"] \\
\omega(Y) \otimes \omega(Y)^* \arrow[r, "1\otimes\omega(f)^*"'] & \omega(Y) \otimes \omega(X)^*.
\end{tikzcd}$$

commutes by Corollary 3.3.9. Thus the following diagram commutes

$$\begin{tikzcd}
\omega(X) \arrow[rr, "\operatorname{db}\otimes 1"] \arrow[dr, "\operatorname{db}\otimes 1"] \arrow[dd, "\omega(f)"'] & & \omega(X) \otimes \omega(X)^* \otimes \omega(X) \arrow[rr, "1\otimes\iota(X,X)"] \arrow[dr, "\omega(f)\otimes 1\otimes 1"] & & \omega(X) \otimes \operatorname{coend}(\omega) \arrow[dd, "\omega(f)\otimes 1"] \\
& \omega(Y) \otimes \omega(Y)^* \otimes \omega(X) \arrow[rr, "1\otimes\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes 1\otimes\omega(f)"'] & & \omega(Y) \otimes \omega(X)^* \otimes \omega(X) \arrow[dr, "1\otimes\iota(X,X)"'] & \\
\omega(Y) \arrow[rr, "\operatorname{db}\otimes 1"] & & \omega(Y) \otimes \omega(Y)^* \otimes \omega(Y) \arrow[rr, "1\otimes\iota(Y,Y)"] & & \omega(Y) \otimes \operatorname{coend}(\omega).
\end{tikzcd}$$

Now let $M \in \mathcal{C}$ be an object and $\varphi : \omega \longrightarrow \omega \otimes M$ a natural transformation. Observe that

$$\begin{tikzcd}
\omega(Y)^* \otimes \omega(X) \arrow[r, "\omega(f)^*\otimes 1"] \arrow[d, "1\otimes\omega(f)"'] & \omega(X)^* \otimes \omega(X) \arrow[d, "\operatorname{ev}"] \\
\omega(Y)^* \otimes \omega(Y) \arrow[r, "\operatorname{ev}"'] & I
\end{tikzcd}$$

commutes by Corollary 3.3.10. Thus also the diagram

$$\begin{tikzcd}
\omega(X)^* \otimes \omega(X) \arrow[rr, "1\otimes\varphi(X)"] & & \omega(X)^* \otimes \omega(X) \otimes M \arrow[dr, "\mathrm{ev}\otimes 1"] & \\
\omega(Y)^* \otimes \omega(X) \arrow[u, "\omega(f)^*\otimes 1"] \arrow[r, "1\otimes\varphi(X)"] \arrow[d, "1\otimes\omega(f)"'] & \omega(Y)^* \otimes \omega(X) \otimes M \arrow[ur, "\omega(f)^*\otimes 1\otimes 1"] \arrow[dr, "1\otimes\omega(f)\otimes 1"'] & & M \\
\omega(Y)^* \otimes \omega(Y) \arrow[rr, "1\otimes\varphi(Y)"] & & \omega(Y)^* \otimes \omega(Y) \otimes M \arrow[ur, "\mathrm{ev}\otimes 1"'] &
\end{tikzcd}$$

commutes. We define $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow M$ from the colimit property as universal factorization

$$\begin{tikzcd}
 & \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"] \arrow[drr, "(\mathrm{ev}\otimes 1)(1\otimes\varphi(X))"] & & \\
\omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes\omega(f)"'] & & \mathrm{coend}(\omega) \arrow[r, "\widetilde{\varphi}"] & M \\
 & \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"] \arrow[urr, "(\mathrm{ev}\otimes 1)(1\otimes\varphi(Y))"'] & &
\end{tikzcd}$$

Hence the diagram

$$\begin{tikzcd}
\omega(X) \arrow[rr, "\delta(X)"] \arrow[dr, "\mathrm{db}\otimes 1"] \arrow[ddd, "\varphi(X)"'] & & \omega(X) \otimes \mathrm{coend}(\omega) \arrow[ddd, "1\otimes\widetilde{\varphi}"] \\
 & \omega(X) \otimes \omega(X)^* \otimes \omega(X) \arrow[ur, "1\otimes\iota(X,X)"] \arrow[d, "1\otimes 1\otimes\varphi(X)"] & \\
 & \omega(X) \otimes \omega(X)^* \otimes \omega(X) \otimes M \arrow[dr, "1\otimes\mathrm{ev}\otimes 1"] & \\
\omega(X) \otimes M \arrow[ur, "\mathrm{db}\otimes 1\otimes 1"] \arrow[rr, "1\otimes 1"'] & & \omega(X) \otimes M
\end{tikzcd}$$

commutes. The exterior portion of this diagram yields

$$\begin{tikzcd}
\omega(X) \arrow[r, "\delta(X)"] \arrow[dr, "\varphi(X)"'] & \omega(X) \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\
 & \omega(X) \otimes M.
\end{tikzcd}$$

It remains to show that $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow M$ is uniquely determined. Let $\widetilde{\varphi}_0 : \mathrm{coend}(\omega) \longrightarrow M$ be another morphism with $\varphi(X) = (1 \otimes \widetilde{\varphi}_0)\delta(X)$ for all $X \in \mathcal{D}$.

Then the following diagram commutes

$$\begin{tikzcd}
\omega(X)^* \otimes \omega(X) \arrow[rr, "\iota(X,X)"] \arrow[dr, "1\otimes\delta(X)"] \arrow[ddr, "1\otimes\varphi(X)"'] & & \mathrm{coend}(\omega) \arrow[dd, "\widetilde{\varphi}_0"] \\
& \omega(X)^* \otimes \omega(X) \otimes \mathrm{coend}(\omega) \arrow[ur, "\mathrm{ev}\otimes 1"] \arrow[d, "1\otimes 1\otimes\widetilde{\varphi}_0"] & \\
& \omega(X)^* \otimes \omega(X) \otimes M \arrow[r, "\mathrm{ev}\otimes 1"] & M,
\end{tikzcd}$$

hence we have $\widetilde{\varphi}_0 = \widetilde{\varphi}$. □

Corollary 3.4.4. *The functor* $\mathrm{Nat}(\omega, \omega \otimes M)$ *is a representable functor in* M *represented by* $\mathrm{coend}(\omega)$.

Proof. The universal problem implies the isomorphism

$$\mathrm{Nat}(\omega, \omega \otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\mathrm{coend}(\omega), M)$$

and the universal natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$ is mapped to the identity under this isomorphism. □

It is also possible to construct an isomorphism

$$\mathrm{Nat}(\omega, \omega' \otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\mathrm{cohom}(\omega', \omega), M)$$

for different functors $\omega, \omega' : \mathcal{D} \longrightarrow \mathcal{C}$ and thus define *cohomomorphism objects*. Observe that only ω' has to take values in $\mathcal{C}_0$ since then we can build objects $\omega'(X)^* \otimes \omega(X)$.