CHAPTER 3

Representation Theory, Reconstruction and Tannaka Duality

Introduction

One of the most interesting properties of quantum groups is their representation theory. It has deep applications in theoretical physics. The mathematical side has to distinguish between the representation theory of quantum groups and the representation theory of Hopf algebras. In both cases the particular structure allows to form tensor products of representations such that the category of representations becomes a monoidal (or tensor) category.

The problem we want to study in this chapter is, how much structure of the quantum group or Hopf algebra can be found in the category of representations. We will show that a quantum monoid can be uniquely reconstructed (up to isomorphism) from its representations. The additional structure given by the antipode is itimitely connected with a certain duality of representations. We will also generalize this process of reconstruction.

On the other hand we will show that the process of reconstruction can also be used to obtain the Tambara construction of the universal quantum monoid of a noncommutative geometrical space (from chapter 1.). Thus we will get another perspective for this theorem.

At the end of the chapter you should

- understand representations of Hopf algebras and of quantum groups,
- know the definition and first fundamental properties of monoidal or tensor categories,
- be familiar with the monoidal structure on the category of representations of Hopf algebras and of quantum groups,
- understand why the category of representations contains the full information about the quantum group resp. the Hopf algebra (Theorem of Tannaka-Krein),
- know the process of reconstruction and examples of bialgebras reconstructed from certain diagrams of finite dimensional vector spaces,
- understand better the Tambara construction of a universal algebra for a finite dimensional algebra.

1. Representations of Hopf Algebras

Let A be an algebra over a commutative ring $\mathbb{K}$. Let A-**Mod** be the category of A-modules. An A-module is also called a *representation* of A.

Observe that the action $A \otimes M \longrightarrow M$ satisfying the module axioms and an algebra homomorphism $A \longrightarrow \mathrm{End}(M)$ are equivalent descriptions of an A-module structure on the $\mathbb{K}$-module M.

The functor $\mathcal{U} : A\text{-}\mathbf{Mod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}$ with $\mathcal{U}({}_AM) = M$ and $\mathcal{U}(f) = f$ is called the *forgetful functor* or the *underlying functor*.

If B is a bialgebra then a *representation of* B is also defined to be a B-module. It will turn out that the property of being a bialgebra leads to the possibility of building tensor products of representations in a canonical way.

Let C be a coalgebra over a commutative ring $\mathbb{K}$. Let C-**Comod** be the category of C-comodules. A C-comodule is also called a *corepresentation* of C.

The functor $\mathcal{U} : C\text{-}\mathbf{Comod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}$ with $\mathcal{U}({}^CM) = M$ and $\mathcal{U}(f) = f$ is called the *forgetful functor* or the *underlying functor*.

If B is a bialgebra then a *corepresentation of* B is also defined to be a B-comodule. It will turn out that the property of being a bialgebra leads to the possibility of building tensor products of corepresentations in a canonical way.

Usually representations of a ring are considered to be modules over the given ring. The role of comodules certainly arises in the context of coalgebras. But it is not quite clear what the good definition of a representation of a quantum group or its representing Hopf algebra is.

For this purpose consider representations M of an ordinary group G. Assume for the simplicity of the argument that G is finite. Representations of G are vector spaces together with a group action $G \times M \longrightarrow M$. Equivalently they are vector spaces together with a group homomorphism $G \longrightarrow \mathrm{Aut}(M)$ or modules over the group algebra: $\mathbb{K}[G] \otimes M \longrightarrow M$. In the situation of quantum groups we consider the representing Hopf algebra H as algebra of functions on the quantum group G.

Then the algebra of functions on G is the Hopf algebra $\mathbb{K}^G$, the dual of the group algebra $\mathbb{K}[G]$. An easy exercise shows that the module structure $\mathbb{K}[G] \otimes M \longrightarrow M$ translates to the structure of a comodule $M \longrightarrow \mathbb{K}^G \otimes M$ and conversely. (Observe that G is finite.) So we should define representations of a quantum group as comodules over the representing Hopf algebra.

Definition 3.1.1. Let G be a quantum group with representing Hopf algebra H. A *representation* of G is a comodule over the representing Hopf algebra H.

From this definition we obtain immediately that we may form tensor products of representations of quantum groups since the representing algebra is a bialgebra.

We come now to the canonical construction of tensor products of (co-)representations.

Lemma 3.1.2. *Let B be a bialgebra. Let $M, N \in B$-**Mod** be two B-modules. Then $M \otimes N$ is a B-module by the action $b(m \otimes n) = \sum b_{(1)}m \otimes b_{(2)}n$. If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are homomorphisms of B-modules in B-**Mod** then $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ is a homomorphism of B-modules.*

Proof. We have homomorphisms of $\mathbb{K}$-algebras $\alpha : B \longrightarrow \mathrm{End}(M)$ and $\beta : B \longrightarrow \mathrm{End}(N)$ defining the B-module structure on M and N. Thus we get a homomorphism of algebras $\mathrm{can}(\alpha \otimes \beta)\Delta : B \longrightarrow B \otimes B \longrightarrow \mathrm{End}(M) \otimes \mathrm{End}(N) \longrightarrow \mathrm{End}(M \otimes N)$. Thus $M \otimes N$ is a B-module. The structure is $b(m \otimes n) = \mathrm{can}(\alpha \otimes \beta)(\sum b_{(1)}) \otimes b_{(2)})(m \otimes n) = \mathrm{can}(\sum \alpha(b_{(1)}) \otimes \beta(b_{(2)}))(m \otimes n) = \sum \alpha(b_{(1)})(m) \otimes \beta(b_{(2)})(n) = \sum b_{(1)}m \otimes b_{(2)}n$.

Furthermore we have $1(m \otimes n) = 1m \otimes 1m = m \otimes n$.

If f, g are homomorphisms of B-modules, then we have $(f \otimes g)(b(m \otimes n)) = (f \otimes g)(\sum b_{(1)}m \otimes b_{(2)}n) = \sum f(b_{(1)}m) \otimes g(b_{(2)}n) = \sum b_{(1)}f(m) \otimes b_{(2)}g(n) = b(f(m) \otimes g(n)) = b(f \otimes g)(m \otimes n)$. Thus $f \otimes g$ is a homomorphism of B-modules. $\square$

Corollary 3.1.3. *Let B be a bialgebra. Then $\otimes : B$-**Mod** $\times B$-**Mod** $\longrightarrow B$-**Mod** with $\otimes(M, N) = M \otimes N$ and $\otimes(f, g) = f \otimes g$ is a functor.*

Proof. The following are obvious from the ordinary properties of the tensor product over $\mathbb{K}$. $1_M \otimes 1_N = 1_{M \otimes N}$ and $(f \otimes g)(f' \otimes g') = ff' \otimes gg'$ for $M, N, f, f', g, g' \in B$-**Mod**. $\square$

Lemma 3.1.4. *Let B be a bialgebra. Let $M, N \in B$-**Comod** be two B-comodules. Then $M \otimes N$ is a B-comodule by the coaction $\delta_{M \otimes N}(m \otimes n) = \sum m_{(1)}n_{(1)} \otimes m_{(M)} \otimes n_{(N)}$.*

*If $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are homomorphisms of B-comodules in B-**Comod** then $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ is a homomorphism of B-comodules.*

Proof. The coaction on $M \otimes N$ may also be described by $(\nabla_B \otimes 1_M \otimes 1_N)(1_B \otimes \tau \otimes 1_N)(\delta_M \otimes \delta_N) : M \otimes N \longrightarrow B \otimes M \otimes B \otimes N \longrightarrow B \otimes B \otimes M \otimes N \longrightarrow B \otimes M \otimes N$. Although a diagrammatic proof of the coassociativity of the coaction and the law of the counit is quite involved it allows generalization so we give it here.

Consider the next diagram.

Square (1) commutes since M and N are comodules.

Squares (2) and (3) commute since $\tau : M \otimes N \longrightarrow N \otimes M$ for $\mathbb{K}$-modules M and N is a natural transformation.

Square (4) represents an interesting property of τ namely

$$(1 \otimes 1 \otimes \tau)(\tau_{B \otimes M, B} \otimes 1) = (1 \otimes 1 \otimes \tau)(\tau \otimes 1 \otimes 1)(1 \otimes \tau \otimes 1) = \\ (\tau \otimes 1 \otimes 1)(1 \otimes 1 \otimes \tau)(1 \otimes \tau \otimes 1) = (\tau \otimes 1 \otimes 1)(1 \otimes \tau_{M, B \otimes B})$$

that uses the fact that $(1 \otimes g)(f \otimes 1) = (f \otimes 1)(1 \otimes g)$ holds and that $\tau_{B \otimes M, B} = (\tau \otimes 1)(1 \otimes \tau)$ and $\tau_{M, B \otimes B} = (1 \otimes \tau)(\tau \otimes 1)$.

Square (5) and (6) commute by the properties of the tensor product.

Square (7) commutes since B is a bialgebra.

$$\begin{tikzcd}
M\otimes N \arrow[r, "\delta\otimes\delta"] \arrow[d, "\delta\otimes\delta"] \arrow[dr, phantom, "(1)"] & B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\Delta\otimes 1\otimes\Delta\otimes 1"] \arrow[dr, phantom, "(2)"] & B\otimes B\otimes M\otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "\Delta\otimes\Delta\otimes 1\otimes 1"] \arrow[ddr, phantom, "(7)"] & B\otimes M\otimes N \arrow[ddd, "\Delta\otimes 1\otimes 1"] \\
B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\delta\otimes 1\otimes\delta"] \arrow[d, "1\otimes\tau\otimes 1"] \arrow[dr, phantom, "(3)"] & B\otimes B\otimes M\otimes B\otimes B\otimes N \arrow[r, "1^2\otimes\tau_{M,B\otimes B}\otimes 1"] \arrow[d, "1\otimes\tau_{B\otimes M,B}\otimes 1\otimes 1"] \arrow[dr, phantom, "(4)"] & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[d, "1\otimes\tau\otimes 1\otimes 1\otimes 1"] & \\
B\otimes B\otimes M\otimes N \arrow[r, "1\otimes 1\otimes\delta\otimes\delta"] \arrow[d, "\nabla\otimes 1\otimes 1"] \arrow[dr, phantom, "(5)"] & B\otimes B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes 1\otimes 1\otimes\tau\otimes 1"] \arrow[d, "\nabla\otimes 1\otimes 1\otimes 1\otimes 1"] \arrow[dr, phantom, "(6)"] & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[d, "\nabla\otimes 1\otimes 1\otimes 1\otimes 1"] & \\
B\otimes M\otimes N \arrow[r, "1\otimes\delta\otimes\delta"] & B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes 1\otimes\tau\otimes 1"] & B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes\nabla\otimes 1\otimes 1"] & B\otimes B\otimes M\otimes N
\end{tikzcd}$$

The law of the counit is

$$\begin{tikzcd}
M\otimes N \arrow[r, "\delta\otimes\delta"] \arrow[dr, "1"] & B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\varepsilon\otimes 1\otimes\varepsilon\otimes 1"] & B\otimes B\otimes M\otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "\varepsilon\otimes\varepsilon\otimes 1\otimes 1"] & B\otimes M\otimes N \arrow[d, "\varepsilon\otimes 1\otimes 1"] \\
 & M\otimes N \arrow[r, "1"] & M\otimes N \arrow[r, "1"] & M\otimes N
\end{tikzcd}$$

where the last square commutes since ε is a homomorphism of algebras.

Now let f and g be homomorphisms of B-comodules. Then the diagram

$$\begin{tikzcd}
M\otimes N \arrow[r, "\delta\otimes\delta"] \arrow[d, "f\otimes g"] & B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "1\otimes f\otimes 1\otimes g"] & B\otimes B\otimes M\otimes N \arrow[r, "\nabla\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes f\otimes g"] & B\otimes M\otimes N \arrow[d, "1\otimes f\otimes g"] \\
M'\otimes N' \arrow[r, "\delta\otimes\delta"] & B\otimes M'\otimes B\otimes N' \arrow[r, "1\otimes\tau\otimes 1"] & B\otimes B\otimes M'\otimes N' \arrow[r, "\nabla\otimes 1\otimes 1"] & B\otimes M'\otimes N'
\end{tikzcd}$$

commutes. Thus $f\otimes g$ is a homomorphism of B-comodules. $\square$

Corollary 3.1.5. *Let B be a bialgebra. Then $\otimes : B$-**Comod** $\times$ B-**Comod** $\longrightarrow$ B-**Comod** with $\otimes(M,N) = M\otimes N$ and $\otimes(f,g) = f\otimes g$ is a functor.*

Proposition 3.1.6. *Let B be a bialgebra. Then the tensor product $\otimes : B$-**Mod** $\times$ B-**Mod** $\longrightarrow B$-**Mod** satisfies the following properties:*

1. *The associativity isomorphism $\alpha : (M_1\otimes M_2)\otimes M_3 \longrightarrow M_1\otimes(M_2\otimes M_3)$ with $\alpha((m\otimes n)\otimes p) = m\otimes(n\otimes p)$ is a natural transformation from the functor*

$\otimes \circ (\otimes \times \mathrm{Id})$ to the functor $\otimes \circ (\mathrm{Id} \times \otimes)$ in the variables M_1, M_2, and M_3 in B-**Mod**.

2. *The counit isomorphisms $\lambda : \mathbb{K} \otimes M \longrightarrow M$ with $\lambda(\kappa \otimes m) = \kappa m$ and $\rho : M \otimes \mathbb{K} \longrightarrow M$ with $\rho(m \otimes \kappa) = \kappa m$ are natural transformations in the variable M in* B-**Mod** *from the functor $\mathbb{K} \otimes$ - resp. - $\otimes \mathbb{K}$ to the identity functor* Id.
3. *The following diagrams of natural transformations are commutative*

$$\begin{tikzcd}
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[r, "\alpha(M_1,M_2,M_3)\otimes 1"] \arrow[d, "\alpha(M_1\otimes M_2,M_3,M_4)"] & (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[r, "\alpha(M_1,M_2\otimes M_3,M_4)"] & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[d, "1\otimes\alpha(M_2,M_3,M_4)"'] \\
(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[rr, "\alpha(M_1,M_2,M_3\otimes M_4)"] & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))
\end{tikzcd}$$

$$\begin{tikzcd}
(M_1 \otimes \mathbb{K}) \otimes M_2 \arrow[rr, "\alpha(M_1,\mathbb{K},M_2)"] \arrow[dr, "\rho(M_1)\otimes 1"'] & & M_1 \otimes (\mathbb{K} \otimes M_2) \arrow[dl, "1\otimes\lambda(M_2)"] \\
& M_1 \otimes M_2 &
\end{tikzcd}$$

Proof. The homomorphisms α, λ, and ρ are already defined in the category $\mathbb{K}$-**Mod** and satisfy the claimed properties. So we have to show, that these are homomorphisms in B-**Mod** and that $\mathbb{K}$ is a B-module. $\mathbb{K}$ is a B-module by $\varepsilon \otimes 1_{\mathbb{K}} : B \otimes \mathbb{K} \longrightarrow \mathbb{K}$. The easy verification uses the coassociativity and the counital property of B. $\square$

Similarly we get

Proposition 3.1.7. *Let B be a bialgebra. Then the tensor product*

$$\otimes : B\text{-}\mathbf{Comod} \times B\text{-}\mathbf{Comod} \longrightarrow B\text{-}\mathbf{Comod}$$

satisfies the following properties:

1. *The associativity isomorphism $\alpha : (M_1 \otimes M_2) \otimes M_3 \longrightarrow M_1 \otimes (M_2 \otimes M_3)$ with $\alpha((m \otimes n) \otimes p) = m \otimes (n \otimes p)$ is a natural transformation from the functor $\otimes \circ (\otimes \times \mathrm{Id})$ to the functor $\otimes \circ (\mathrm{Id} \times \otimes)$ in the variables M_1, M_2, and M_3 in* B-**Comod**.
2. *The counit isomorphisms $\lambda : \mathbb{K} \otimes M \longrightarrow M$ with $\lambda(\kappa \otimes m) = \kappa m$ and $\rho : M \otimes \mathbb{K} \longrightarrow M$ with $\rho(m \otimes \kappa) = \kappa m$ are natural transformations in the variable M in* B-**Comod** *from the functor $\mathbb{K} \otimes$ - resp. - $\otimes \mathbb{K}$ to the identity functor* Id.
3. *The following diagrams of natural transformations are commutative*

$$\begin{tikzcd}
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[r, "\alpha(M_1,M_2,M_3)\otimes 1"] \arrow[d, "\alpha(M_1\otimes M_2,M_3,M_4)"] & (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[r, "\alpha(M_1,M_2\otimes M_3,M_4)"] & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[d, "1\otimes\alpha(M_2,M_3,M_4)"'] \\
(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[rr, "\alpha(M_1,M_2,M_3\otimes M_4)"] & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))
\end{tikzcd}$$

$$\begin{tikzcd}
(M_1 \otimes \mathbb{K}) \otimes M_2 \arrow[rr, "\alpha(M_1,\mathbb{K},M_2)"] \arrow[dr, "\rho(M_1)\otimes 1"'] & & M_1 \otimes (\mathbb{K} \otimes M_2) \arrow[dl, "1\otimes\lambda(M_2)"] \\
& M_1 \otimes M_2 &
\end{tikzcd}$$

Remark 3.1.8. We now get some simple properties of the underlying functors $\mathcal{U} : B\text{-}\mathbf{Mod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}$ resp. $\mathcal{U} : B\text{-}\mathbf{Comod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}$ that are easily verified.

$$\begin{aligned}
&\mathcal{U}(M \otimes N) = \mathcal{U}(M) \otimes \mathcal{U}(N),\\
&\mathcal{U}(f \otimes g) = f \otimes g,\\
&\mathcal{U}(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}(\alpha) = \alpha,\ \mathcal{U}(\lambda) = \lambda,\ \mathcal{U}(\rho) = \rho.
\end{aligned}$$

Problem 3.1.1. We have seen that in representation theory and in corepresentation theory of quantum groups such as $\mathbb{K}G$, $U(\mathfrak{g})$, $SL_q(2)$, $U_q(sl(2))$ the ordinary tensor product (in $\mathbb{K}$-$\mathbf{Mod}$) of two (co-)reprensentations is in a canonical way again a (co-)reprensentation. For two $\mathbb{K}G$-modules M and N the structure is $g(m \otimes n) = gm \otimes gn$ for $g \in G$. For $U(\mathfrak{g})$-modules it is $g(m \otimes n) = gm \otimes n + m \otimes gn$ for $g \in \mathfrak{g}$. For $U_q(sl(2))$-modules it is $E(m \otimes n) = m \otimes En + Em \otimes Kn$, $F(m \otimes n) = K^{-1}m \otimes Fn + Fm \otimes n$, $K(m \otimes n) = Km \otimes Kn$.

Remark 3.1.9. Let A and B be algebras over a commutative ring $\mathbb{K}$. Let $f : A \longrightarrow B$ be a homomorphism of algebras. Then we have a functor $\mathcal{U}_f : B\text{-}\mathbf{Mod} \longrightarrow A\text{-}\mathbf{Mod}$ with $\mathcal{U}_f({}_BM) = {}_AM$ and $\mathcal{U}_f(g) = g$ where $am := f(a)m$ for $a \in A$ and $m \in M$. The functor $\mathcal{U}_f$ is also called *forgetful* or *underlying functor*.

The action of A on a B-module M can also be seen as the homomorphism $A \longrightarrow B \longrightarrow \mathrm{End}(M)$.

We denote the underlying functors previously discussed by

$$\mathcal{U}_A : A\text{-}\mathbf{Mod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod} \text{ resp. } \mathcal{U}_B : B\text{-}\mathbf{Mod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}.$$

Proposition 3.1.10. *Let $f : B \longrightarrow C$ be a homomorphism of bialgebras. Then $\mathcal{U}_f$ satisfies the following properties:*

$$\begin{aligned}
&\mathcal{U}_f(M \otimes N) = \mathcal{U}_f(M) \otimes \mathcal{U}_f(N),\\
&\mathcal{U}_f(g \otimes h) = g \otimes h,\\
&\mathcal{U}_f(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}_f(\alpha) = \alpha,\ \mathcal{U}_f(\lambda) = \lambda,\ \mathcal{U}_f(\rho) = \rho,\\
&\mathcal{U}_B\mathcal{U}_f(M) = \mathcal{U}_C(M),\\
&\mathcal{U}_B\mathcal{U}_f(g) = \mathcal{U}_C(g).
\end{aligned}$$

Proof. This is clear since the underlying $\mathbb{K}$-modules and the $\mathbb{K}$-linear maps stay unchanged. The only thing to check is that U_f generates the correct B-module structure on the tensor product. For $U_f(M \otimes N) = M \otimes N$ we have $b(m \otimes n) = f(b)(m \otimes n) = \sum f(b)_{(1)}m \otimes f(b)_{(2)}n = \sum f(b_{(1)})m \otimes f(b_{(2)})n = \sum b_{(1)}m \otimes b_{(2)}n$. $\square$

Remark 3.1.11. Let C and D be coalgebras over a commutative ring $\mathbb{K}$. Let $f : C \longrightarrow D$ be a homomorphism of coalgebras. Then we have a functor $\mathcal{U}_f : C$-**Comod** $\longrightarrow D$-**Comod** with $\mathcal{U}_f({}^C M) = {}^D M$ and $\mathcal{U}_f(g) = g$ where $\delta_D = (f \otimes 1)\delta_C : M \longrightarrow C \otimes M \longrightarrow D \otimes M$. Again the functor $\mathcal{U}_f$ is called *forgetful* or *underlying functor*.

We denote the underlying functors previously discussed by

$$\mathcal{U}_C : C\text{-}\mathbf{Comod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod} \text{ resp. } \mathcal{U}_D : D\text{-}\mathbf{Comod} \longrightarrow \mathbb{K}\text{-}\mathbf{Mod}.$$

Proposition 3.1.12. *Let $f : B \longrightarrow C$ be a homomorphism of bialgebras. Then $\mathcal{U}_f : C$-**Comod** $\longrightarrow D$-**Comod** satisfies the following properties:*

$$\begin{aligned}
&\mathcal{U}_f(M \otimes N) = \mathcal{U}_f(M) \otimes \mathcal{U}_f(N),\\
&\mathcal{U}_f(g \otimes h) = g \otimes h,\\
&\mathcal{U}_f(\mathbb{K}) = \mathbb{K},\\
&\mathcal{U}_f(\alpha) = \alpha,\ \mathcal{U}_f(\lambda) = \lambda,\ \mathcal{U}_f(\rho) = \rho,\\
&\mathcal{U}_C\mathcal{U}_f(M) = \mathcal{U}_B(M),\\
&\mathcal{U}_C\mathcal{U}_f(g) = \mathcal{U}_B(g).
\end{aligned}$$

PROOF. We leave the proof to the reader. □

Proposition 3.1.13. *Let H be a Hopf algebra. Let M and N be be H-modules. Then* $\mathrm{Hom}(M, N)$*, the set $\mathbb{K}$-linear maps from M to N, becomes an H-module by $(hf)(m) = \sum h_{(1)} f(S(h_{(2)} m)$. This structure makes*

$$\mathrm{Hom} : H\text{-}\mathbf{Mod} \times H\text{-}\mathbf{Mod} \longrightarrow H\text{-}\mathbf{Mod}$$

a functor contravariant in the first variable and covariant in the second variable.

PROOF. The main part to be proved is that the action $H \otimes \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, N)$ satisfies the associativity law. Let $f \in \mathrm{Hom}(M, N)$, $h, k \in H$, and $m \in M$. Then $((hk)f)(m) = \sum (hk)_{(1)} f(S((hk)_{(2)}) = \sum h_{(1)} k_{(1)} f(S(k_{(2)}) S(h_{(2)}) m) = \sum h_{(1)} (kf)(S(h_{(2)}) m) = (h(kf))(m)$.

We leave the proof of the other properties, in particular the functorial properties, to the reader. □

Corollary 3.1.14. *Let M be an H-module. Then the dual $\mathbb{K}$-module $M^* =$* $\mathrm{Hom}(M, \mathbb{K})$ *becomes an H-module by $(hf)(m) = f(S(h)m)$.*

PROOF. The space $\mathbb{K}$ is an H-module via $\varepsilon : H \longrightarrow \mathbb{K}$. Hence we have $(hf)(m) = \sum h_{(1)} f(S(h_{(2)} m) = \sum \varepsilon(h_{(1)}) f(S(h_{(2)} m) = f(S(h)m)$. □

2. Monoidal Categories

For our further investigations we need a generalized version of the tensor product that we are going to introduce in this section. This will give us the possibility to study more general versions of the notion of algebras and representations.

Definition 3.2.1. A *monoidal category* (or *tensor category*) consists of
a category $\mathcal{C}$,
a covariant functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the *tensor product*,
an object $I \in \mathcal{C}$, called the *unit*,
natural isomorphisms

$$\begin{aligned}
&\alpha(A,B,C) : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C),\\
&\lambda(A) : I \otimes A \longrightarrow A,\\
&\rho(A) : A \otimes I \longrightarrow A,
\end{aligned}$$

called *associativity, left unit* and *right unit*, such that the following diagrams commute:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\alpha(A\otimes B,C,D)"] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha(B,C,D)"'] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha(A,I,B)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1\otimes\lambda(B)"] \\
& A \otimes B &
\end{tikzcd}$$

These diagrams are called *coherence diagrams* or *constraints*.

A monoidal category is called a *strict monoidal category*, if the morphisms α, λ, ρ are the identity morphisms.

Remark 3.2.2. We define $A_1 \otimes \ldots \otimes A_n := (\ldots(A_1 \otimes A_2) \otimes \ldots) \otimes A_n$.

There is an important theorem of S. MacLane that says that all diagrams whose morphisms are constructed by using copies of α, λ, ρ, identities, inverses, tensor products and compositions of such commute. We will not prove this theorem. It implies that each monoidal category can be replaced by (is monoidally equivalent to) a strict monoidal category. That means that we may omit in diagrams the morphisms α, λ, ρ or replace them by identities. In particular there is only one automorphism of $A_1 \otimes \ldots \otimes A_n$ formed by coherence morphisms namely the identity.

Remark 3.2.3. For each monoidal category $\mathcal{C}$ we can construct the monoidal category $\mathcal{C}^{symm}$ symmetric to $\mathcal{C}$ that coincides with $\mathcal{C}$ as a category and has tensor product $A \boxtimes B := B \otimes A$ and the coherence morphisms

$$\begin{aligned}
&\alpha(C,B,A)^{-1} : (A \boxtimes B) \boxtimes C \longrightarrow A \boxtimes (B \boxtimes C),\\
&\rho(A) : I \boxtimes A \longrightarrow A,\\
&\lambda(A) : A \boxtimes I \longrightarrow A.
\end{aligned}$$

Then the coherence diagrams are commutative again, so that $\mathcal{C}^{symm}$ is a monoidal category.

Example 3.2.4. 1. Let R be an arbitrary ring. The category ${}_R\mathcal{M}_R$ of R-R-bimodules with the tensor product $M \otimes_R N$ is a monoidal category. In particular the $\mathbb{K}$-modules form a monoidal category. This is our most important example of a monoidal category.

2. Let B be a bialgebra and B-**Mod** be the category of left B-modules. We define the structure of a B-module on the tensor product $M \otimes N = M \otimes_{\mathbb{K}} N$ by

$$B \otimes M \otimes N \xrightarrow{\Delta \otimes 1_M \otimes 1_N} B \otimes B \otimes M \otimes N \xrightarrow{1_B \otimes \tau \otimes 1_N} B \otimes M \otimes B \otimes N \xrightarrow{\rho_M \otimes \rho_N} M \otimes N$$

as in the previous section. So B-**Mod** is a monoidal category by 3.1.7

3. Let B be a bialgebra and B-**Comod** be the category of B-comodules. The tensor product $M \otimes N = M \otimes_{\mathbb{K}} N$ carries the structure of a B-comodule by

$$M \otimes N \xrightarrow{\delta_M \otimes \delta_N} B \otimes M \otimes B \otimes N \xrightarrow{1_B \otimes \tau \otimes 1_N} B \otimes B \otimes M \otimes N \xrightarrow{\nabla \otimes 1_M \otimes 1_N} B \otimes M \otimes N.$$

as in the previous section. So B-**Comod** is a monoidal category by 3.1.8

4. Let G be a monoid. A $\mathbb{K}$-module together with a family of submodules $(V_g | g \in G)$ is called *G-graded* if $V = \oplus_{g \in G} V_g$.

Let V and W be G-graded $\mathbb{K}$-modules. A homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow W$ is called G-graded if $f(V_g) \subseteq W_g$ for all $g \in G$.

The G-graded $\mathbb{K}$-modules and their homomorphisms form the category $(\mathbb{K}\text{-}\mathbf{Mod})^G$ of *G-graded $\mathbb{K}$-modules.*

There is a monoidal structure on $(\mathbb{K}\text{-}\mathbf{Mod})^G$ given by the ordinary tensor product $V \otimes W$. The submodules on the tensor product $V \otimes W$ are given by $(V \otimes W)_g := \sum_{h \in G} V_h \otimes W_{h^{-1}g} = \sum_{h,k \in G, hk=g} V_h \otimes W_k$.

5. A *chain complex* of $\mathbb{K}$-modules

$$M = (\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0)$$

consists of a family of a family of $\mathbb{K}$-modules M_i and a family of homomorphisms $\partial_n : M_n \longrightarrow M_{n-1}$ with $\partial_{n-1}\partial_n = 0$. This chain complex is indexed by the monoid $\mathbb{N}_0$. One may also consider more general chain complexes indexed by an arbitrary cyclic monoid. Chain complexes indexed by $\mathbb{N}_0 \times \mathbb{N}_0$ are called double complexes. So much more general chain complexes may be considered. We restrict ourselves to chain complexes over $\mathbb{N}_0$.

Let M and N be chain complexes. A *homomorphism of chain complexes* $f : M \longrightarrow N$ consists of a family of homomorphisms of $\mathbb{K}$-modules $f_n : M_n \longrightarrow N_n$ such that $f_n \partial_{n+1} = \partial_{n+1} f_{n+1}$ for all $n \in \mathbb{N}_0$.

The chain complexes with these homomorphisms form the category of chain complexes $\mathbb{K}$-**Comp**.

If M and N are chain complexes then we form a new chain complex $M \otimes N$ with $(M \otimes N)_n := \bigoplus_{i=0}^n M_i \otimes N_{n-i}$ and $\partial : (M \otimes N)_n \longrightarrow (M \otimes N)_{n-1}$ given by

$\partial(m_i \otimes n_{n-i}) := (-1)^i \partial_M(m_i) \otimes n_{n-i} + m_i \otimes \partial(n_{n-i})$. This is often called the total complex associated with the double complex of the tensor product of M and N. Then it is easily checked that $\mathbb{K}$-**Comp** is a monoidal category with this tensor product.

Problem 3.2.2. 1. Prove that the category $(\mathbb{K}\text{-}\mathbf{Mod})^G$ of G-graded $\mathbb{K}$-modules is equivalent to the category $\mathbb{K}G$-**Comod** of $\mathbb{K}G$-comodules by the following construction. If V is a G-graded $\mathbb{K}$-module the V becomes a $\mathbb{K}G$-comodule by the map $\delta : V \longrightarrow \mathbb{K}G \otimes V$, $\delta(v) := g \otimes v$ for all $v \in V_g$ and all $g \in G$. Conversely if $V, \delta : V \longrightarrow \mathbb{K}G \otimes V$ is a $\mathbb{K}G$-comodule then V together with the submodules $V_g := \{v \in V | \delta(v) = g \otimes v\}$ is a G-graded $\mathbb{K}$-module.

Since $\mathbb{K}G$ is a bialgebra the category of $\mathbb{K}G$-comodules is a monoidal category. Show that the equivalence defined above between $(\mathbb{K}\text{-}\mathbf{Mod})^G$ and $\mathbb{K}G$-**Comod** preserves the tensor products, hence that it is a monoidal equivalence.

2. Let $B = \mathbb{K}\langle x, y\rangle / I$ where I is generated by $x^2, xy + yx$. Then B is a bialgebra with the diagonal $\Delta(y) = y \otimes y$, $\Delta(x) = x \otimes 1 + y \otimes x$. The counit is $\varepsilon(y) = 1, \varepsilon(x) = 0$. We introduced (the coopposite bialgebra of) this bialgebra in A.7 2.

Show that the category $\mathbb{K}$-**Comp** of chain complexes is equivalent to the category B-**Comod** of B-comodules by the following construction. If M is a chain complex then define a B-comodule on $M = \oplus_{i \in \mathbb{N}} M_i$ with the structure map $\delta : M \longrightarrow B \otimes M$, $\delta(m) := y^i \otimes m + xy^{i-1} \otimes \partial_i(m)$ for all $m \in M_i$ and for all $i \in \mathbb{N}$ resp. $\delta(m) := 1 \otimes m$ for $m \in M_0$. Conversely if $M, \delta : M \longrightarrow B \otimes M$ is a B-comodule then we define $\mathbb{K}$-modules $M_i := \{m \in M | \exists m' \in M[\delta(m) = y^i \otimes m + xy^{i-1} \otimes m']\}$ and $\mathbb{K}$-linear maps $\partial_i : M_i \longrightarrow M_{i-1}$ by $\partial_i(m) := m'$ for $\delta(m) = y^i \otimes m + xy^{i-1} \otimes m'$. Check that this defines an equivalence of categories.

(Hint: Let $m \in M \in B$-**Comod**. Since y^i, xy^i form a basis of B we have $\delta(m) = \sum_i y^i \otimes m_i + \sum_i xy^i \otimes m'_i$. We apply to this the equation $(1 \otimes \delta)\delta = (\Delta \otimes 1)\delta$ and compare coefficients to get

$$\delta(m_i) = y^i \otimes m_i + xy^{i-1} \otimes m'_{i-1}, \quad \delta(m'_i) = y^i \otimes m'_i$$

for all $i \in \mathbb{N}_0$ (with $m'_{-1} = 0$). Consequently for each $m_i \in M_i$ there is exactly one $\partial(m_i) = m'_{i-1} \in M$ such that

$$\delta(m_i) = y^i \otimes m_i + xy^{i-1} \otimes \partial(m_i).$$

Since $\delta(m'_{i-1}) = y^{i-1} \otimes m'_{i-1}$ for all $i \in \mathbb{N}$ we see that $\partial(m_i) \in M_{i-1}$. So we have defined $\partial : M_i \longrightarrow M_{i-1}$. Furthermore we see from this equation that $\partial^2(m_i) = 0$ for all $i \in \mathbb{N}$. Hence we have obtained a chain complex from (M, δ).

If we apply $(\epsilon \otimes 1)\delta(m) = m$ then we get $m = \sum m_i$ with $m_i \in M_i$ hence $M = \bigoplus_{i \in \mathbb{N}} M_i$. This together with the inverse construction leads to the required equivalence.)

3. A cochain complex has the form

$$M = (M_0 \xrightarrow{\partial_0} M_1 \xrightarrow{\partial_1} M_2 \xrightarrow{\partial_2} \dots)$$

with $\partial_{i+1}\partial_i = 0$. Show that the category $\mathbb{K}$-**Cocomp** of cochain complexes is equivalent to **Comod**-B where B is chosen as in example 5.

Lemma 3.2.5. *Let $\mathcal{C}$ be a monoidal category. Then the following diagrams commute*

$$\begin{tikzcd}
(I\otimes A)\otimes B \arrow[rr, "\alpha"] \arrow[dr, "\lambda(A)\otimes 1_B"'] & & I\otimes(A\otimes B) \arrow[dl, "\lambda(A\otimes B)"] \\
& A\otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A\otimes B)\otimes I \arrow[rr, "\alpha"] \arrow[dr, "\rho(A\otimes B)"'] & & A\otimes(B\otimes I) \arrow[dl, "1_A\otimes\rho(B)"] \\
& A\otimes B &
\end{tikzcd}$$

and we have $\lambda(I) = \rho(I)$.

Proof. First we observe that the identity functor $\mathrm{Id}_{\mathcal{C}}$ and the functor $I\otimes$ - are isomorphic by the natural isomorphism λ. Thus we have $I\otimes f = I\otimes g \Longrightarrow f = g$. In the following diagram

$$\begin{tikzcd}
((I\otimes I)\otimes A)\otimes B \arrow[rr, "\alpha\otimes 1"] \arrow[dr, "(\rho\otimes 1)\otimes 1"] \arrow[ddd, "\alpha"'] & & (I\otimes(I\otimes A))\otimes B \arrow[rr, "\alpha"] \arrow[dl, "(1\otimes\lambda)\otimes 1"'] & & I\otimes((I\otimes A)\otimes B) \arrow[dl, "1\otimes(\lambda\otimes 1)"'] \arrow[ddd, "1\otimes\alpha"] \\
& (I\otimes A)\otimes B \arrow[rr, "\alpha"] \arrow[d, "\alpha"] & & I\otimes(A\otimes B) \arrow[d, "1"'] & \\
& I\otimes(A\otimes B) \arrow[rr, "1"] & & I\otimes(A\otimes B) \arrow[dr, "1\otimes\lambda"'] & \\
(I\otimes I)\otimes(A\otimes B) \arrow[ur, "\rho\otimes(1\otimes 1)"] \arrow[rrrr, "\alpha"] & & & & I\otimes(I\otimes(A\otimes B))
\end{tikzcd}$$

all subdiagrams commute except for the right hand trapezoid. Since all morphisms are isomorphisms the right hand trapezoid must commute also. Hence the first diagram of the Lemma commutes.

In a similar way one shows the commutativity of the second diagram.

Furthermore the following diagram commutes

$$\begin{tikzcd}
I\otimes(I\otimes I) \arrow[dr, "1\otimes\rho"'] & & (I\otimes I)\otimes I \arrow[ll, "\alpha"'] \arrow[rr, "\alpha"] \arrow[dl, "\rho"] \arrow[dr, "\rho\otimes 1"'] & & I\otimes(I\otimes I) \arrow[dl, "1\otimes\lambda"] \\
& I\otimes I \arrow[dr, "\rho"'] & & I\otimes I \arrow[dl, "\rho"] & \\
& & I & &
\end{tikzcd}$$

Here the left hand triangle commutes by the previous property. The commutativity of the right hand diagram is given by the axiom. The lower square commutes since ρ is a natural transformation. In particular $\rho(1 \otimes \rho) = \rho(1 \otimes \lambda)$. Since ρ is an isomorphism and $I \otimes \text{-} \cong \mathrm{Id}_{\mathcal{C}}$ we get $\rho = \lambda$. $\square$

Problem 3.2.3. For morphisms $f : I \longrightarrow M$ and $g : I \longrightarrow N$ in a monoidal category we define $(f \otimes 1 : N \longrightarrow M \otimes N) := (f \otimes 1_I)\rho(I)^{-1}$ and $(1 \otimes g : M \longrightarrow M \otimes N) := (1 \otimes g)\lambda(I)^{-1}$. Show that the diagram

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "1\otimes g"] \\
N \arrow[r, "f\otimes 1"'] & M \otimes N
\end{tikzcd}$$

commutes.

Definition 3.2.6. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ be monoidal categories. A functor

$$\mathcal{F} : \mathcal{C} \to \mathcal{D}$$

together with a natural transformation

$$\xi(M, N) : \mathcal{F}(M) \otimes \mathcal{F}(N) \to \mathcal{F}(M \otimes N)$$

and a morphism

$$\xi_0 : I_{\mathcal{D}} \to \mathcal{F}(I_{\mathcal{C}})$$

is called *weakly monoidal* if the following diagrams commute

$$\begin{tikzcd}
(\mathcal{F}(M) \otimes \mathcal{F}(N)) \otimes \mathcal{F}(P) \arrow[r, "\xi\otimes 1"] \arrow[d, "\alpha"'] & \mathcal{F}(M \otimes N) \otimes \mathcal{F}(P) \arrow[r, "\xi"] & \mathcal{F}((M \otimes N) \otimes P) \arrow[d, "\mathcal{F}(\alpha)"] \\
\mathcal{F}(M) \otimes (\mathcal{F}(N) \otimes \mathcal{F}(P)) \arrow[r, "1\otimes\xi"] & \mathcal{F}(M) \otimes \mathcal{F}(N \otimes P) \arrow[r, "\xi"] & \mathcal{F}(M \otimes (N \otimes P))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes \mathcal{F}(M) \arrow[r, "\xi_0\otimes 1"] \arrow[dr, "\lambda"'] & \mathcal{F}(I) \otimes \mathcal{F}(M) \arrow[r, "\xi"] & \mathcal{F}(I \otimes M) \arrow[dl, "\mathcal{F}(\lambda)"] \\
& \mathcal{F}(M) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(M) \otimes I \arrow[r, "1\otimes\xi_0"] \arrow[dr, "\rho"'] & \mathcal{F}(M) \otimes \mathcal{F}(I) \arrow[r, "\xi"] & \mathcal{F}(M \otimes I) \arrow[dl, "\mathcal{F}(\rho)"] \\
& \mathcal{F}(M). &
\end{tikzcd}$$

If, in addition, the morphisms ξ and ξ_0 are isomorphisms then the functor is called a *monoidal functor*. The functor is called a *strict monoidal functor* if ξ and ξ_0 are the identity morphisms.

A natural transformation $\zeta : \mathcal{F} \longrightarrow \mathcal{F}'$ between weakly monoidal functors is called a *monoidal natural transformation* if the diagrams

$$\begin{tikzcd}
\mathcal{F}(M) \otimes \mathcal{F}(N) \arrow[r, "\xi"] \arrow[d, "\zeta\otimes\zeta"'] & \mathcal{F}(M \otimes N) \arrow[d, "\zeta"] \\
\mathcal{F}'(M) \otimes \mathcal{F}'(N) \arrow[r, "\xi'"'] & \mathcal{F}'(M \otimes N)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{F}(I) \arrow[dd, "\zeta"] \\
I \arrow[ur, "\xi_0"] \arrow[dr, "\xi_0'"'] & \\
 & \mathcal{F}'(I)
\end{tikzcd}$$

commute.

We can generalize the notions of an algebra or of a coalgebra in the context of a monoidal category. We define

Definition 3.2.7. Let $\mathcal{C}$ be a monoidal category. An *algebra* or a *monoid* in $\mathcal{C}$ consists of an object A together with a multiplication $\nabla : A \otimes A \longrightarrow A$ that is associative

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\nabla"] \arrow[d, "\nabla\otimes 1"'] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}$$

or more precisely

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha"] \arrow[d, "\nabla\otimes\mathrm{id}"'] & A \otimes (A \otimes A) \arrow[r, "\mathrm{id}\otimes\nabla"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[rr, "\nabla"] & & A
\end{tikzcd}$$

and has a unit $\eta : I \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd}
I \otimes A \cong A \cong A \otimes I \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, "\eta\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A.
\end{tikzcd}$$

Let A and B be algebras in $\mathcal{C}$. A *morphism of algebras* $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\nabla_A"'] & B \otimes B \arrow[d, "\nabla_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & I \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

commute.

Remark 3.2.8. It is obvious that the composition of two morphisms of algebras is again a morphism of algebras. The identity also is a morphism of algebras. Thus we obtain the category $\mathbf{Alg}(\mathcal{C})$ of algebras in $\mathcal{C}$.

Definition 3.2.9. Let $\mathcal{C}$ be a monoidal category. A *coalgebra* or a *comonoid* in $\mathcal{C}$ consists of an object C together with a comultiplication $\Delta : A \longrightarrow A \otimes A$ that is coassociative

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id} \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

or more precisely

$$\begin{tikzcd} C \arrow[rr, "\Delta"] \arrow[d, "\Delta"] & & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"'] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & (C \otimes C) \otimes C \arrow[r, "\alpha"] & C \otimes (C \otimes C) \end{tikzcd}$$

and has a counit $\varepsilon : C \longrightarrow I$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\ C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"'] & I \otimes C \cong C \cong C \otimes I. \end{tikzcd}$$

Let C and D be coalgebras in $\mathcal{C}$. A *morphism of coalgebras* $f : C \longrightarrow D$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & I & \end{tikzcd}$$

commute.

Remark 3.2.10. It is obvious that the composition of two morphisms of coalgebras is again a morphism of coalgebras. The identity also is a morphism of coalgebras. Thus we obtain the category $\mathbf{Coalg}(\mathcal{C})$ of coalgebras in $\mathcal{C}$.

Remark 3.2.11. Observe that the notions of bialgebra, Hopf algebra, and comodule algebra cannot be generalized to an arbitrary monoidal category since we need to have an algebra structure on the tensor product of two algebras and this requires us to interchange the middle tensor factors. These interchanges or flips are known under the name symmetry, quasisymmetry or braiding and will be discussed later on.

3. Dual Objects

At the end of the first section in Corollary 3.1.15 we saw that the dual of an H-module can be constructed. We did not show the corresponding result for comodules. In fact such a construction for comodules needs some finiteness conditions. With this restriction the notion of a dual object can be introduced in an arbitrary monoidal category.

Definition 3.3.1. Let $(\mathcal{C}, \otimes)$ be a monoidal category $M \in \mathcal{C}$ be an object. An object $M^* \in \mathcal{C}$ together with a morphism $\mathrm{ev} : M^* \otimes M \longrightarrow I$ is called a *left dual* for M if there exists a morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$ in $\mathcal{C}$ such that

$$(M \xrightarrow{\mathrm{db} \otimes 1} M \otimes M^* \otimes M \xrightarrow{1 \otimes \mathrm{ev}} M) = 1_M$$
$$(M^* \xrightarrow{1 \otimes \mathrm{db}} M^* \otimes M \otimes M^* \xrightarrow{\mathrm{ev} \otimes 1} M^*) = 1_{M^*}.$$

A monoidal category is called *left rigid* if each object $M \in \mathcal{C}$ has a left dual.

Symmetrically we define: an object ${}^*M \in \mathcal{C}$ together with a morphism $\mathrm{ev} : M \otimes {}^*M \longrightarrow I$ is called a *right dual* for M if there exists a morphism $\mathrm{db} : I \longrightarrow {}^*M \otimes M$ in $\mathcal{C}$ such that

$$(M \xrightarrow{1 \otimes \mathrm{db}} M \otimes {}^*M \otimes M \xrightarrow{\mathrm{ev} \otimes 1} M) = 1_M$$
$$({}^*M \xrightarrow{\mathrm{db} \otimes 1} {}^*M \otimes M \otimes {}^*M \xrightarrow{1 \otimes \mathrm{ev}} {}^*M) = 1_{{}^*M}.$$

A monoidal category is called *right rigid* if each object $M \in \mathcal{C}$ has a left dual.

The morphisms ev and db are called the *evaluation* respectively the *dual basis*.

Remark 3.3.2. If (M^*, ev) is a left dual for M then obviously (M, ev) is a right dual for M^* and conversely. One uses the same morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$.

Lemma 3.3.3. *Let (M^*, ev) be a left dual for M. Then there is a natural isomorphism*

$$\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, \text{-}) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, \text{-} \otimes M^*),$$

i. e. the functor $\text{-} \otimes M : \mathcal{C} \longrightarrow \mathcal{C}$ is left adjoint to the functor $\text{-} \otimes M^* : \mathcal{C} \longrightarrow \mathcal{C}$.

Proof. We give the unit and the counit of the pair of adjoint functors. We define $\Phi(A) := 1_A \otimes \mathrm{db} : A \longrightarrow A \otimes M \otimes M^*$ and $\Psi(B) := 1_B \otimes \mathrm{ev} : B \otimes M^* \otimes M \longrightarrow B$. These are obviously natural transformations. We have commutative diagrams

$$(A \otimes M \xrightarrow[1_A \otimes \mathrm{db} \otimes 1_M]{\mathcal{F}\Phi(A)=} A \otimes M \otimes M^* \otimes M \xrightarrow[1_A \otimes 1_M \otimes \mathrm{ev}]{\Psi\mathcal{F}(A)=} A \otimes M) = 1_{A \otimes M}$$

and

$$(B \otimes M^* \xrightarrow[1_B \otimes 1_{M^*} \otimes \mathrm{db}]{\Phi\mathcal{G}(B)=} B \otimes M^* \otimes M \otimes M^* \xrightarrow[1_B \otimes \mathrm{ev} \otimes 1_{M^*}]{\mathcal{G}\Psi(B)=} B \otimes M^*) = 1_{B \otimes M^*}$$

thus the Lemma has been proved by Corollary A.9.11. $\square$

The converse holds as well. If $\text{-} \otimes M$ is left adjoint to $\text{-} \otimes M^*$ then the unit Φ gives a morphism $\mathrm{db} := \Phi(I) : I \longrightarrow M \otimes M^*$ and the counit Ψ gives a morphism $\mathrm{ev} := \Psi(I) : M^* \otimes M \longrightarrow I$ satisfying the required properties. Thus we have

Corollary 3.3.4. *If* $-\otimes M : \mathcal{C} \longrightarrow \mathcal{C}$ *is left adjoint to* $-\otimes M^* : \mathcal{C} \longrightarrow \mathcal{C}$ *then* M^* *is a left dual for* M.

Corollary 3.3.5. (M^*, ev) *is a left dual for* M *if and only if there is a natural isomorphism*

$$\mathrm{Mor}_{\mathcal{C}}(M^* \otimes -, -) \cong \mathrm{Mor}_{\mathcal{C}}(-, M \otimes -),$$

i. e. the functor $M^* \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ *is left adjoint to the functor* $M \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$. *The natural isomorphism if given by*

$$(f : M^* \otimes N \longrightarrow P) \mapsto ((1_M \otimes f)(\mathrm{db} \otimes 1_N) : N \longrightarrow M \otimes M^* \otimes N \longrightarrow M \otimes P)$$

and

$$(g : N \longrightarrow M \otimes P) \mapsto ((\mathrm{ev} \otimes 1_P)(1_{M^*} \otimes g) : M^* \otimes N \longrightarrow M^* \otimes M \otimes P \longrightarrow P).$$

Proof. We have a natural isomorphism

$$\mathrm{Mor}_{\mathcal{C}}(M^* \otimes -, -) \cong \mathrm{Mor}_{\mathcal{C}}(-, M \otimes -),$$

iff (M, ev) is a right dual for M^* (as a symmetric statement to Lemma 3.3.3) iff (M^*, ev) is a left dual for M. $\square$

Corollary 3.3.6. *If* M *has a left dual then this is unique up to isomorphism.*

Proof. Let (M^*, ev) and $(M^!, \mathrm{ev}^!)$ be left duals for M. Then the functors $-\otimes M^*$ and $-\otimes M^!$ are isomorphic by Lemma A.9.7. In particular we have $M^* \cong I \otimes M^* \cong I \otimes M^! \cong M^!$. If we consider the construction of the isomorphism then we get in particular that $(\mathrm{ev}^! \otimes 1)(1 \otimes \mathrm{db}) : M^! \longrightarrow M^! \otimes M \otimes M^* \longrightarrow M^*$ is the given isomorphism. $\square$

Problem 3.3.4. Let (M^*, ev) be a left dual for M. Then there is a *unique* morphism $\mathrm{db} : I \longrightarrow M \otimes M^*$ satisfying the conditions of Definition 3.3.1.

Definition 3.3.7. Let (M^*, ev_M) and (N^*, ev_N) be left duals for M resp. N. For each morphism $f : M \longrightarrow N$ we define the *transposed morphism*

$$(f^* : N^* \longrightarrow M^*) := (N^* \xrightarrow{1 \otimes \mathrm{db}_M} N^* \otimes M \otimes M^* \xrightarrow{1 \otimes f \otimes 1} N^* \otimes N \otimes M^* \xrightarrow{\mathrm{ev}_N \otimes 1} M^*).$$

With this definition we get that the left dual is a contravariant functor, since we have

Lemma 3.3.8. *Let* (M^*, ev_M), (N^*, ev_N), *and* (P^*, ev_P) *be left duals for* M, N *and* P *respectively.*

1. We have $(1_M)^* = 1_{M^*}$.

2. If $f : M \longrightarrow N$ *and* $g : N \longrightarrow P$ *are given then* $(gf)^* = f^* g^*$ *holds.*

Proof. 1. $(1_M)^* = (\mathrm{ev} \otimes 1)(1 \otimes 1 \otimes 1)(1 \otimes \mathrm{db}) = 1_{M^*}$.

2. The following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "\mathrm{db}_N \otimes 1"] \arrow[d, "f"] & N \otimes N^* \otimes M \arrow[d, "1\otimes 1\otimes f"] & \\
N \arrow[r, "\mathrm{db}_N \otimes 1"] & N \otimes N^* \otimes N \arrow[r, "1\otimes \mathrm{ev}_N"] \arrow[d, "g\otimes 1\otimes 1"] & N \arrow[d, "g"] \\
 & P \otimes N^* \otimes N \arrow[r, "1\otimes \mathrm{ev}_N"] & P
\end{tikzcd}$$

Hence we have $gf = (1 \otimes \mathrm{ev}_N)(g \otimes 1 \otimes f)(\mathrm{db}_N \otimes 1)$. Thus the following diagram commutes

$$\begin{tikzcd}
P^* \arrow[r, "1\otimes \mathrm{db}"] \arrow[d, "1\otimes \mathrm{db}"] & P^* \otimes N \otimes N^* \arrow[r, "1\otimes g\otimes 1"] \arrow[d, "1\otimes \mathrm{db}"] & P^* \otimes P \otimes N^* \arrow[dd, "1\otimes \mathrm{db}"] \arrow[ddr, "\mathrm{ev} \otimes 1"] & \\
P^* \otimes M \otimes M^* \arrow[r, "1\otimes \mathrm{db} \otimes 1"] \arrow[dd, "1\otimes gf\otimes 1"] & P^* \otimes N \otimes N^* \otimes M \otimes M^* \arrow[dd, "1\otimes g\otimes 1\otimes f\otimes 1"] \arrow[dr, "1\otimes g\otimes 1"] & & \\
 & & P^* \otimes P \otimes N^* \otimes M \otimes M^* \arrow[dl, "1\otimes f\otimes 1"] \arrow[dd, "\mathrm{ev} \otimes 1"] & N^* \arrow[ddl, "1\otimes \mathrm{db}"] \\
P^* \otimes P \otimes M^* \arrow[d, "\mathrm{ev} \otimes 1"] & P^* \otimes P \otimes N^* \otimes N \otimes M^* \arrow[l, "1\otimes \mathrm{ev} \otimes 1"'] \arrow[d, "\mathrm{ev} \otimes 1"] & & \\
M^* & N^* \otimes N \otimes M^* \arrow[l, "1\otimes \mathrm{ev}"'] & N^* \otimes M \otimes M^* \arrow[l, "1\otimes f\otimes 1"'] &
\end{tikzcd}$$

$\square$

Problem 3.3.5. 1. In the category of $\mathbb{N}$-graded vector spaces determine all objects M that have a left dual.

2. In the category of chain complexes $\mathbb{K}$-**Comp** determine all objects M that have a left dual.

3. In the category of cochain complexes $\mathbb{K}$-**Cocomp** determine all objects M that have a left dual.

4. Let (M^*, ev) be a left dual for M. Show that $\mathrm{db} : I \longrightarrow M \otimes M^*$ is uniquely determined by M, M^*, and ev. (Uniqueness of the dual basis.)

5. Let (M^*, ev) be a left dual for M. Show that $\mathrm{ev} : M^* \otimes M \longrightarrow I$ is uniquely determined by M, M^*, and db.

Corollary 3.3.9. *Let M, N have the left duals (M^*, ev_M) and (N^*, ev_N) and let $f : M \longrightarrow N$ be a morphism in $\mathcal{C}$. Then the following diagram commutes*

$$\begin{tikzcd} I \arrow[r, "\mathrm{db}_M"] \arrow[d, "\mathrm{db}_N"'] & M \otimes M^* \arrow[d, "f\otimes 1"] \\ N \otimes N^* \arrow[r, "1\otimes f^*"'] & N \otimes M^*. \end{tikzcd}$$

Proof. The following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\mathrm{db}\otimes 1"] \arrow[d, "f"'] & N \otimes N^* \otimes M \arrow[d, "1\otimes 1\otimes f"] \\ N \arrow[r, "\mathrm{db}\otimes 1"] \arrow[dr, "1"'] & N \otimes N^* \otimes N \arrow[d, "1\otimes \mathrm{ev}"] \\ & N \end{tikzcd}$$

This implies $(f \otimes 1_{M^*}) \mathrm{db}_M = ((1_N \otimes \mathrm{ev}_N)(1_N \otimes 1_{N^*} \otimes f)(\mathrm{db}_N \otimes 1_M) \otimes 1_{M^*}) \mathrm{db}_M = (1_N \otimes \mathrm{ev}_N \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes f \otimes 1_{M^*})(\mathrm{db}_N \otimes 1_M \otimes 1_{M^*}) \mathrm{db}_M = (1_N \otimes \mathrm{ev}_N \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes f \otimes 1_{M^*})(1_N \otimes 1_{N^*} \otimes \mathrm{db}_M) \mathrm{db}_N = (1_N \otimes (\mathrm{ev}_N \otimes 1_{M^*})(1_{N^*} \otimes f \otimes 1_{M^*})(1_{N^*} \otimes \mathrm{db}_M)) \mathrm{db}_N = (1_N \otimes f^*) \mathrm{db}_N$. □

Corollary 3.3.10. *Let M, N have the left duals (M^*, ev_M) and (N^*, ev_N) and let $f : M \longrightarrow N$ be a morphism in $\mathcal{C}$. Then the following diagram commutes*

$$\begin{tikzcd} N^* \otimes M \arrow[r, "f^*\otimes 1"] \arrow[d, "1\otimes f"'] & M^* \otimes M \arrow[d, "\mathrm{ev}_M"] \\ N^* \otimes N \arrow[r, "\mathrm{ev}_N"'] & I. \end{tikzcd}$$

Proof. This statement follows immediately from the symmetry of the definition of a left dual. □

Example 3.3.11. Let $M \in {}_R\mathcal{M}_R$ be an R-R-bimodule. Then $\mathrm{Hom}_R(M., R.)$ is an R-R-bimodule by $(rfs)(x) = rf(sx)$. Furthermore we have the morphism $\mathrm{ev} : \mathrm{Hom}_R(M., R.) \otimes_R M \longrightarrow R$ defined by $\mathrm{ev}(f \otimes_R m) = f(m)$.

(Dual Basis Lemma:) A module $M \in \mathcal{M}_R$ is called *finitely generated and projective* if there are elements $m_1, \ldots, m_n \in M$ und $m^1, \ldots, m^n \in \mathrm{Hom}_R(M., R.)$ such that

$$\forall m \in M : \sum_{i=1}^{n} m_i m^i(m) = m.$$

Actually this is a consequence of the dual basis lemma. But this definition is equivalent to the usual definition.

Let $M \in {}_R\mathcal{M}_R$. M as a right R-module is finitely generated and projective iff M has a left dual. The left dual is isomorphic to $\mathrm{Hom}_R(M., R.)$.

If M_R is finitely generated projective then we use $\mathrm{db} : R \longrightarrow M \otimes_R \mathrm{Hom}_R(M., R.)$ with $\mathrm{db}(1) = \sum_{i=1}^n m_i \otimes_R m^i$. In fact we have $(1 \otimes_R \mathrm{ev})(\mathrm{db} \otimes_R 1)(m) = (1 \otimes_R \mathrm{ev})(\sum m_i \otimes_R m^i \otimes_R m) = \sum m_i m^i(m) = m$. We have furthermore $(\mathrm{ev} \otimes_R 1)(1 \otimes_R \mathrm{db})(f)(m) = (\mathrm{ev} \otimes_R 1)(\sum_{i=1}^n f \otimes_R m_i \otimes_R m^i)(m) = \sum f(m_i)m^i(m) = f(\sum m_i m^i(m))$ $= f(m)$ for all $m \in M$ hence $(\mathrm{ev} \otimes_R 1)(1 \otimes_R \mathrm{db})(f) = f$.

Conversely if M has a left dual M^* then $\mathrm{ev} : M^* \otimes_R M \longrightarrow R$ defines a homomorphism $\iota : M^* \longrightarrow \mathrm{Hom}_R(M., R.)$ in ${}_R\mathcal{M}_R$ by $\iota(m^*)(m) = \mathrm{ev}(m^* \otimes_R m)$. We define $\sum_{i=1}^n m_i \otimes m^i := \mathrm{db}(1) \in M \otimes M^*$, then $m = (1 \otimes \mathrm{ev})(\mathrm{db} \otimes 1)(m) = (1 \otimes \mathrm{ev})(\sum m_i \otimes m^i \otimes m) = \sum m_i \iota(m^i)(m)$ so that $m_1, \ldots, m_n \in M$ and $\iota(m^1), \ldots, \iota(m^n) \in \mathrm{Hom}_R(M., R.)$ form a dual basis for M, i. e. M is finitely generated and projective as an R-module. Thus M^* and $\mathrm{Hom}_R(M., R.)$ are isomorphic by the map ι.

Analogously $\mathrm{Hom}_R(.M, .R)$ is a right dual for M iff M is finitely generated and projective as a left R-module.

Problem 3.3.6. Find an example of an object M in a monoidal category $\mathcal{C}$ that has a left dual but no right dual.

Definition 3.3.12. Given objects M, N in $\mathcal{C}$. An object $[M, N]$ is called a *left inner Hom* of M and N if there is a natural isomorphism $\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, N) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, [M, N])$, i. e. if it represents the functor $\mathrm{Mor}_{\mathcal{C}}(\text{-} \otimes M, N)$.

If there is an isomorphism $\mathrm{Mor}_{\mathcal{C}}(P \otimes M, N) \cong \mathrm{Mor}_{\mathcal{C}}(P, [M, N])$ natural in the three variable M, N, P then the category $\mathcal{C}$ is called *monoidal and left closed.*

If there is an isomorphism $\mathrm{Mor}_{\mathcal{C}}(M \otimes P, N) \cong \mathrm{Mor}_{\mathcal{C}}(P, [M, N])$ natural in the three variable M, N, P then the category $\mathcal{C}$ is called *monoidal and right closed.*

If M has a left dual M^* in $\mathcal{C}$ then there are inner Homs $[M, \text{-}]$ defined by $[M, N] := N \otimes M^*$. In particular left rigid monoidal categories are left closed.

Example 3.3.13.
1. The category of finite dimensional vector spaces is a monoidal category where each object has a (left and right) dual. Hence it is (left and right) closed and rigid.
2. Let **Ban** be the category of (complex) Banach spaces where the morphisms satisfy $\| f(m) \| \leq \| m \|$ i. e. the maps are bounded by 1 or contracting. **Ban** is a monoidal category by the complete tensor product $M \widehat{\otimes} N$. In **Ban** exists an inner Hom functor $[M, N]$ that consists of the set of bounded linear maps from M to N made into a Banach space by an appropriate topology. Thus **Ban** is a monoidal closed category.
3. Let H be a Hopf algebra. The category H-**Mod** of left H-modules is a monoidal category (see Example 3.2.4 2.). Then $\mathrm{Hom}_{\mathbb{K}}(M, N)$ is an object in H-**Mod** by the multiplication

$$(hf)(m) := \sum h_{(1)} f(mS(h_{(2)})$$

as in Proposition 3.1.14.

$\mathrm{Hom}_{\mathbb{K}}(M,N)$ is an inner Hom functor in the monoidal category H-**Mod**. The isomorphism $\phi : \mathrm{Hom}_{\mathbb{K}}(P, \mathrm{Hom}_{\mathbb{K}}(M,N)) \cong \mathrm{Hom}_{\mathbb{K}}(P \otimes M, N)$ can be restricted to an isomorphism

$$\mathrm{Hom}_H(P, \mathrm{Hom}_{\mathbb{K}}(M,N)) \cong \mathrm{Hom}_H(P \otimes M, N),$$

because $\phi(f)(h(p \otimes m)) = \phi(f)(\sum h_{(1)}p \otimes h_{(2)}m) = \sum f(h_{(1)}p)(h_{(2)}m) = \sum (h_{(1)}(f(p)))(h_{(2)}m) = \sum h_{(1)}(f(p)(S(h_{(2)})h_{(3)}m)) = h(f(p)(m)) = h(\phi(f)(p \otimes m))$ and conversely $(h(f(p)))(m) = \sum h_{(1)}(f(p)(S(h_{(2)})m)) = \sum h_{(1)}(\phi(f)(p \otimes S(h_{(2)})m)) = \sum \phi(f)(h_{(1)}p \otimes h_{(2)}S(h_{(3)})m) = \phi(f)(hp \otimes m) = f(hp)(m)$. Thus H-**Mod** is left closed.

If $M \in H$-**Mod** is a finite dimensional vector space then the dual vector space $M^* := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ again is an H-module by $(hf)(m) := f(S(h)m)$. Furthermore M^* is a left dual for M with the morphisms

$$\mathrm{db} : \mathbb{K} \ni 1 \mapsto \sum_i m_i \otimes m^i \in M \otimes M^*$$

and

$$\mathrm{ev} : M^* \otimes M \ni f \otimes m \mapsto f(m) \in \mathbb{K}$$

where m_i and m^i are a dual basis of the vector space M. Clearly we have $(1 \otimes \mathrm{ev})(\mathrm{db} \otimes 1) = 1_M$ and $(\mathrm{ev} \otimes 1)(1 \otimes \mathrm{db}) = 1_{M^*}$ since M is a finite dimensional vector space. We have to show that db and ev are H-module homomorphisms. We have

$$\begin{aligned}&(h\,\mathrm{db}(1))(m) = (h(\textstyle\sum m_i \otimes m^i))(m) = (\sum h_{(1)}m_i \otimes h_{(2)}m^i)(m) = \\ &\textstyle\sum (h_{(1)}m_i)((h_{(2)}m^i)(m)) = \sum (h_{(1)}m_i)(m^i(S(h_{(2)})m)) = \\ &\textstyle\sum h_{(1)}S(h_{(2)})m = \varepsilon(h)m = \varepsilon(h)(\sum m_i \otimes m^i)(m) = \varepsilon(h)\,\mathrm{db}(1)(m) = \\ &\mathrm{db}(\varepsilon(h)1)(m) = \mathrm{db}(h1)(m),\end{aligned}$$

hence $h\,\mathrm{db}(1) = \mathrm{db}(h1)$. Furthermore we have

$$\begin{aligned}&h\,\mathrm{ev}(f \otimes m) = hf(m) = \textstyle\sum h_{(1)}f(S(h_{(2)})h_{(3)}m) = \sum (h_{(1)}f)(h_{(2)}m) = \\ &\textstyle\sum \mathrm{ev}(h_{(1)}f \otimes h_{(2)}m) = \mathrm{ev}(h(f \otimes m)).\end{aligned}$$

4. Let H be a Hopf algebra. Then the category of left H-comodules (see Example 3.2.4 3.) is a monoidal category. Let $M \in H$-**Comod** be a finite dimensional vector space. Let m_i be a basis for M and let the comultiplication of the comodule be $\delta(m_i) = \sum h_{ij} \otimes m_j$. Then we have $\Delta(h_{ik}) = \sum h_{ij} \otimes h_{jk}$. $M^* := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ becomes a left H-comodule $\delta(m^j) := \sum S(h_{ij}) \otimes m^i$. One verifies that M^* is a left dual for M.

Lemma 3.3.14. *Let $M \in \mathcal{C}$ be an object with left dual (M^*, ev). Then*
1. *$M \otimes M^*$ is an algebra with multiplication*

$$\nabla := 1_M \otimes \mathrm{ev} \otimes 1_{M^*} : M \otimes M^* \otimes M \otimes M^* \longrightarrow M \otimes M^*$$

and unit

$$u := \mathrm{db} : I \longrightarrow M \otimes M^*;$$

2. $M^* \otimes M$ *is a coalgebra with comultiplication*

$$\Delta := 1_{M^*} \otimes \mathrm{db} \otimes 1_M : M^* \otimes M \longrightarrow M^* \otimes M \otimes M^* \otimes M$$

and counit

$$\varepsilon := \mathrm{ev} : M^* \otimes M \longrightarrow I.$$

PROOF. 1. The associativity is given by $(\nabla \otimes 1)\nabla = (1_M \otimes \mathrm{ev} \otimes 1_{M^*} \otimes 1_M \otimes 1_{M^*})(1_M \otimes \mathrm{ev} \otimes 1_{M^*}) = 1_M \otimes \mathrm{ev} \otimes \mathrm{ev} \otimes 1_{M^*} = (1_M \otimes 1_{M^*} \otimes 1_M \otimes \mathrm{ev} \otimes 1_{M^*})(1_M \otimes \mathrm{ev} \otimes 1_{M^*}) = (1 \otimes \nabla)\nabla$. The axiom for the left unit is $\nabla(u \otimes 1) = (1_M \otimes \mathrm{ev} \otimes 1_{M^*})(\mathrm{db} \otimes 1_M \otimes 1_{M^*}) = 1_M \otimes 1_{M^*}$.

2. is dual to the statement for algebras. $\square$

Lemma 3.3.15. 1. *Let A be an algebra in $\mathcal{C}$ and left $M \in \mathcal{C}$ be a left rigid object with left dual (M^*, ev). There is a bijection between the set of morphisms $f : A \otimes M \longrightarrow M$ making M a left A-module and the set of algebra morphisms $\widetilde{f} : A \longrightarrow M \otimes M^*$.*
2. *Let C be a coalgebra in $\mathcal{C}$ and left $M \in \mathcal{C}$ be a left rigid object with left dual (M^*, ev). There is a bijection between the set of morphisms $f : M \longrightarrow M \otimes C$ making M a right C-comodule and the set of coalgebra morphisms $\widetilde{f} : M^* \otimes M \longrightarrow C$.*

PROOF. 1. By Lemma 3.3.14 the object $M \otimes M^*$ is an algebra. Given $f : A \otimes M \longrightarrow M$ such that M becomes an A-module. By Lemma 3.3.3 we associate $\widetilde{f} := (f \otimes 1)(1 \otimes \mathrm{db}) : A \longrightarrow A \otimes M \otimes M^* \longrightarrow M \otimes M^*$. The compatibility of $\widetilde{f}$ with the multiplication is given by the commutative diagram

$$\begin{tikzcd}
A \otimes A \arrow[rrrr, "\nabla"] \arrow[dr, "1\otimes 1\otimes \mathrm{db}"] \arrow[ddr, "1\otimes\widetilde{f}"'] \arrow[dddd, "\widetilde{f}\otimes\widetilde{f}"'] & & & & A \arrow[dl, "1\otimes \mathrm{db}"'] \arrow[dddd, "\widetilde{f}"] \\
& A \otimes A \otimes M \otimes M^* \arrow[rr, "\nabla\otimes 1\otimes 1"] \arrow[d, "1\otimes f\otimes 1"] & & A \otimes M \otimes M^* \arrow[d, "f\otimes 1"'] & \\
& A \otimes M \otimes M^* \arrow[rr, "f\otimes 1"] \arrow[d, "1\otimes \mathrm{db}\otimes 1\otimes 1"] \arrow[drr, "1"] \arrow[ddl, "\widetilde{f}\otimes 1\otimes 1"'] & & M \otimes M^* \arrow[ddr, "1"] & \\
& A \otimes M \otimes M^* \otimes M \otimes M^* \arrow[rr, "1\otimes 1\otimes \mathrm{ev}\otimes 1"'] \arrow[dl, "f\otimes 1\otimes 1"] & & A \otimes M \otimes M^* \arrow[dr, "f\otimes 1"'] & \\
M \otimes M^* \otimes M \otimes M^* \arrow[rrrr, "1\otimes \mathrm{ev}\otimes 1"'] & & & & M \otimes M^*
\end{tikzcd}$$

The unit axiom is given by

$$\begin{tikzcd}
I \arrow[r, "\mathrm{db}"] \arrow[d, "u"'] & M \otimes M^* \arrow[d, "u\otimes 1"'] \arrow[dr, "1\otimes 1"] & \\
A \arrow[r, "1\otimes \mathrm{db}"] & A \otimes M \otimes M^* \arrow[r, "f\otimes 1"] & M \otimes M^*
\end{tikzcd}$$

Conversely let $g : A \longrightarrow M \otimes M^*$ be an algebra homomorphism and consider $\widetilde{g} := (1 \otimes \mathrm{ev})(g \otimes 1) : A \otimes M \longrightarrow M \otimes M^* \otimes M \longrightarrow M$. Then M becomes a left A-module since

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[rrrr, "\nabla\otimes 1"] \arrow[dr, "1\otimes g\otimes 1"] \arrow[drr, "g\otimes g\otimes 1"] \arrow[ddd, "1\otimes\widetilde{g}"'] & & & & A \otimes M \arrow[dl, "g\otimes 1"'] \arrow[ddd, "\widetilde{g}"] \\
& A \otimes M \otimes M^* \otimes M \arrow[r, "g\otimes 1\otimes 1\otimes 1"'] \arrow[ddl, "1\otimes 1\otimes \mathrm{ev}"] & M \otimes M^* \otimes M \otimes M^* \otimes M \arrow[r, "1\otimes \mathrm{ev} \otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes 1\otimes \mathrm{ev}"] & M \otimes M^* \otimes M \arrow[ddr, "1\otimes \mathrm{ev}"'] & \\
& & M \otimes M^* \otimes M \arrow[drr, "1\otimes \mathrm{ev}"] & & \\
A \otimes M \arrow[urr, "g\otimes 1"] \arrow[rrrr, "\widetilde{g}"] & & & & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \arrow[d, "u\otimes 1"'] \arrow[dr, "\mathrm{db}\otimes 1"] \arrow[drr, "1"] & & \\
A \otimes M \arrow[r, "g\otimes 1"'] & M \otimes M^* \otimes M \arrow[r, "1\otimes \mathrm{ev}"'] & M
\end{tikzcd}$$

commute.

2. (M^*, ev) is a left dual for M in the category $\mathcal{C}$ if and only if (M^*, db) is the right dual for M in the dual category $\mathcal{C}^{op}$. So if we dualize the result of part 1. we have to change sides, hence 2. $\square$

4. Finite reconstruction

The endomorphism ring of a vector space enjoys the following universal property. It is a vector space itself and allows a homomorphism $\rho : \mathrm{End}(V) \otimes V \longrightarrow V$. It is universal with respect to this property, i. e. if Z is a vector space and $f : Z \otimes V \longrightarrow V$ is a homomorphism, then there is a unique homomorphism $g : Z \longrightarrow \mathrm{End}(V)$ such that

$$\begin{tikzcd} Z \otimes V \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ \mathrm{End}(V) \otimes V \arrow[r, "\rho"'] & V \end{tikzcd}$$

commutes.

The algebra structure of $\mathrm{End}(V)$ comes for free from this universal property.

If we replace the vector space V by a diagram of vector spaces $\omega : \mathcal{D} \longrightarrow \mathbf{Vec}$ we get a similar universal object $\mathrm{End}(\omega)$. Again the universal property induces a unique algebra structure on $\mathrm{End}(\omega)$.

Problem 3.4.7. 1. Let V be a vector space. Show that there is a universal vector space E and homomorphism $\rho : E \otimes V \longrightarrow V$ (such that for each vector space Z and each homomorphism $f : Z \otimes V \longrightarrow V$ there is a unique homomorphism $g : Z \longrightarrow E$ such that

$$\begin{tikzcd} Z \otimes V \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ E \otimes V \arrow[r, "\rho"'] & V \end{tikzcd}$$

commutes). We call E and $\rho : E \otimes V \longrightarrow V$ a *vector space acting universally on* V.

2. Let E and $\rho : E \otimes V \longrightarrow V$ be a vector space acting universally on V. Show that E uniquely has the structure of an algebra such that V becomes a left E-module.

3. Let $\omega : \mathcal{D} \longrightarrow \mathbf{Vec}$ be a diagram of vector spaces. Show that there is a universal vector space E and natural transformation $\rho : E \otimes \omega \longrightarrow \omega$ (such that for each vector space Z and each natural transformation $f : Z \otimes \omega \longrightarrow \omega$ there is a unique homomorphism $g : Z \longrightarrow E$ such that

$$\begin{tikzcd} Z \otimes \omega \arrow[d, "g\otimes 1"'] \arrow[dr, "f"] & \\ E \otimes \omega \arrow[r, "\rho"'] & \omega \end{tikzcd}$$

commutes). We call E and $\rho : E \otimes \omega \longrightarrow \omega$ a *vector space acting universally on* ω.

4. Let E and $\rho : E \otimes \omega \longrightarrow \omega$ be a vector space acting universally on ω. Show that E uniquely has the structure of an algebra such that ω becomes a diagram of left E-modules.

Similar considerations can be carried out for coactions $V \longrightarrow V \otimes C$ or $\omega \longrightarrow \omega \otimes C$ and a coalgebra structure on C. There is one restriction, however. We can only use finite dimensional vector spaces V or diagrams of finite dimensional vector spaces. This will be done further down.

As we have seen, ???

We want to find a universal natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$. For this purpose we consider the isomorphisms

$$\mathrm{Mor}_{\mathcal{C}}(\omega(X), \omega(X) \otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\omega(X)^* \otimes \omega(X), M)$$

that are given by $f \mapsto (\mathrm{ev} \otimes 1)(1 \otimes f)$ and as inverse $g \mapsto (1 \otimes g)(\mathrm{db} \otimes 1)$. We first develop techniques to describe the properties of a natural transformation $\phi : \omega \longrightarrow \omega \otimes M$ as properties of the associated family $g(X) : \omega(X)^* \otimes \omega(X) \longrightarrow M$. We will see that $g : \omega^* \otimes \omega \longrightarrow M$ will be a *cone*. Then we will show that ϕ is a universal natural transformation if and only if its associated cone is universal. In the literature this is called a coend.

Throughout this section assume the following. Let $\mathcal{D}$ be an arbitrary diagram scheme. Let $\mathcal{C}$ be a cocomplete monoidal category such that the tensor product preserves colimits in both arguments. Let $\mathcal{C}_0$ be the full subcategory of those objects in $\mathcal{C}$ that have a left dual. Let $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$ such that $\omega(X) \in \mathcal{C}_0$ for all $X \in \mathcal{D}$, i. e. ω is given by a functor $\omega_0 : \mathcal{D} \longrightarrow \mathcal{C}_0$. We call such a diagram a *finite diagram* in $\mathcal{C}$. Finally for an object $M \in \mathcal{C}$ let $\omega \otimes M : \mathcal{D} \longrightarrow \mathcal{C}$ be the functor with $(\omega \otimes M)(X) = \omega(X) \otimes M$.

Remark 3.4.1. Consider the following category $\widetilde{\mathcal{D}}$. For each morphism $f : X \longrightarrow Y$ there is an object $\widetilde{f} \in \widetilde{D}$. The object corresponding to the identity $1_X : X \longrightarrow X$ is denoted by $\widetilde{X} \in \widetilde{\mathcal{D}}$. For each morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ there are two morphisms $f_1 : \widetilde{f} \longrightarrow \widetilde{X}$ and $f_2 : \widetilde{f} \longrightarrow \widetilde{Y}$ in $\widetilde{\mathcal{D}}$. Furthermore there are the identities $1_f : \widetilde{f} \longrightarrow \widetilde{f}$ in $\widetilde{\mathcal{D}}$.

Since there are no morphisms with $\widetilde{X}$ as domain other than $(1_X)_i : \widetilde{X} \longrightarrow \widetilde{X}$ and $1_f : \widetilde{f} \longrightarrow \widetilde{f}$ we only have to define the following compositions $(1_X)_i \circ f_j := f_j$. Then $\widetilde{\mathcal{D}}$ becomes a category and we have $1_{\widetilde{X}} = (1_X)_1 = (1_X)_2$.

We define a diagram $\omega^* \otimes \omega : \widetilde{\mathcal{D}} \longrightarrow \mathcal{C}$ as follows. If $f : X \longrightarrow Y$ is given then

$$(\omega^* \otimes \omega)(\widetilde{f}) := \omega(Y)^* \otimes \omega(X)$$

and

$$\omega(f_1) := \omega(f)^* \otimes \omega(1_X),$$
$$\omega(f_2) := \omega(1_Y)^* \otimes \omega(f).$$

The colimit of $\omega^* \otimes \omega$ consists of an object $\operatorname{coend}(\omega) \in \mathcal{C}$ together with a family of morphisms $\iota(X,X) : \omega(X)^* \otimes \omega(X) \longrightarrow \operatorname{coend}(\omega)$ such that the diagrams

$$\begin{tikzcd}
 & \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"] & \\
(\omega^* \otimes \omega)(\widetilde{f}) = \omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes\omega(f)"'] & & \operatorname{coend}(\omega) \\
 & \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"'] &
\end{tikzcd}$$

commute for all $f : X \longrightarrow Y$ in $\mathcal{D}$. Indeed, such a family $\iota(\widetilde{X}) := \iota(X,X)$ can be uniquely extended to a natural transformation by defining $\iota(\widetilde{f}) := \iota(X,X)(\omega(f)^* \otimes \omega(X)) = \iota(Y,Y)(\omega(Y)^* \otimes \omega(f))$. In addition the pair $(\operatorname{coend}(\omega), \iota)$ is universal with respect to this property.

In the literature such a universal object is called a *coend* of the bifunctor $\omega^* \otimes \omega : \mathcal{D}^{op} \times \mathcal{D} \longrightarrow \mathcal{C}$.

Corollary 3.4.2. *The following is a coequalizer*

$$\coprod_{f \in \operatorname{Mor}\mathcal{D}} \omega(\operatorname{Zi}(f))^* \otimes \omega(\operatorname{Qu}(f)) \underset{q}{\overset{p}{\rightrightarrows}} \coprod_{X \in \operatorname{Ob}\mathcal{D}} \omega(X)^* \otimes \omega(X) \longrightarrow \operatorname{coend}(\omega)$$

Proof. This is just a reformulation of Remark A.10.11, since the colimit may also be built from the commutative squares given above.

Observe that for the construction of the colimit not all objects of the diagram have to be used but only those of the form $\omega(X)^* \otimes \omega(X)$. □

Theorem 3.4.3. *(***Tannaka-Krein***)*

Let $\omega : \mathcal{D} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ be a finite diagram. Then there exists an object $\operatorname{coend}(\omega) \in \mathcal{C}$ *and a natural transformation $\delta : \omega \longrightarrow \omega \otimes \operatorname{coend}(\omega)$ such that for each object $M \in \mathcal{C}$ and each natural transformation $\varphi : \omega \longrightarrow \omega \otimes M$ there exists a unique morphism $\widetilde{\varphi} : \operatorname{coend}(\omega) \longrightarrow M$ such that the diagram*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \operatorname{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\
 & \omega \otimes M
\end{tikzcd}$$

commutes.

Proof. Let $\operatorname{coend}(\omega) \in \mathcal{C}$ together with morphisms $\iota(\widetilde{f}) : \omega(Y)^* \otimes \omega(X) \longrightarrow \operatorname{coend}(\omega)$ be the colimit of the diagram $\omega^* \otimes \omega : \widetilde{\mathcal{D}} \longrightarrow \mathcal{C}$. So we get commutative diagrams

$$\begin{tikzcd}
 & \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"] & \\
\omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[rr, "\iota(\widetilde{f})"] \arrow[dr, "1\otimes\omega(f)"'] & & \mathrm{coend}(\omega) \\
 & \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"'] &
\end{tikzcd}$$

for each $f : X \longrightarrow Y$ in $\mathcal{C}$.

For $X \in \mathcal{C}$ we define a morphism $\delta(X) : \omega(X) \longrightarrow \omega(X) \otimes \mathrm{coend}(\omega)$ by $(1 \otimes \iota(X,X))(\mathrm{db} \otimes 1) : \omega(X) \longrightarrow \omega(X) \otimes \omega(X)^* \otimes \omega(X) \longrightarrow \omega(X) \otimes \mathrm{coend}(\omega)$. Then we get as in Corollary 3.3.5 $\quad \iota(X,X) = (1 \otimes \mathrm{ev})(1 \otimes \delta(X))$.

We show that δ is a natural transformation. For each $f : X \longrightarrow Y$ the square

$$\begin{tikzcd}
I \arrow[r, "\mathrm{db}_X"] \arrow[d, "\mathrm{db}_Y"'] & \omega(X) \otimes \omega(X)^* \arrow[d, "\omega(f)\otimes 1"] \\
\omega(Y) \otimes \omega(Y)^* \arrow[r, "1\otimes\omega(f)^*"'] & \omega(Y) \otimes \omega(X)^*.
\end{tikzcd}$$

commutes by Corollary 3.3.9. Thus the following diagram commutes

$$\begin{tikzcd}
\omega(X) \arrow[rr, "\mathrm{db}\otimes 1"] \arrow[dr, "\mathrm{db}\otimes 1"] \arrow[dd, "\omega(f)"'] & & \omega(X) \otimes \omega(X)^* \otimes \omega(X) \arrow[rr, "1\otimes\iota(X,X)"] \arrow[dr, "\omega(f)\otimes 1\otimes 1"] & & \omega(X) \otimes \mathrm{coend}(\omega) \arrow[dd, "\omega(f)\otimes 1"] \\
 & \omega(Y) \otimes \omega(Y)^* \otimes \omega(X) \arrow[rr, "1\otimes\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes 1\otimes\omega(f)"'] & & \omega(Y) \otimes \omega(X)^* \otimes \omega(X) \arrow[dr, "1\otimes\iota(X,X)"'] & \\
\omega(Y) \arrow[rr, "\mathrm{db}\otimes 1"] & & \omega(Y) \otimes \omega(Y)^* \otimes \omega(Y) \arrow[rr, "1\otimes\iota(Y,Y)"] & & \omega(Y) \otimes \mathrm{coend}(\omega).
\end{tikzcd}$$

Now let $M \in \mathcal{C}$ be an object and $\varphi : \omega \longrightarrow \omega \otimes M$ a natural transformation. Observe that

$$\begin{tikzcd}
\omega(Y)^* \otimes \omega(X) \arrow[r, "\omega(f)^*\otimes 1"] \arrow[d, "1\otimes\omega(f)"'] & \omega(X)^* \otimes \omega(X) \arrow[d, "\mathrm{ev}"] \\
\omega(Y)^* \otimes \omega(Y) \arrow[r, "\mathrm{ev}"'] & I
\end{tikzcd}$$

commutes by Corollary 3.3.10. Thus also the diagram

$$\begin{tikzcd}
\omega(X)^* \otimes \omega(X) \arrow[rr, "1\otimes\varphi(X)"] & & \omega(X)^* \otimes \omega(X) \otimes M \arrow[dr, "\mathrm{ev}\otimes 1"] & \\
\omega(Y)^* \otimes \omega(X) \arrow[u, "\omega(f)^*\otimes 1"'] \arrow[d, "1\otimes\omega(f)"'] \arrow[r, "1\otimes\varphi(X)"] & \omega(Y)^* \otimes \omega(X) \otimes M \arrow[ur, "\omega(f)^*\otimes 1\otimes 1"] \arrow[dr, "1\otimes\omega(f)\otimes 1"'] & & M \\
\omega(Y)^* \otimes \omega(Y) \arrow[rr, "1\otimes\varphi(Y)"] & & \omega(Y)^* \otimes \omega(Y) \otimes M \arrow[ur, "\mathrm{ev}\otimes 1"'] &
\end{tikzcd}$$

commutes. We define $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow M$ from the colimit property as universal factorization

$$\begin{tikzcd}
 & \omega(X)^* \otimes \omega(X) \arrow[dr, "\iota(X,X)"'] \arrow[drr, "(\mathrm{ev}\otimes 1)(1\otimes\varphi(X))"] & & \\
\omega(Y)^* \otimes \omega(X) \arrow[ur, "\omega(f)^*\otimes 1"] \arrow[dr, "1\otimes\omega(f)"'] & & \mathrm{coend}(\omega) \arrow[r, "\widetilde{\varphi}"] & M \\
 & \omega(Y)^* \otimes \omega(Y) \arrow[ur, "\iota(Y,Y)"] \arrow[urr, "(\mathrm{ev}\otimes 1)(1\otimes\varphi(Y))"'] & &
\end{tikzcd}$$

Hence the diagram

$$\begin{tikzcd}
\omega(X) \arrow[rr, "\delta(X)"] \arrow[dr, "\mathrm{db}\otimes 1"] \arrow[ddd, "\varphi(X)"'] & & \omega(X) \otimes \mathrm{coend}(\omega) \arrow[ddd, "1\otimes\widetilde{\varphi}"] \\
 & \omega(X) \otimes \omega(X)^* \otimes \omega(X) \arrow[ur, "1\otimes\iota(X,X)"'] \arrow[d, "1\otimes 1\otimes\varphi(X)"'] & \\
 & \omega(X) \otimes \omega(X)^* \otimes \omega(X) \otimes M \arrow[dr, "1\otimes\mathrm{ev}\otimes 1"] & \\
\omega(X) \otimes M \arrow[ur, "\mathrm{db}\otimes 1\otimes 1"] \arrow[rr, "1\otimes 1"'] & & \omega(X) \otimes M
\end{tikzcd}$$

commutes. The exterior portion of this diagram yields

$$\begin{tikzcd}
\omega(X) \arrow[r, "\delta(X)"] \arrow[dr, "\varphi(X)"'] & \omega(X) \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\
 & \omega(X) \otimes M.
\end{tikzcd}$$

It remains to show that $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow M$ is uniquely determined. Let $\widetilde{\varphi}_0 : \mathrm{coend}(\omega) \longrightarrow M$ be another morphism with $\varphi(X) = (1 \otimes \widetilde{\varphi}_0)\delta(X)$ for all $X \in \mathcal{D}$.

Then the following diagram commutes

$$\begin{tikzcd}
\omega(X)^* \otimes \omega(X) \arrow[rr, "\iota(X,X)"] \arrow[dr, "1\otimes\delta(X)"] \arrow[ddr, "1\otimes\varphi(X)"'] & & \mathrm{coend}(\omega) \arrow[dd, "\widetilde{\varphi}_0"] \\
& \omega(X)^* \otimes \omega(X) \otimes \mathrm{coend}(\omega) \arrow[ur, "\mathrm{ev}\otimes 1"] \arrow[d, "1\otimes 1\otimes\widetilde{\varphi}_0"] & \\
& \omega(X)^* \otimes \omega(X) \otimes M \arrow[r, "\mathrm{ev}\otimes 1"] & M,
\end{tikzcd}$$

hence we have $\widetilde{\varphi}_0 = \widetilde{\varphi}$. □

Corollary 3.4.4. *The functor* $\mathrm{Nat}(\omega, \omega \otimes M)$ *is a representable functor in* M *represented by* $\mathrm{coend}(\omega)$.

PROOF. The universal problem implies the isomorphism

$$\mathrm{Nat}(\omega, \omega \otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\mathrm{coend}(\omega), M)$$

and the universal natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$ is mapped to the identity under this isomorphism. □

It is also possible to construct an isomorphism

$$\mathrm{Nat}(\omega, \omega' \otimes M) \cong \mathrm{Mor}_{\mathcal{C}}(\mathrm{cohom}(\omega', \omega), M)$$

for different functors $\omega, \omega' : \mathcal{D} \longrightarrow \mathcal{C}$ and thus define *cohomomorphism objects*. Observe that only ω' has to take values in $\mathcal{C}_0$ since then we can build objects $\omega'(X)^* \otimes \omega(X)$.

5. The coalgebra coend

Proposition 3.5.1. *Let $\mathcal{C}$ be a monoidal category and $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$. Assume that there is a universal object* $\mathrm{coend}(\omega)$ *and natural transformation* $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$.

Then there is exactly one coalgebra structure on $\mathrm{coend}(\omega)$ *such that the diagrams*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\Delta"] \\
\omega \otimes \mathrm{coend}(\omega) \arrow[r, "\delta\otimes 1"'] & \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_\omega"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\epsilon"] \\
& \omega \otimes I
\end{tikzcd}$$

commute.

Proof. Because of the universal property of $\mathrm{coend}(\omega)$ there are structure morphisms $\Delta : \mathrm{coend}(\omega) \longrightarrow \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$ and $\epsilon : \mathrm{coend}(\omega) \longrightarrow I$. This implies the coalgebra property similar to the proof of Corollary 3.3.8. □

Observe that by this construction all objects and all morphisms of the diagram $\omega : \mathcal{D} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ are comodules or morphisms of comodules over the coalgebra $\mathrm{coend}(\omega)$. In fact $C := \mathrm{coend}(\omega)$ is the universal coalgebra over which the given diagram becomes a diagram of comodules.

Corollary 3.5.2. *Let $(\mathcal{D}, \omega)$ be a diagram $\mathcal{C}$ with objects in $\mathcal{C}_0$. Then all objects of the diagram are comodules over the coalgebra $C :=$* $\mathrm{coend}(\omega)$ *and all morphisms are morphisms of comodules. If D is another coalgebra and all objects of the diagram are D-comodules by $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ and all morphisms of the diagram are morphisms of D-comodules then there exists a unique morphism of coalgebras* $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow D$ *such that the diagram*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes\widetilde{\varphi}"] \\
& \omega \otimes D
\end{tikzcd}$$

commutes.

Proof. The morphisms $\varphi(X) : \omega(X) \longrightarrow \omega(X) \otimes D$ define a natural transformation since all morphisms of the diagram are morphisms of comodules. So the existence and the uniqueness of a morphism $\widetilde{\varphi} : \mathrm{coend}(\omega) \longrightarrow D$ is clear. The only thing to show is that this is a morphism of coalgebras. This follows from the universal property of $C = \mathrm{coend}(\omega)$ and the diagram

$$\begin{tikzcd}
\omega \arrow[rr, "\delta"] \arrow[dr, "\delta"] \arrow[dd, "1"'] & & \omega \otimes C \arrow[dr, "1\otimes\Delta"] \arrow[dd, "1\otimes\widetilde{\varphi}"' near start] & \\
& \omega \otimes C \arrow[rr, "\delta\otimes 1" near start] \arrow[dd, "1\otimes\widetilde{\varphi}"' near start] & & \omega \otimes C \otimes C \arrow[dd, "1\otimes\widetilde{\varphi}\otimes\widetilde{\varphi}"] \\
\omega \arrow[rr, "\varphi"' near start] \arrow[dr, "\varphi"'] & & \omega \otimes D \arrow[dr, "1\otimes\Delta"'] & \\
& \omega \otimes D \arrow[rr, "\varphi\otimes 1"'] & & \omega \otimes D \otimes D
\end{tikzcd}$$

where the right side of the cube commutes by the universal property. Similarly we get that $\widetilde{\varphi}$ preserves the counit since the following diagram commutes

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "1"'] \arrow[drr, "1" near start] & \omega \otimes C \arrow[d, "1\otimes\widetilde{\varphi}"'] \arrow[dr, "1\otimes\varepsilon"] & \\
\omega \arrow[r, "\varphi"] \arrow[drr, "1"'] & \omega \otimes D \arrow[dr, "1\otimes\varepsilon"] & \omega \otimes \mathbb{K} \arrow[d, "1"] \\
& & \omega \otimes \mathbb{K}
\end{tikzcd}$$

$\square$

6. The bialgebra coend

Let $\omega : \mathcal{D} \longrightarrow \mathcal{C}$ and $\omega' : \mathcal{D}' \longrightarrow \mathcal{C}$ be diagrams in $\mathcal{C}$. We call the diagram $(\mathcal{D}, \omega) \otimes (\mathcal{D}', \omega') := (\mathcal{D} \times \mathcal{D}', \omega \otimes \omega')$ with $(\omega \otimes \omega')(X, Y) := \omega(X) \otimes \omega'(Y)$ the *tensor product* of these two diagrams. The new diagram consists of all possible tensor products of all objects and all morphisms of the original diagrams.

From now on we assume that the category $\mathcal{C}$ is the category of vector spaces and we use the symmetry $\tau : V \otimes W \longrightarrow W \otimes V$ in **Vec**.

Proposition 3.6.1. *Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}', \omega')$ be finite diagrams in* **Vec**. *Then*

$$\mathrm{coend}(\omega \otimes \omega') \cong \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega').$$

PROOF. First observe the following. If two diagrams $\omega : \mathcal{D} \longrightarrow \mathbf{Vec}$ and $\omega' : \mathcal{D}' \longrightarrow \mathbf{Vec}$ are given then $\varinjlim_{\mathcal{D}} \varinjlim_{\mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega \otimes \omega') \cong \varinjlim_{\mathcal{D}} (\omega) \otimes \varinjlim_{\mathcal{D}'} (\omega')$ since the tensor product preserves colimits and colimits commute with colimits. For this consider the diagram

$$\begin{array}{ccc} \omega(X) \otimes \omega'(Y) & \longrightarrow & \omega(X) \otimes \varinjlim_{\mathcal{D}'}(\omega') \\ \downarrow & & \downarrow \\ \varinjlim_{\mathcal{D}}(\omega) \otimes \omega'(Y) & \longrightarrow & \varinjlim_{\mathcal{D}}(\omega) \otimes \varinjlim_{\mathcal{D}'}(\omega') \qquad \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'}(\omega \otimes \omega'). \end{array}$$

The maps in the diagram are the injections for the corresponding colimits. In particular we have $\mathrm{coend}(\omega \otimes \omega') \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} ((\omega \otimes \omega')^* \otimes (\omega \otimes \omega')) \cong \varinjlim_{\mathcal{D} \times \mathcal{D}'} (\omega^* \otimes \omega \otimes \omega'^* \otimes \omega') \cong \varinjlim_{\mathcal{D}} (\omega^* \otimes \omega) \otimes \varinjlim_{\mathcal{D}'} (\omega'^* \otimes \omega') \cong \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega')$.

The (universal) morphism

$$(\iota(X) \otimes \iota'(Y))(1 \otimes \tau \otimes 1) : \omega(X)^* \otimes \omega'(Y)^* \otimes \omega(X) \otimes \omega'(Y) \longrightarrow \varinjlim(\omega^* \otimes \omega) \otimes \varinjlim(\omega'^* \otimes \omega')$$

can be identified with the universal morphism

$$\iota(X, Y) : \omega(X)^* \otimes \omega'(Y)^* \otimes \omega(X) \otimes \omega'(Y) \longrightarrow \varinjlim((\omega \otimes \omega')^* \otimes (\omega \otimes \omega')).$$

Hence the induced morphisms

$$(1 \otimes \tau \otimes 1)(\delta \otimes \delta') : \omega(X) \otimes \omega'(Y) \longrightarrow \omega(X) \otimes \omega'(Y) \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega')$$

and

$$\delta : \omega(X) \otimes \omega'(Y) \longrightarrow \omega(X) \otimes \omega'(Y) \otimes \mathrm{coend}(\omega \otimes \omega')$$

can be identified. □

Corollary 3.6.2. *For all finite diagrams $(\mathcal{D}, \omega)$ and $(\mathcal{D}', \omega')$ in $\mathcal{D}$ there is a universal natural transformation $\delta : \omega \otimes \omega' \longrightarrow \omega \otimes \omega' \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega')$ so that for each object M and each natural transformation $\varphi : \omega \otimes \omega' \longrightarrow \omega \otimes \omega' \otimes M$ there exists* *a unique morphism* $\widetilde{\varphi} : \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega') \longrightarrow M$ *such that*

$$\begin{tikzcd}
\omega \otimes \omega' \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes \omega' \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega') \arrow[d, "1\otimes 1\otimes\widetilde{\varphi}"] \\
& \omega \otimes \omega' \otimes M
\end{tikzcd}$$

commutes.

Definition 3.6.3. Let $(\mathcal{D}, \omega)$ be a diagram in $\mathcal{C} = \mathbf{Vec}$. Then ω is called *reconstructive*

- if there is an object $\mathrm{coend}(\omega)$ in $\mathcal{C}$ and a universal natural transformation $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$
- and if $(1 \otimes \tau \otimes 1)(\delta \otimes \delta) : \omega \otimes \omega \longrightarrow \omega \otimes \omega \otimes \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$ is a univesarl natural transformation of bifunctors.

Definition 3.6.4. Let $(\mathcal{D}, \omega)$ be a diagram in $\mathbf{Vec}$. Let $\mathcal{D}$ be a monoidal category and ω be a monoidal functor. Then $(\mathcal{D}, \omega)$ is called a *monoidal diagram.*

Let $(\mathcal{D}, \omega)$ be a monoidal diagram $\mathbf{Vec}$. Let $A \in \mathbf{Vec}$ be an algebra. A natural transformation $\varphi : \omega \longrightarrow \omega \otimes B$ is called monoidal *monoidal* if the diagrams

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r, "\varphi(X)\otimes\varphi(Y)"] \arrow[d, "\rho"'] & \omega(X) \otimes \omega(Y) \otimes B \otimes B \arrow[d, "\rho\otimes m"] \\
\omega(X \otimes Y) \arrow[r, "\varphi(X\otimes Y)"] & \omega(X \otimes Y) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d] \\
\omega(I) \arrow[r, "\varphi(I)"] & \omega(I) \otimes B
\end{tikzcd}$$

commute.

We denote the set of monoidal natural transformations by $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$.

Problem 3.6.8. Show that $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$ is a functor in B.

Theorem 3.6.5. *Let* $(\mathcal{D}, \omega)$ *be a reconstructive, monoidal diagram in* $\mathbf{Vec}$. *Then* $\mathrm{coend}(\omega)$ *is a bialgebra and* $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$ *is a monoidal natural transformation.*

If B *is a bialgebra and* $\partial : \omega \longrightarrow \omega \otimes B$ *is a monoidal natural transformation, then there is a unique homomorphism of bialgebras* $f : coend(\omega) \longrightarrow B$ *such that the* *diagram*

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\partial"'] & \omega \otimes \mathrm{coend}(\omega) \arrow[d, "1\otimes f"] \\ & \omega \otimes B \end{tikzcd}$$

commutes.

PROOF. The multiplication of $\mathrm{coend}(\omega)$ arises from the following diagram

$$\begin{tikzcd} \omega(X)\otimes\omega(Y) \arrow[r, "\delta\otimes\delta"] \arrow[d] & \omega(X)\otimes\omega(Y)\otimes\mathrm{coend}(\omega)\otimes\mathrm{coend}(\omega) \arrow[d] \arrow[dr] & \\ \omega(X\otimes Y) \arrow[r, "\delta"] & \omega(X\otimes Y)\otimes\mathrm{coend}(\omega) \cong & \omega(X)\otimes\omega(Y)\otimes\mathrm{coend}(\omega) \end{tikzcd}$$

For the construction of the unit we consider the diagram $\mathcal{D}_0 = (\{I\}, \{\mathrm{id}\})$ together with $\omega_0 : \mathcal{D}_0 \longrightarrow \mathbf{Vec}$, $\omega_0(I) = \mathbb{K}$, the monoidal unit object in the monoidal category of diagrams in $\mathbf{Vec}$. Then $(\mathbb{K} \longrightarrow \mathbb{K}\otimes\mathbb{K}) = (\omega_0 \longrightarrow \omega_0 \otimes \mathrm{coend}(\omega_0))$ is the universal map. The following diagram then induced the unit for $\mathrm{coend}(\omega)$

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K}\otimes\mathbb{K} \arrow[d] \arrow[dr] & \\ \omega(I) \arrow[r] & \omega(I)\otimes\mathrm{coend}(\omega) \cong & \mathbb{K}\otimes\mathrm{coend}(\omega) \end{tikzcd}$$

By using the universal property one checks the laws for bialgebras.

The above diagrams show in particular that the natural transformation $\delta : \omega \longrightarrow \omega\otimes\mathrm{coend}(\omega)$ is monoidal. $\square$

7. The quantum monoid of a quantum space

Problem 3.7.9. If A is a finite dimensional algebra and $\delta : A \longrightarrow M(A) \otimes A$ the universal cooperation of the Tambara bialgebra on A from the left then $\tau\delta : A \longrightarrow A \otimes M(A)$ (with the same multiplication on $M(A)$) is a universal cooperation of $M(A)$ on A from the right. The comultiplication defined by this cooperation is $\tau\Delta : M(A) \longrightarrow M(A) \otimes M(A)$. Thus we have to distinguish between the left and the right Tambara bialgebra on A and we have $M_r(A) = M_l(A)^{cop}$.

Now consider the special monoidal diagram scheme $\mathcal{D} := \mathcal{D}[X; m, u]$. To make things simpler we assume that **Vec** is strict monoidal. The category $\mathcal{D}[X; m, u]$ has the objects $X \otimes \ldots \otimes X = X^{\otimes n}$ for all $n \in \mathbb{N}$ (and $I := X^{\otimes 0}$) and the morphisms $m : X \otimes X \longrightarrow X$, $u : I \longrightarrow X$ and all morphisms formally constructed from m, u, id by taking tensor products and composition of morphisms.

Let A be an algebra with multiplication $m_A : A \otimes A \longrightarrow A$ and unit $u_A : \mathbb{K} \longrightarrow A$. Then $\omega_A : \mathcal{D} \longrightarrow \mathcal{C}$ defined by $\omega(X) = A$, $\omega(X^{\otimes n}) = A^{\otimes n}$, $\omega(m) = m_A$ and $\omega(u) = u_A$ is a strict monoidal functor. If A is finite dimensional then the diagram is finite. We get

Theorem 3.7.1. *Let A be a finite dimensional algebra. Then the algebra $M(A)$ coacting universally from the right on A (the right Tambara bialgebra) $M(A)$ and* $\mathrm{coend}(\omega_A)$ *are isomorphic as bialgebras.*

Proof. We have studied the Tambara bialgebra for left coaction $f : A \longrightarrow M(A) \otimes A$ but here we need the analogue for universal right coaction $f : A \longrightarrow A \otimes M(A)$ (see Problem 3.9).

Let B be an algebra and $f : A \longrightarrow A \otimes B$ be a homomorphism of algebras. For $\omega = \omega_A$ we define

$$\varphi(X^{\otimes n}) : \omega(X^{\otimes n}) = A^{\otimes n} \xrightarrow{f^{\otimes n}} A^{\otimes n} \otimes B^{\otimes n} \xrightarrow{1 \otimes m_B^n} A^{\otimes n} \otimes B = \omega(X^{\otimes n}) \otimes B,$$

where $m_B^n : B^{\otimes n} \longrightarrow B$ is the n-fold multiplication on B. The map φ is a natural transformation since the diagrams

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\varphi(I)"] \arrow[d, "u"'] & \mathbb{K} \otimes B \arrow[d, "1 \otimes u"] \\ A \arrow[r, "\varphi(X)"'] & A \otimes B \end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\varphi(X \otimes X)"] \arrow[dr, "f \otimes f"] \arrow[ddd, "m"] & & A \otimes A \otimes B \arrow[ddd, "m \otimes 1"] \\
& A \otimes A \otimes B \otimes B \arrow[ur, "1 \otimes m"] \arrow[d, "m \otimes m"] & \\
& A \otimes B \arrow[dr, "1 \otimes 1"] & \\
A \arrow[ur, "f"] \arrow[rr, "\varphi(X)"'] & & A \otimes B
\end{tikzcd}$$

commute. Furthermore the following commute

$$\begin{tikzcd}
A^{\otimes r} \otimes A^{\otimes s} \arrow[rr, "\varphi(X^{\otimes r}) \otimes \varphi(X^{\otimes s})"] \arrow[dr] \arrow[ddd] & & A^{\otimes r} \otimes A^{\otimes s} \otimes B \otimes B \arrow[ddd] \\
& A^{\otimes r} \otimes A^{\otimes s} \otimes B^{\otimes r} \otimes B^{\otimes s} \arrow[ur] \arrow[d] & \\
& A^{\otimes (r+s)} \otimes B^{\otimes (r+s)} \arrow[dr] & \\
A^{\otimes (r+s)} \arrow[ur] \arrow[rr, "\varphi(X^{\otimes (r+s)})"'] & & A^{\otimes (r+s)} \otimes B
\end{tikzcd}$$

so that $\varphi : \omega_A \longrightarrow \omega_A \otimes B$ is a monoidal natural transformation.

Conversely let $\varphi : \omega_A \longrightarrow \omega_A \otimes B$ be a natural transformation. Let $f := \varphi(X) : A \longrightarrow A \otimes B$. Then the following commute

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "="'] & A \otimes A \otimes B \otimes B \arrow[d, "1 \otimes m"] \\
A \otimes A \arrow[r, "\varphi(X \otimes X)"] \arrow[d, "m"'] & A \otimes A \otimes B \arrow[d, "m \otimes 1"] \\
A \arrow[r, "f"] & A \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d, "="'] & \mathbb{K} \otimes \mathbb{K} \arrow[d] \\
\mathbb{K} \arrow[r] \arrow[d, "u"'] & \mathbb{K} \otimes B \arrow[d, "u \otimes 1"] \\
A \arrow[r, "f"] & A \otimes B.
\end{tikzcd}$$

Hence $f : A \longrightarrow A \otimes B$ is a homomorphism of algebras.

Thus we have defined an isomorphism

$$\mathbb{K}\mathbf{Alg}(A, A \otimes B) \cong \mathrm{Nat}^{\otimes}(\omega_A, \omega_A \otimes B)$$

that is natural in B. If A is finite dimensional then the left hand side is represented by the Tambara bialgebra $M_r(A)$ and the right hand side by the bialgebra $\mathrm{coend}(\omega_A)$. Thus both bialgebras must be isomorphic. $\square$

Corollary 3.7.2. *There is a unique isomorphism of bialgebras* $M_r(A) \cong \mathrm{coend}(\omega_A)$ *such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A \otimes M_r(A) \arrow[d] \\ & A \otimes \mathrm{coend}(\omega_A) \end{tikzcd}$$

commutes

PROOF. This is a direct consequence of the universal property. $\square$

Thus the Tambara bialgebra that represents the universal quantum monoid acting on a finite quantum space may be reconstructed by the Tannaka-Krein reconstruction from representation theory. Similar reconstructions can be given for more complicated quantum spaces such as so called quadratic quantum spaces.

8. Reconstruction and $\mathcal{C}$-categories

Now we show that an arbitrary coalgebra C can be reconstructed by the methods introduced above from its (co-)representations or more precisely from the underlying functor $\omega : \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$. In this case one can not use the usual construction of $\mathrm{coend}(\omega)$ that is restricted to finite dimensional comodules.

The following Theorem is an example that shows that the restriction to finite dimensional comodules in general is too strong for Tannaka reconstruction. There may be universal coendomorphism bialgebras for more general diagrams. On the other hand the following Theorem also holds if one only considers finite dimensional corepresentations of C. However the proof then becomes somewhat more complicated.

Definition 3.8.1. Let $\mathcal{C}$ be a monoidal category. A category $\mathcal{D}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D}$ and natural isomorphisms $\beta : (A \otimes B) \otimes M \longrightarrow A \otimes (B \otimes M)$, $\eta : I \otimes M \longrightarrow M$ is called a *$\mathcal{C}$-category* if the following diagrams commute

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes M \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\beta(A\otimes B,C,M)"] & (A \otimes (B \otimes C)) \otimes M \arrow[r, "\beta(A,B\otimes C,M)"] & A \otimes ((B \otimes C) \otimes M) \arrow[d, "1\otimes\beta(B,C,M)"'] \\
(A \otimes B) \otimes (C \otimes M) \arrow[rr, "\beta(A,B,C\otimes M)"] & & A \otimes (B \otimes (C \otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes M \arrow[rr, "\beta(A,I,M)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (I \otimes M) \arrow[dl, "1\otimes\eta(M)"] \\
& A \otimes M &
\end{tikzcd}$$

A $\mathcal{C}$-category is called *strict* if the morphisms β, η are the identities.

Let $(\mathcal{D}, \otimes)$ and $(\mathcal{D}', \otimes)$ be $\mathcal{C}$-categories. A functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ together with a natural transformation $\zeta(A, M) : A \otimes \mathcal{F}(M) \longrightarrow \mathcal{F}(A \otimes M)$ is called a *weak $\mathcal{C}$-functor* if the following diagrams commute

$$\begin{tikzcd}
(A \otimes B) \otimes \mathcal{F}(M) \arrow[rr, "\zeta"] \arrow[d, "\beta"'] & & \mathcal{F}((A \otimes B) \otimes M) \arrow[d, "\mathcal{F}(\beta)"] \\
A \otimes (B \otimes \mathcal{F}(M)) \arrow[r, "1\otimes\zeta"] & A \otimes \mathcal{F}(B \otimes M) \arrow[r, "\zeta"] & \mathcal{F}(A \otimes (B \otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes \mathcal{F}(M) \arrow[rr, "\zeta"] \arrow[dr, "\eta"'] & & \mathcal{F}(I \otimes M) \arrow[dl, "\mathcal{F}(\eta)"] \\
& \mathcal{F}(M) &
\end{tikzcd}$$

If, in addition, ζ is an isomorphism then we call $\mathcal{F}$ a $\mathcal{C}$-functor. The functor is called a *strict $\mathcal{C}$-functor* if ζ is the identity morphism.

A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{F}'$ between (weak) $\mathcal{C}$-functors is called a $\mathcal{C}$-*transformation* if

$$\begin{tikzcd} A \otimes \mathcal{F}(M) \arrow[r, "\zeta"] \arrow[d, "1_A \otimes \varphi(M)"'] & \mathcal{F}(A \otimes M) \arrow[d, "\varphi(A \otimes M)"] \\ A \otimes \mathcal{F}'(M) \arrow[r, "\zeta'"'] & \mathcal{F}'(A \otimes M) \end{tikzcd}$$

commutes.

Example 3.8.2. Let C be a coalgebra and $\mathcal{C} := \mathbf{Vec}$. Then the category **Comod**-C of right C-comodules is a $\mathcal{C}$-category since $N \in \mathbf{Comod}\text{-}C$ and $V \in \mathcal{C} = \mathbf{Vec}$ implies that $V \otimes N$ is a comodules with the comodule structure of N.

The underlying functor $\omega : \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ is a strict $\mathcal{C}$-functor since we have $V \otimes \omega(N) = \omega(V \otimes N)$. Similarly $\omega \otimes M : \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ is a $\mathcal{C}$-functor since $V \otimes (\omega(N) \otimes M) \cong \omega(V \otimes N) \otimes M$.

Lemma 3.8.3. *Let C be a coalgebra. Let $\omega : \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ be the underlying functor. Let $\varphi : \omega \longrightarrow \omega \otimes M$ be a natural transformation. Then φ is a $\mathcal{C}$-transformation with $\mathcal{C} = \mathbf{Vec}$.*

PROOF. It suffices to show $1_V \otimes \varphi(N) = \varphi(V \otimes N)$ for an arbitrary comodule N. We show that the diagram

$$\begin{tikzcd} V \otimes N \arrow[r, "\varphi(V \otimes N)"] \arrow[d, "1"'] & V \otimes N \otimes M \arrow[d, "1"] \\ V \otimes N \arrow[r, "1_V \otimes \varphi(N)"'] & V \otimes N \otimes M \end{tikzcd}$$

commutes. Let (v_i) be a basis of V. For an arbitrary vector space W let $p_i : V \otimes W \longrightarrow W$ be the projections defined by $p_i(t) = p_i(\sum_j v_j \otimes w_j) = w_i$ where $\sum_j v_j \otimes w_j$ is the unique representation of an arbitrary tensor in $V \otimes W$. So we get

$$t = \sum_i v_i \otimes p_i(t)$$

for all $t \in V \otimes W$. Now we consider $V \otimes N$ as a comodule by the comodule structure of N. Then the $p_i : V \otimes N \longrightarrow N$ are homomorphisms of comodules. Hence all diagrams of the form

$$\begin{tikzcd} V \otimes N \arrow[r, "\varphi(V \otimes N)"] \arrow[d, "p_i"'] & V \otimes N \otimes M \arrow[d, "p_i \otimes M"] \\ N \arrow[r, "\varphi(N)"'] & N \otimes M. \end{tikzcd}$$

commute. Expressed in formulas this means $\varphi(N)p_i(t) = p_i\varphi(V \otimes N)(t)$ for all $t \in V \otimes N$. Hence we have

$$\begin{aligned}(1_V \otimes \varphi(N))(t) &= (1_V \otimes \varphi(N))(\sum v_i \otimes p_i(t)) = \sum v_i \otimes \varphi(N)p_i(t) \\ &= \sum_i v_i \otimes p_i\varphi(V \otimes N)(t) = \varphi(V \otimes N)(t)\end{aligned}$$

So we have $1_V \otimes \varphi(N) = \varphi(V \otimes N)$ as claimed. $\square$

We prove the following Theorem only for the category $\mathcal{C} = \mathbf{Vec}$ of vector spaces. The Theorem holds in general and says that in an arbitrary symmetric monoidal category $\mathcal{C}$ the coalgebra C represents the functor $\mathcal{C}\text{-}\operatorname{Nat}(\omega, \omega \otimes M) \cong \operatorname{Mor}_{\mathcal{C}}(C, M)$ of natural $\mathcal{C}$-transformations.

Theorem 3.8.4. (Reconstruction of coalgebras) *Let C be a coalgebra. Let ω : $\mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ be the underlying functor. Then $C \cong \operatorname{coend}(\omega)$.*

Proof. Let M in $\mathbf{Vec}$ and let $\varphi : \omega \longrightarrow \omega \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C \longrightarrow M$ by $\widetilde{\varphi} = (\epsilon \otimes 1)\varphi(C)$ using the fact that C is a comodule.

Let N be a C-comodule. Then N is a subcomodule of $N \otimes C$ by $\delta : N \longrightarrow N \otimes C$ since the diagram

$$\begin{tikzcd} N \arrow[r, "\delta"] \arrow[d, "\delta"'] & N \otimes C \arrow[d, "1\otimes\Delta"] \\ N \otimes C \arrow[r, "\delta\otimes 1"'] & N \otimes C \otimes C \end{tikzcd}$$

commutes. Thus the following diagram commutes

$$\begin{tikzcd} N \arrow[r, "\delta"] \arrow[d, "\varphi(N)"'] & N \otimes C \arrow[d, "\varphi(N\otimes C)=1_N\otimes\varphi(C)"'] \arrow[dr, "1\otimes\widetilde{\varphi}"] & \\ N \otimes M \arrow[r, "\delta\otimes 1"] \arrow[rr, bend right, "1"] & N \otimes C \otimes M \arrow[r, "1\otimes\epsilon\otimes 1"] & N \otimes M \end{tikzcd}$$

In particular we have shown that the diagram

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes C \arrow[d, "1\otimes\widetilde{\varphi}"] \\ & \omega \otimes M \end{tikzcd}$$

commutes.

To show the uniqueness of $\widetilde{\varphi}$ let $g : C \longrightarrow M$ be another homomorphism with $(1 \otimes g)\delta = \varphi$. For $c \in C$ we have $g(c) = g(\epsilon \otimes 1)\Delta(c) = (\epsilon \otimes 1)(1 \otimes g)\Delta(c) = (\epsilon \otimes 1)\varphi(C)(c) = \widetilde{\varphi}(c)$.

The coalgebra structure from Corollary 3.5.1 is the original coalgebra structure of C. This can be seen as follows. The comultiplication $\delta : \omega \longrightarrow \omega \otimes C$ is a natural transformation hence $(\delta \otimes 1_C)\delta : \omega \longrightarrow \omega \otimes C \otimes C$ is also a natural transformation. As in Corollary 3.5.1 this induced a unique homomorphism $\Delta : C \longrightarrow C \otimes C$ so that the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes \operatorname{coend}(\omega) \arrow[d, "1\otimes\Delta"] \\
\omega \otimes \operatorname{coend}(\omega) \arrow[r, "\delta\otimes 1"'] & \omega \otimes \operatorname{coend}(\omega) \otimes \operatorname{coend}(\omega)
\end{tikzcd}$$

commutes. In a similar way the natural isomorphism $\omega \cong \omega \otimes \mathbb{K}$ induces a unique homomorphism $\epsilon : C \longrightarrow \mathbb{K}$ so that the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_\omega"'] & \omega \otimes \operatorname{coend}(\omega) \arrow[d, "1\otimes\epsilon"] \\
& \omega \otimes I
\end{tikzcd}$$

commutes. Because of the uniqueness these must be the structure homomorphisms of C. $\square$

We need a more general version of this Theorem in the next chapter. So let C be a coalgebra. Let $\omega : \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ be the underlying functor and $\delta : \omega \longrightarrow \omega \otimes C$ the universal natural transformation for $C \cong \operatorname{coend}(\omega)$.

We use the permutation map τ on the tensor product that gives the natural isomorphism

$$\tau : N_1 \otimes T_1 \otimes N_2 \otimes T_2 \otimes \ldots \otimes N_n \otimes T_n \cong N_1 \otimes N_2 \otimes \ldots \otimes N_n \otimes T_1 \otimes T_2 \ldots \otimes \otimes T_n$$

which is uniquely determined by the coherence theorems and is constructed by suitable applications of the flip $\tau : N \otimes T \cong T \otimes N$.

Let $\omega^n : \mathbf{Comod}\text{-}C \times \mathbf{Comod}\text{-}C \times \ldots \times \mathbf{Comod}\text{-}C \longrightarrow \mathbf{Vec}$ be the functor $\omega^n(N_1, N_2, \ldots, N_n) = \omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n)$. For notational convenience we abbreviate $\{N\}^n := N_1 \otimes N_2 \otimes \ldots \otimes N_n$, similarly $\{C\}^n = C \otimes C \otimes \ldots \otimes C$ and $\{f\}^n := f_1 \otimes f_2 \otimes \ldots \otimes f_n$. So we get $\tau : \{N \otimes T\}^n \cong \{N\}^n \otimes \{T\}^n$.

Lemma 3.8.5. *Let $\varphi : \omega^n \longrightarrow \omega^n \otimes M$ be a natural transformation. Then φ is a $\mathcal{C}$-transformation in the sense that the diagrams*

$$\begin{tikzcd}
\{V \otimes N\}^n \arrow[r, "\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)"] \arrow[d, "\tau"'] & \{V \otimes N\}^n \otimes M \arrow[d, "\tau \otimes M"] \\
\{V\}^n \otimes \{N\}^n \arrow[r, "\{V\}^n \otimes \varphi(N_1, \ldots, N_n)"'] & \{V\}^n \otimes \{N\}^n \otimes M
\end{tikzcd}$$

commute for all vector spaces V_i and C-comodules N_i.

Proof. Choose bases $\{v_{ij}\}$ of the vector spaces V_i with corresponding projections $p_{ij} : V_i \otimes N_i \longrightarrow N_i$. Then we have $\tau(t_1 \otimes \ldots \otimes t_n) = \sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes p_{1i_1}(t_1) \otimes \ldots \otimes p_{ni_n}(t_n)$ so $\tau = \sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes \{p\}^n$.

The $p_{ij_i} : V_i \otimes N_i \longrightarrow N_i$ are homomorphisms of C-comodules. Hence the diagrams

$$\begin{tikzcd}
\{V \otimes N\}^n \arrow[r, "\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)"] \arrow[d, "\{p\}^n"'] & \{V \otimes N\}^n \otimes M \arrow[d, "\{p\}^n \otimes M"] \\
\{N\}^n \arrow[r, "\varphi(N_1, \ldots, N_n)"'] & \{N\}^n \otimes M
\end{tikzcd}$$

commute for all choices of $\{p\}^n = p_{1i_1} \otimes \ldots \otimes p_{ni_n}$.

So we get for all $t_i \in V_i \otimes N_i$

$$\begin{aligned}
&(\{V\}^n \otimes \varphi(N_1, \ldots, N_n))\tau(t_1 \otimes \ldots \otimes t_n) = \\
&= (\{V\}^n \otimes \varphi(N_1, \ldots, N_n))(\textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes p_{1i_1}(t_1) \otimes \ldots \otimes p_{ni_n}(t_n)) \\
&= \textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes \varphi(N_1, \ldots, N_n)\{p\}^n(t_1 \otimes \ldots \otimes t_n) \\
&= \textstyle\sum v_{1i_1} \otimes \ldots \otimes v_{ni_n} \otimes (\{p\}^n \otimes M)\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)(t_1 \otimes \ldots \otimes t_n) \\
&= (\tau \otimes M)\varphi(V_1 \otimes N_1, \ldots, V_n \otimes N_n)(t_1 \otimes \ldots \otimes t_n).
\end{aligned}$$

$\square$

Theorem 3.8.6. *With the notation given above we have*

$$\operatorname{coend}(\omega^n) \cong C \otimes C \otimes \ldots \otimes C$$

with the universal natural transformation

$$\begin{gathered}
\delta^{(n)}(N_1, N_2, \ldots, N_n) := \tau(\delta(N_1) \otimes \delta(N_2) \otimes \ldots \otimes \delta(N_n)) : \\
\omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n) \longrightarrow \omega(N_1) \otimes C \otimes \omega(N_2) \otimes C \otimes \ldots \otimes \omega(N_n) \otimes C \\
\cong \omega(N_1) \otimes \omega(N_2) \otimes \ldots \otimes \omega(N_n) \otimes C \otimes C \otimes \ldots \otimes C.
\end{gathered}$$

Proof. We proceed as in the proof of the previous Theorem.

Let M in **Vec** and let $\varphi : \omega^n \longrightarrow \omega^n \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C^n = \omega(C) \otimes \omega(C) \otimes \ldots \otimes \omega(C) = C \otimes C \otimes \ldots \otimes C \longrightarrow M$ by $\widetilde{\varphi} = (\varepsilon^n \otimes 1_M)\varphi(C, \ldots, C)$ using the fact that C is a comodule.

As in the preceding proof we get that $\delta : N_i \longrightarrow N_i \otimes C$ are homomorphisms of C-comodules. Thus the following diagram commutes

$$\begin{tikzcd}
N_1 \otimes \ldots \otimes N_n \arrow[r, "\delta\otimes\ldots\otimes\delta"] \arrow[d, "\varphi(N_1\otimes\ldots\otimes N_n)"'] & N_1 \otimes C \otimes \ldots \otimes N_n \otimes C \arrow[r, "\tau"] \arrow[d, "\varphi(N_1\otimes C,\ldots,N_n\otimes C)"'] & N_1 \otimes \ldots \otimes N_n \otimes C \otimes \ldots \otimes C \arrow[d, "N_1\otimes\ldots\otimes N_n\otimes\varphi(C,\ldots,C)"'] \\
N_1 \otimes \ldots \otimes N_n \otimes M \arrow[r, "\delta\otimes\ldots\otimes\delta\otimes M"] \arrow[drr, "1"] & N_1 \otimes C \otimes \ldots \otimes N_n \otimes C \otimes M \arrow[r, "\tau\otimes M"] \arrow[dr, "\{1\otimes\varepsilon\}^n\otimes 1"] & N_1 \otimes \ldots \otimes N_n \otimes C \otimes \ldots \otimes C \otimes M \arrow[d, "1\otimes\{\epsilon\}^n\otimes 1"] \\
& & N_1 \otimes \ldots \otimes N_n \otimes M
\end{tikzcd}$$

Hence we get the commutative diagram

$$\begin{tikzcd}
\omega^n \arrow[r, "\delta^{(n)}"] \arrow[dr, "\varphi"'] & \omega^n \otimes \{C\}^n \arrow[d, "1\otimes\widetilde{\varphi}"] \\
& \omega^n \otimes M
\end{tikzcd}$$

To show the uniqueness of $\widetilde{\varphi}$ let $g : C^n \longrightarrow M$ be another homomorphism with $(1_{\omega^n} \otimes g)\delta^{(n)} = \varphi$. We have $g = g(\varepsilon^n \otimes 1_{C^n})\tau\Delta^n = g(\varepsilon^n \otimes 1_{C^n})\delta^{(n)}(C, \ldots, C) = (\varepsilon^n \otimes 1_M)(1_{C^n} \otimes g)\delta^{(n)}(C, \ldots, C) = (\varepsilon^n \otimes 1_M)\varphi(C, \ldots, C) = \widetilde{\varphi}$. $\square$

Now we prove the finite dimensional case of reconstruction of coalgebras.

Proposition 3.8.7. (Reconstruction) *Let C be a coalgebra. Let* **Comod**$_0$-C *be the category of finite dimensional C-comodules and ω :* **Comod**$_0$-$C \longrightarrow$ **Vec** *be the underlying functor. Then we have $C \cong \mathrm{coend}(\omega)$.*

Proof. Let M be in **Vec** and let $\varphi : \omega \longrightarrow \omega \otimes M$ be a natural transformation. We define the homomorphism $\widetilde{\varphi} : C \longrightarrow M$ as follows. Let $c \in C$. Let N be a finite dimensional C-subcomodule of C containing c. Then we define $g(c) := (\epsilon|_N \otimes 1)\varphi(N)(c)$. If N' is another finite dimensional subcomodule of C with $c \in N'$ and with $N \subseteq N'$ then the following commutes

$$\begin{tikzcd}
N \arrow[r, "\varphi(N)"] \arrow[d] & N \otimes M \arrow[d] \arrow[dr] & & \\
N' \arrow[r, "\varphi(N')"] & N' \otimes M \arrow[r] & C \otimes M \arrow[r, "\epsilon\otimes 1"] & M
\end{tikzcd}$$

Thus the definition of $\widetilde{\varphi}(c)$ is independent of the choice of N. Furthermore $\widetilde{\varphi} : N \longrightarrow M$ is obviously a linear map. For any two elements $c, c' \in C$ there is a finite dimensional subcomodule $N \subseteq C$ with $c, c' \in N$ e. g. the sum of the finite dimensional subcomodules containing c and c' separately. Thus $\widetilde{\varphi}$ can be extended to all of C.

The rest of the proof is essentially the same as the proof of the first reconstruction theorem. □

The representations allow to reconstruct further structure of the coalgebra. We prove a reconstruction theorem about bialgebras. Recall that the category of B-comodules over a bialgebra B is a monoidal category, furthermore that the underlying functor $\omega : \mathbf{Comod}\text{-}B \longrightarrow \mathbf{Vec}$ is a monoidal functor. From this information we can reconstruct the full bialgebra structure of B. We have

Theorem 3.8.8. *Let B be a coalgebra. Let* $\mathbf{Comod}\text{-}B$ *be a monoidal category such that the underlying functor* $\omega : \mathbf{Comod}\text{-}B \longrightarrow \mathbf{Vec}$ *is a monoidal functor. Then there is a unique bialgebra structure on B that induces the given monoidal structure on the corepresentations.*

Proof. First we prove the uniqueness of the multiplication $\nabla : B \otimes B \longrightarrow B$ and of the unit $\eta : \mathbb{K} \longrightarrow B$. The natural transformation $\delta : \omega \longrightarrow \omega \otimes B$ becomes a monoidal natural transformation with $\nabla : B \otimes B \longrightarrow B$ and $\eta : \mathbb{K} \longrightarrow B$ We show that ∇ and η are uniquely determined by ω and δ.

Let $\nabla' : B \otimes B \longrightarrow B$ and $\eta' : B \longrightarrow \mathbb{K}$ be morphisms that make δ a monoidal natural transformation. The diagrams

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r, "\delta(X)\otimes\delta(Y)"] \arrow[d, "\rho"'] & \omega(X) \otimes \omega(Y) \otimes B \otimes B \arrow[d, "\rho\otimes\nabla'"] \\
\omega(X \otimes Y) \arrow[r, "\delta(X\otimes Y)"] & \omega(X \otimes Y) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d, "1 \otimes \eta'"] \\
\omega(\mathbb{K}) \arrow[r, "\delta(\mathbb{K})"] & \omega(\mathbb{K}) \otimes B
\end{tikzcd}$$

commute. In particular the following diagrams commute

$$\begin{tikzcd}
\omega(B) \otimes \omega(B) \arrow[r, "\delta(B)\otimes\delta(B)"] \arrow[d, "\rho"'] & \omega(B) \otimes \omega(B) \otimes B \otimes B \arrow[d, "\rho\otimes\nabla'"] \\
\omega(B \otimes B) \arrow[r, "\delta(B\otimes B)"] & \omega(B \otimes B) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d, "1 \otimes \eta'"] \\
\omega(\mathbb{K}) \arrow[r, "\delta(\mathbb{K})"] & \omega(\mathbb{K}) \otimes B
\end{tikzcd}$$

Hence we get $\sum b_{(1)} \otimes c_{(1)} \otimes b_{(2)} c_{(2)} = \sum b_{(1)} \otimes c_{(1)} \otimes \nabla'(b_{(2)} \otimes c_{(2)})$ and $1 \otimes 1 = 1 \otimes \eta'(1)$. This implies $bc = \sum \epsilon(b_{(1)})\epsilon(c_{(1)}) b_{(2)} c_{(2)} = \sum \epsilon(b_{(1)})\epsilon(c_{(1)})\nabla'(b_{(2)} \otimes c_{(2)}) = \nabla'(b \otimes c)$ and $1 = \eta'(1)$.

Now we show the existence of a bialgebra structure. Let B be a coalgebra only and let $\omega : \mathbf{Comod}\text{-}B \longrightarrow \mathbf{Vec}$ be a monoidal functor with $\xi : \omega(M) \otimes \omega(N) \longrightarrow \omega(M \otimes N)$ and $\xi_0 : \mathbb{K} \longrightarrow \omega(\mathbb{K})$. First we observe that the new tensor product between the comodules M and N coincides with the tensor product of the underlying vector spaces (up to an isomorphism ξ). Because of the coherence theorems for monoidal categories (that also hold in our situation) we may identify along the maps ξ and ξ_0.

We define $\eta := (\mathbb{K} \stackrel{\delta(\mathbb{K})}{\longrightarrow} \mathbb{K} \otimes B \cong B)$ and $\nabla := (B \otimes B \stackrel{\delta(B \otimes B)}{\longrightarrow} B \otimes B \otimes B \stackrel{\epsilon \otimes \epsilon \otimes 1_B}{\longrightarrow} \mathbb{K} \otimes \mathbb{K} \otimes B \cong B)$.

Since the structural morphism for the comodule $\delta : M \longrightarrow M \otimes B$ is a homomorphism of of B comodules where the comodule structure on $M \otimes B$ is only given by the diagonal of B that is the $\mathcal{C}$-structure on $\omega : \mathbf{Comod}\text{-}B \longrightarrow \mathbf{Vec}$ we get that also $\delta(M) \otimes \delta(N) : M \otimes N \longrightarrow M \otimes N \otimes B$ is a comodule homomorphism. Hence the first square in the following diagram commutes

$$\begin{tikzcd}
M \otimes N \arrow[r, "\delta(M)\otimes\delta(N)"] \arrow[d, "\delta(M\otimes N)"'] & M \otimes B \otimes N \otimes B \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\delta(M\otimes B\otimes N\otimes B)"'] & M \otimes N \otimes B \otimes B \arrow[d, "1\otimes 1\otimes\delta(B\otimes B)"'] \\
M \otimes N \otimes B \arrow[r, "\delta(M)\otimes\delta(N)\otimes 1_B"] & M \otimes B \otimes N \otimes B \otimes B \arrow[r, "1\otimes\tau\otimes 1\otimes 1"] & M \otimes N \otimes B \otimes B \otimes B
\end{tikzcd}$$

The second square commutes by a similar reasoning since the comodule structure on $M \otimes B$ resp. $N \otimes B$ is given by the diagonal on B hence $M \otimes N$ can be factored out of the natural ($\mathcal{C}$-)transformation. Now we attach

$$1_M \otimes 1_N \otimes \epsilon \otimes \epsilon \otimes 1_B : M \otimes N \otimes B \otimes B \otimes B \longrightarrow M \otimes N \otimes B$$

to the commutative rectangle and obtain $\delta(M \otimes N) = (1_M \otimes 1_N \otimes \nabla)(1 \otimes \tau \otimes 1)(\delta(M) \otimes \delta(N)$. Thus the comodule structure on $M \otimes N$ is induced by the multiplication $\nabla : B \otimes B \longrightarrow B$ defined above.

So the following diagrams commute

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\Delta\otimes\Delta"] \arrow[d, "1_B\otimes 1_B"] & & B \otimes B \otimes B \otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & B \otimes B \otimes B \otimes B \arrow[d, "1\otimes\nabla"] \\
B \otimes B \arrow[rrrr, "\delta(B\otimes B)"] \arrow[dddd, "\nabla"'] \arrow[dr, "\delta(B\otimes B)"] & & & & B \otimes B \otimes B \arrow[dl, "\delta(B\otimes B)\otimes 1"'] \arrow[dddd, "\nabla\otimes 1"] \\
& B \otimes B \otimes B \arrow[rr, "1\otimes 1\otimes\Delta"] \arrow[dl, "\epsilon\otimes\epsilon\otimes 1"] & & B \otimes B \otimes B \otimes B \arrow[dr, "\epsilon\otimes\epsilon\otimes 1\otimes 1"'] & \\
B \arrow[rrrr, "\Delta"] & & & & B \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\nabla"] \arrow[dd, "1"'] \arrow[dr, "\delta(B\otimes B)"] & & B \arrow[ddl, "\epsilon"] \\
& B \otimes B \otimes B \arrow[ur, "\epsilon\otimes\epsilon\otimes 1"] \arrow[dl, "1\otimes 1\otimes\epsilon"] & \\
B \otimes B \arrow[r, "\epsilon\otimes\epsilon"] & \mathbb{K} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} \arrow[rrr, "\eta"] \arrow[drr, "\delta(\mathbb{K})"] \arrow[ddr, "\delta(\mathbb{K})"] \arrow[ddd, "\eta"'] & & & B \arrow[dd, "\Delta"] \\
& & \mathbb{K} \otimes B \arrow[ur, "\cong"'] \arrow[d, "1\otimes\Delta"] & \\
& \mathbb{K} \otimes B \arrow[r, "\delta(\mathbb{K})\otimes 1"] \arrow[dl, "\cong"] \arrow[dr, "\eta\otimes 1"] & \mathbb{K} \otimes B \otimes B \arrow[r, "\cong"] \arrow[d, "\cong"] & B \otimes B \arrow[dl, "1_B"] \\
B \arrow[rr, "\eta\otimes 1"] & & B \otimes B &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[rr, "1"] \arrow[dr, "\delta(\mathbb{K})"] \arrow[ddr, "\eta"'] & & \mathbb{K} \\
& \mathbb{K} \otimes B \arrow[ur, "1\otimes\epsilon"] \arrow[d, "\cong"] & \\
& B \arrow[uur, "\epsilon"'] &
\end{tikzcd}$$

Hence η and ∇ are coalgebra homomorphisms.

To show the associativity of ∇ we identify along the maps $\alpha : (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ and furthermore simplify the relevant diagram by fixing that σ represents a suitable permutation of the tensor factors. Then the following commute

$$\begin{tikzcd}
B \otimes B \otimes B \arrow[r, "\sigma(\delta(B)\otimes\delta(B)\otimes\delta(B))"] \arrow[d, "1"] & B \otimes B \otimes B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] \arrow[d, shift right, "1\otimes(\nabla\otimes 1)"'] \arrow[d, shift left, "1\otimes(1\otimes\nabla)"] & B \otimes B \otimes B \arrow[d, shift right, "\nabla\otimes 1"'] \arrow[d, shift left, "1\otimes\nabla"] \\
B \otimes B \otimes B \arrow[r, shift left, "\sigma(\delta(B\otimes B)\otimes\delta(B))"] \arrow[r, shift right, "\sigma(\delta(B)\otimes\delta(B\otimes B))"'] \arrow[d, "1"] & B \otimes B \otimes B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] \arrow[d, "1\otimes\nabla"] & B \otimes B \arrow[d, "\nabla"] \\
B \otimes B \otimes B \arrow[r, "\delta(B\otimes B\otimes B)"] & B \otimes B \otimes B \otimes B \arrow[r, "\epsilon\otimes\epsilon\otimes\epsilon\otimes 1"] & B
\end{tikzcd}$$

The upper row is the identity hence we get the associative law.

For the proof that η has the properties of a unit we must explicitly consider the coherence morphisms λ and ρ By reasons of symmetry we will only show one half of of the unit axiom. This axiom follows from the commutativity of the following diagram

$$\begin{tikzcd}
B \arrow[r, "\delta(B)"] \arrow[d, "\rho^{-1}"'] & B \otimes B \arrow[r, "\rho^{-1}"] & B \otimes B \otimes \mathbb{K} \arrow[rr, "\rho"] \arrow[d, "1\otimes 1\otimes\delta(\mathbb{K})"] & & B \otimes B \arrow[r, "\epsilon\otimes 1"] \arrow[d, "1\otimes 1\otimes\eta"'] & B \arrow[d, "1\otimes\eta"'] \\
B \otimes \mathbb{K} \arrow[urr, "\delta(B)\otimes 1"] \arrow[rr, "\delta(B)\otimes\delta(\mathbb{K})"] \arrow[d, "="'] & & B \otimes B \otimes \mathbb{K} \otimes B \arrow[r, "1\otimes\tau\otimes 1"] & B \otimes \mathbb{K} \otimes B \otimes B \arrow[r, "\rho\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes\nabla"] & B \otimes B \otimes B \arrow[r, "\epsilon\otimes 1\otimes 1"] \arrow[dd, "1\otimes\nabla"'] & B \otimes B \arrow[dd, "\nabla"'] \\
B \otimes \mathbb{K} \arrow[rrr, "\delta(B\otimes\mathbb{K})"] \arrow[d, "\rho"'] & & & B \otimes \mathbb{K} \otimes B \arrow[dr, "\rho\otimes 1"] & & \\
B \arrow[rrrr, "\delta(B)"] & & & & B \otimes B \arrow[r, "\epsilon\otimes 1"] & B
\end{tikzcd}$$

$\square$