CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

7. Duality of Hopf Algebras

In 2.4.8 we have seen that the dual Hopf algebra H^* of a finite dimensional Hopf algebra H satisfies certain relations w.r.t. the evaluation map. The multiplication of H^* is derived from the comultiplication of H and the comultiplication of H^* is derived from the multiplication of H.

This kind of duality is restricted to the finite-dimensional situation. Nevertheless one wants to have a process that is close to the finite-dimensional situation. This short section is devoted to several approaches of duality for Hopf algebras.

First we use the relations of the finite-dimensional situation to give a general definition.

Definition 2.7.1. Let H and L be Hopf algebras. Let

$$\mathrm{ev} : L \otimes H \ni a \otimes h \mapsto \langle a, h \rangle \in \mathbb{K}$$

be a bilinear form satisfying

$$\langle a \otimes b, \sum h_{(1)} \otimes h_{(2)} \rangle = \langle ab, h \rangle, \quad \langle 1, h \rangle = \varepsilon(h) \tag{1}$$

$$\langle \sum a_{(1)} \otimes a_{(2)}, h \otimes j \rangle = \langle a, hj \rangle, \quad \langle a, 1 \rangle = \varepsilon(a) \tag{2}$$

$$\langle a, S(h) \rangle = \langle S(a), h \rangle \tag{3}$$

Such a map is called a *weak duality of Hopf algebras*. The bilinear form is called *left* (*right*) *nondegenerate* if $\langle a, H \rangle = 0$ implies $a = 0$ $(\langle L, h \rangle = 0$ implies $h = 0)$. A *duality of Hopf algebras* is a weak duality that is left and right nondegenerate.

Remark 2.7.2. If H is a finite dimensional Hopf algebra then the usual evaluation $\mathrm{ev} : H^* \otimes H \longrightarrow \mathbb{K}$ defines a duality of Hopf algebras.

Remark 2.7.3. Assume that $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ defines a weak duality. By A.4.15 we have isomorphisms $\mathrm{Hom}(L \otimes H, \mathbb{K}) \cong \mathrm{Hom}(L, \mathrm{Hom}(H, \mathbb{K}))$ and $\mathrm{Hom}(L \otimes H, \mathbb{K}) \cong \mathrm{Hom}(H, \mathrm{Hom}(L, \mathbb{K}))$. Denote the homomorphisms associated with $\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ by $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ resp. $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$. They satisfy $\varphi(a)(h) = \mathrm{ev}(a \otimes h) = \psi(h)(a)$.

$\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ is left nondegenerate iff $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ is injective. $\mathrm{ev} : L \otimes K \longrightarrow \mathbb{K}$ is right nondegenerate iff $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$ is injective.

Lemma 2.7.4. 1. *The bilinear form* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *satisfies (1) if and only if* $\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K})$ *is a homomorphism of algebras.*

2. *The bilinear form* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *satisfies (2) if and only if* $\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K})$ *is a homomorphism of algebras.*

Proof. $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ satisfies the right equation of (1) iff $\varphi(ab)(h) = \langle ab, h \rangle = \langle a \otimes b, \sum h_{(1)} \otimes h_{(2)} \rangle = \sum \langle a, h_{(1)} \rangle \langle b, h_{(2)} \rangle = \sum \varphi(a)(h_{(1)}) \varphi(b)(h_{(2)}) = (\varphi(a) * \varphi(b))(h)$ by the definition of the algebra structure on $\mathrm{Hom}(H, \mathbb{K})$.

$\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ satisfies the left equation of (1) iff $\varphi(1)(h) = \langle 1, h\rangle = \varepsilon(h)$. The second part of the Lemma follows by symmetry. □

Example 2.7.5. There is a weak duality between the quantum groups $\mathbb{SL}_q(2)$ and $U_q(sl(2))$. (Kassel: Chapter VII.4).

Proposition 2.7.6. *Let* $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ *be a weak duality of Hopf algebras. Let* $I := \mathrm{Ker}(\varphi : L \longrightarrow \mathrm{Hom}(H, \mathbb{K}))$ *and* $J := \mathrm{Ker}(\psi : H \longrightarrow \mathrm{Hom}(L, \mathbb{K}))$. *Let* $\overline{L} := L/I$ *and* $\overline{H} := H/J$. *Then* $\overline{L}$ *and* $\overline{H}$ *are Hopf algebras and the induced bilinear form* $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow \mathbb{K}$ *is a duality.*

Proof. First observe that I and J are two sided ideals hence $\overline{L}$ and $\overline{H}$ are algebras. Then $\mathrm{ev} : L \otimes H \longrightarrow \mathbb{K}$ can be factored through $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow \mathbb{K}$ and the equations (1) and (2) are still satisfied for the residue classes.

The ideals I and J are biideals. In fact, let $x \in I$ then $\langle \Delta(x), a \otimes b\rangle = \langle x, ab\rangle = 0$ hence $\Delta(x) \in \mathrm{Ker}(\varphi \otimes \varphi : L \otimes L \longrightarrow \mathrm{Hom}(H \otimes H, \mathbb{K}) = I \otimes L + L \otimes I$ (the last equality is an easy exercise in linear algebra) and $\varepsilon(x) = \langle x, 1\rangle = 0$. Hence as in the proof of Theorem 2.6.3 we get that $\overline{L} = L/I$ and $\overline{H} = H/J$ are bialgebras. Since $\langle S(x), a\rangle = \langle x, S(a)\rangle = 0$ we have an induced homomorphism $\overline{S} : \overline{L} \longrightarrow \overline{L}$. The identities satisfied in L hold also for the residue classes in $\overline{L}$ so that L and similarly $\overline{H}$ become Hopf algebras. Finally we have by definition of I that $\langle \overline{x}, \overline{a}\rangle = \langle x, a\rangle = 0$ for all $a \in H$ iff $a \in I$ or $\overline{a} = 0$. Thus the bilinear form $\overline{\mathrm{ev}} : \overline{L} \otimes \overline{H} \longrightarrow K$ defines a duality. □

Problem 2.7.1. (in Linear Algebra)

1. For $U \subseteq V$ define $U^\perp := \{f \in V^* | f(U) = 0\}$. For $Z \subseteq V^*$ define $Z^\perp := \{v \in V | Z(v) = 0\}$. Show that the following hold:
   (a) $U \subseteq V \Longrightarrow U = U^{\perp\perp}$;
   (b) $Z \subseteq V^*$ and $\dim Z < \infty \Longrightarrow Z = Z^{\perp\perp}$;
   (c) $\{U \subseteq V | \dim V/U < \infty\} \cong \{Z \subseteq V^* | \dim Z < \infty\}$ under the maps $U \mapsto U^\perp$ and $Z \mapsto Z^\perp$.
2. Let $V = \bigoplus_{i=1}^{\infty} \mathbb{K}x_i$ be an infinite-dimensional vector space. Find an element $g \in (V \otimes V)^*$ that is not in $V^* \otimes V^* \quad (\subseteq (V \otimes V)^*)$.

Definition 2.7.7. Let A be an algebra. We define $A^o := \{f \in A^* | \exists \text{ ideal } {}_AI_A \subseteq A : \dim(A/I) < \infty \text{ and } f(I) = 0\}$.

Lemma 2.7.8. *Let* A *be an algebra and* $f \in A^*$. *The following are equivalent:*

1. $f \in A^o$*;*
2. *there exists* $I_A \subseteq A$ *such that* $\dim A/I < \infty$ *and* $f(I) = 0$*;*
3. $A \cdot f \subseteq {}_A\mathrm{Hom}_{\mathbb{K}}(.A_A, .\mathbb{K})$ *is finite dimensional;*
4. $A \cdot f \cdot A$ *is finite dimensional;*
5. $\nabla^*(f) \in A^* \otimes A^*$.

Proof. 1. $\Longrightarrow$ 2. and 4. $\Longrightarrow$ 3. are trivial.

2. $\Longrightarrow$ 3. Let $I_A \subseteq A$ with $f(I) = 0$ and $\dim A/I < \infty$. Write $A^* \otimes A \longrightarrow \mathbb{K}$ as $\langle g, a\rangle$. Then $\langle af, i\rangle = \langle f, ia\rangle = 0$ hence $Af \subset I^\perp$ and $\dim Af < \infty$.

3. $\Longrightarrow$ 2. Let $\dim Af < \infty$. Then $I_A := (Af)^\perp$ is an ideal of finite codimension in A and $f(I) = 0$ holds.

2. $\Longrightarrow$ 1. Let $I_A \subset A$ with $\dim A/I_A < \infty$ and $f(I) = 0$ be given. Then right multiplication induces $\varphi : A \longrightarrow \mathrm{Hom}_{\mathbb{K}}(A/I., A/I.)$ and $\dim \mathrm{End}_{\mathbb{K}}(A/I) < \infty$. Thus $J = \mathrm{Ker}(\varphi) \subseteq A$ is a two sided ideal of finite codimension and $J \subset I$ (since $\varphi(j)(\bar{1}) = 0 = \bar{1} \cdot j = \bar{j}$ implies $j \in I$). Furthermore we have $f(J) \subseteq f(I) = 0$.

1. $\Longrightarrow$ 4. $\langle afb, i\rangle = \langle f, bia\rangle = 0$ implies $A \cdot f \cdot A \subseteq {}_AI_A^\perp$ hence $\dim AfA < \infty$.

3. $\Longrightarrow$ 5. We observe that $\nabla^*(f) = f\nabla \in (A \otimes A)^*$. We want to show that $\nabla^*(f) \in A^* \otimes A^*$. Let $g_1, \ldots, g_n$ be a basis of Af. Then there exist $h_1, \ldots, h_n \in A^*$ such that $bf = \sum h_i(b)g_i$. Let $a, b \in A$. Then $\langle \nabla^*(f), a \otimes b\rangle = \langle f, ab\rangle = \langle bf, a\rangle = \sum h_i(b)g_i(a) = \langle \sum g_i \otimes h_i, a \otimes b\rangle$ so that $\nabla^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$.

5. $\Longrightarrow$ 3. Let $\nabla^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$. Then $bf = \sum h_i(b)g_i$ for all $b \in A$ as before. Thus Af is generated by the $g_1, \ldots, g_n$. $\square$

Proposition 2.7.9. *Let (A, m, u) be an algebra. Then we have $m^*(A^o) \subseteq A^o \otimes A^o$. Furthermore $(A^o, \Delta, \varepsilon)$ is a coalgebra with $\Delta = m^*$ and $\varepsilon = u^*$.*

Proof. Let $f \in A^o$ and let $g_1, \ldots, g_n$ be a basis for Af. Then we have $m^*(f) = \sum g_i \otimes h_i$ for suitable $h_i \in A^*$ as in the proof of the previous proposition. Since $g_i \in Af$ we get $Ag_i \subseteq Af$ and $\dim(Ag_i) < \infty$ and hence $g_i \in A^o$. Choose $a_1, \ldots, a_n \in A$ such that $g_i(a_j) = \delta_{ij}$. Then $(fa_j)(a) = f(a_j a) = \langle m^*(f), a_j \otimes a\rangle = \sum g_i(a_j)h_i(a) = h_j(a)$ implies $fa_j = h_j \in fA$. Observe that $\dim(fA) < \infty$ hence $\dim(h_jA) < \infty$, so that $h_j \in A^o$. This proves $m^*(f) \in A^o \otimes A^o$.

One checks easily that counit law and coassociativity hold. $\square$

Theorem 2.7.10. (The Sweedler dual:) *Let $(B, m, u, \Delta, \varepsilon)$ be a bialgebra. Then $(B^o, \Delta^*, \varepsilon^*, m^*, u^*)$ again is a bialgebra. If $B = H$ is a Hopf algebra with antipode S, then S^* is an antipode for $B^o = H^o$.*

Proof. We know that $(B^*, \Delta^*, \varepsilon^*)$ is an algebra and that (B^o, m^*, u^*) is a coalgebra. We show now that $B^o \subseteq B^*$ is a subalgebra. Let $f, g \in B^o$ with $\dim(Bf) < \infty$ and $\dim(Bg) < \infty$. Let $a \in B$. Then we have $(a(fg))(b) = (fg)(ba) = \sum f(b_{(1)}a_{(1)})g(b_{(2)}a_{(2)}) = \sum(a_{(1)}f)(b_{(1)})(a_{(2)}g)(b_{(2)}) = \sum((a_{(1)}f)(a_{(2)}g))(b)$ hence $a(fg) = \sum(a_{(1)}f)(a_{(2)}g) \in (Bf)(Bg)$. Since $\dim(Bf)(Bg) < \infty$ we have $\dim(B(fg)) < \infty$ so that $fg \in B^o$. Furthermore we have $\varepsilon \in B^o$, since $\mathrm{Ker}(\varepsilon)$ has codimension 1. Thus $B^o \subseteq B^*$ is a subalgebra. It is now easy to see that B^o is a bialgebra.

Now let S be the antipode of H. We show $S^*(H^o) \subseteq H^o$. Let $a \in H$, $f \in H^o$. Then $\langle aS^*(f), b\rangle = \langle S^*(f), ba\rangle = \langle f, S(ba)\rangle = \langle f, S(a)S(b)\rangle = \langle fS(a), S(b)\rangle = \langle S^*(fS(a)), b\rangle$. This implies $aS^*(f) = S^*(fS(a))$ and $HS^*(f) = S^*(fS(H)) \subseteq S^*(fH)$. Since $f \in H^o$ we get $\dim(fH) < \infty$ so that $\dim(S^*(fH)) < \infty$ and $\dim(HS^*(f)) < \infty$. This shows $S^*(f) \in H^o$. The rest of the proof is now trivial. $\square$

Definition 2.7.11. Let $G = \mathbb{K}\text{-}\mathbf{cAlg}(H, \text{-})$ be an affine group and $R \in \mathbb{K}\text{-}\mathbf{cAlg}$. We define $G \otimes_{\mathbb{K}} R := G|_{R\text{-}\mathbf{cAlg}}$ to be the restriction to commutative R-algebras. The functor $G \otimes_{\mathbb{K}} R$ is represented by $H \otimes R \in R\text{-}\mathbf{cAlg}$:

$$G|_{R\text{-}\mathbf{cAlg}}(A) = \mathbb{K}\text{-}\mathbf{cAlg}(H, A) \cong R\text{-}\mathbf{cAlg}(H \otimes R, A).$$

Theorem 2.7.12. (The Cartier dual:) *Let H be a finite dimensional commutative cocommutative Hopf algebra. Let $G = \mathbb{K}\text{-}\mathbf{cAlg}(H, \text{-})$ be the associated affine group and let $D(G) := \mathbb{K}\text{-}\mathbf{cAlg}(H^*, \text{-})$ be the dual group. Then we have*

$$D(G) = \mathcal{G}r(G, G_m)$$

where $\mathcal{G}r(G, G_m)(R) = \mathrm{Gr}(G \otimes_{\mathbb{K}} R, G_m \otimes_{\mathbb{K}} R)$ is the set of group (-functor) homomorphisms and G_m is the multiplicative group.

Proof. We have $\mathcal{G}r(G, G_m)(R) = \mathrm{Gr}(G \otimes_{\mathbb{K}} R, G_M \otimes_{\mathbb{K}} R) \cong R\text{-Hopf-Alg}(\mathbb{K}[t, t^{-1}] \otimes R, H \otimes R) \cong R\text{-Hopf-Alg}(R[t, t^{-1}], H \otimes R) \cong \{x \in U(H \otimes R) | \Delta(x) = x \otimes x, \varepsilon(x) = 1\}$, since $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ imply $xS(x) = \varepsilon(x) = 1$.

Consider $x \in \mathrm{Hom}_R((H \otimes R)^*, R) = \mathrm{Hom}_R(H^* \otimes R, R)$. Then $\Delta(x) = x \otimes x$ iff $x(v^* w^*) = \langle x, v^* w^* \rangle = \langle \Delta(x), v^* \otimes w^* \rangle = x(v^*) x(w^*)$ and $\varepsilon(x) = 1$ iff $\langle x, \varepsilon \rangle = 1$. Hence $x \in R\text{-}\mathbf{cAlg}((H \otimes R)^*, R) \cong \mathbb{K}\text{-}\mathbf{cAlg}(H^*, R) = D(G)(R)$. $\square$