CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups

4. Formal Groups

Consider now $\mathbb{K}$-**cCoalg** the category of cocommutative $\mathbb{K}$-coalgebras. Let $C, D \in \mathbb{K}$-**cCoalg**. Then $C \otimes D$ is again a cocommutative $\mathbb{K}$-coalgebra by Problem A.11.4. In fact this holds also for non-commutative $\mathbb{K}$-algebras, but in $\mathbb{K}$-**cCoalg** we have

Proposition 2.4.1. *The tensor product in* $\mathbb{K}$-**cCoalg** *is the (categorical) product.*

Proof. Let $f \in \mathbb{K}$-**cCoalg**$(Z, C), g \in \mathbb{K}$-**cCoalg**(Z, D). The map $(f, g) : Z \longrightarrow C \otimes D$ defined by $(f, g)(z) := \sum f(z_{(1)}) \otimes g(z_{(2)})$ is the unique homomorphism of coalgebras such that $(1 \otimes \varepsilon_D)(f, g)(z) = f(z)$ and $(\varepsilon_C \otimes 1)(f, g)(z) = g(z)$ or such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & \downarrow^{(f,g)} & \searrow^{g} & \\ C & \xleftarrow{p_C} & C \otimes D & \xrightarrow{p_D} & D \end{array}$$

commutes, where $p_C(c \otimes d) = (1 \otimes \varepsilon)(c \otimes d) = c\varepsilon(d)$ and $p_D(c \otimes d) = (\varepsilon \otimes 1)(c \otimes d) = \varepsilon(c)d$ are homomorphisms of coalgebras. $\square$

So the category $\mathbb{K}$-**cCoalg** has finite products and also a final object $\mathbb{K}$.

Definition 2.4.2. A *formal group* is a group in the category of $\mathbb{K}$-**cCoalg** of cocommutative coalgebras.

A formal group G defines a contravariant representable functor from $\mathbb{K}$-**cCoalg** to **Gr**, the category of groups.

Proposition 2.4.3. *Let $H \in \mathbb{K}$-**cCoalg**. H a represents a formal group if and only if there are given morphisms in* $\mathbb{K}$-**cCoalg**

$$\nabla : H \otimes H \longrightarrow H, \quad u : \mathbb{K} \longrightarrow H, \quad S : H \longrightarrow H$$

such that the following diagrams commute

(associativity)
$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{1 \otimes \nabla} & H \otimes H \\ \downarrow{\scriptstyle \nabla \otimes 1} & & \downarrow{\scriptstyle \nabla} \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

(unit)
$$\begin{array}{ccc} \mathbb{K} \otimes H \cong H \cong H \otimes \mathbb{K} & \xrightarrow{u \otimes 1} & H \otimes H \\ \downarrow{\scriptstyle 1 \otimes u} & \searrow^{1} & \downarrow{\scriptstyle \nabla} \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

(inverse)
$$\begin{array}{ccccc} H & \xrightarrow{\varepsilon} & \mathbb{K} & \xrightarrow{\eta} & H \\ \downarrow{\scriptstyle \Delta} & & & & \uparrow{\scriptstyle \nabla} \\ H \otimes H & & \xrightarrow[\mathrm{id} \otimes S]{S \otimes \mathrm{id}} & & H \otimes H \end{array}$$

PROOF. For an arbitrary formal group H we get $\nabla = p_1 * p_2 \in \mathbb{K}\text{-}\mathbf{cCoalg}(H \otimes H, H)$, $u = e \in \mathbb{K}\text{-}\mathbf{cCoalg}(\mathbb{K}, H)$, and $S = (\mathrm{id})^{-1} \in \mathbb{K}\text{-}\mathbf{cCoalg}(H, H)$. These maps, the Yoneda Lemma and Remark 2.2.6 lead to the proof of the proposition. □

Remark 2.4.4. In particular the representing object $(H, \nabla, u, \Delta, \varepsilon, S)$ of a formal group G is a cocommutative Hopf algebra and every such Hopf algebra represents a formal group. Hence the category of formal groups is equivalent to the category of cocommutative Hopf algebras.

Corollary 2.4.5. *A coalgebra $H \in \mathbb{K}\text{-}\mathbf{cCoalg}$ represents a formal group if and only if H is a cocommutative Hopf algebra.*

The category of cocommutative Hopf algebras is equivalent to the category of formal groups.

Corollary 2.4.6. *The following categories are equivalent:*

1. *The category of commutative, cocommutative Hopf algebras.*
2. *The category of commutative formal groups.*
3. *The dual of the category of commutative affine algebraic groups.*

Example 2.4.7. 1. Group algebras $\mathbb{K}G$ are formal groups.

2. Universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$ are formal groups.
3. The tensor algebra $T(V)$ and the symmetric algebra $S(V)$ are formal groups.
4. Let C be a cocommutative coalgebra and G be a group. Then the group $\mathbb{K}G(C) = \mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}G)$ is isomorphic to the set of families $(h_g^* | g \in G)$ of decompositions of the unit of C^* into a sum of orthogonal idempotents $h_g^* \in C^*$ that are locally finite.

   To see this embed $\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}G) \subseteq \mathrm{Hom}(C, \mathbb{K}G)$ and embedthe set $\mathrm{Hom}(C, \mathbb{K}G)$ into the set $(C^*)^G$ of G-families of elements in the algebra C^* by $h \mapsto (h_g^*)$ with $h(c) = \sum_{g \in G} h_g^*(c) g$. The linear map h is a homomorphism of coalgebras iff $(h \otimes h)\Delta = \Delta h$ and $\varepsilon h = \varepsilon$ iff $\sum h(c_{(1)}) \otimes h(c_{(2)}) = \sum h(c)_{(1)} \otimes h(c)_{(2)}$ and $\varepsilon(h(c)) = \varepsilon(c)$ for all $c \in C$ iff $\sum h_g^*(c_{(1)}) g \otimes h_l^*(c_{(2)}) l = \sum h_g^*(c) g \otimes g$ and $\sum h_g^*(c) = \varepsilon(c)$ iff $\sum h_g^*(c_{(1)}) h_l^*(c_{(2)}) = \delta_{gl} h_g^*(c)$ and $\sum h_g^* = \varepsilon$ iff $h_g^* * h_l^* = \delta_{gl} h_g^*$ and $\sum h_g^* = 1_{C^*}$. Furthermore the families must be locally finite, i.e. for each $c \in C$ only finitely many of them give non-zero values.
5. Let C be a cocommutative coalgebra and $\mathbb{K}[x]$ be the Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$ (the symmetric algebra of the one dimensional vector space $\mathbb{K}x$). We embed as before $\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}[x]) \subseteq \mathrm{Hom}(C, \mathbb{K}[x]) = (C^*)^{\{\mathbf{N}_0\}}$, the set of locally finite $\mathbf{N}_0$-families in C^* by $h(c) = \sum_{i=0}^{\infty} h_i^*(c) x^i$. The map h is a homomorphism of coalgebras iff $\Delta(h(c)) = \sum h_i^*(c)(x \otimes 1 + 1 \otimes x)^i = \sum h_i^*(c)\binom{i}{l} x^l \otimes x^{i-l} = (h \otimes h)\Delta(c) = \sum h_i^*(c_{(1)}) h_j^*(c_{(2)}) x^i \otimes x^j$ and $\varepsilon(\sum h_i^*(c) x^i) = \varepsilon(c)$ iff $h_i^* * h_j^* = \binom{i+j}{i} h_{i+j}^*$ and $h_0^* = \varepsilon = 1_{C^*}$.

   Now let $\mathbb{K}$ be a field of characteristic zero. Let $p_i := h_i^*/i!$. Then the conditions simplify to $p_i p_j = p_{i+j}$ and $p_0 = 1$. Hence the series for h is completely determined by the term $p := p_1$ since $p_n = p_1^n$. Since the series must be locally finite we get that for each $c \in C$ there must be an $n \in \mathbf{N}_0$ such that $p^m(c) = 0$ for all $m \geq n$. Hence the element p is *topologically nilpotent* and

$$\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}[x]) \cong \operatorname{rad}_t(C^*)$$

the radial of topologically nilpotent elements of C^*.

It is easy to see that $\operatorname{rad}_t(C^*)$ is a group under addition and that this group structure coincides with the one on $\mathbb{K}\text{-}\mathbf{cCoalg}(C, \mathbb{K}[x])$.

Remark 2.4.8. Let H be a finite dimensional Hopf algebra. Then by A.6.6 and A.6.8 we get that H^* is an algebra and a coalgebra. The commutative diagrams defining the bialgebra property and the antipode can be transferred easily, so H^* is again a Hopf algebra. Hence the functor $\text{-}^* : \mathbf{vec} \longrightarrow \mathbf{vec}$ from finite dimensional vector spaces to itself induces a duality $\text{-}^* : \mathbb{K}\text{-}\mathbf{hopfalg} \longrightarrow \mathbb{K}\text{-}\mathbf{hopfalg}$ from the category of finite dimensional Hopf algebras to itself.

An affine algebraic group is called *finite* if the representing Hopf algebra is finite dimensional. A formal group is called *finite* if the representing Hopf algebra is finite dimensional.

Thus the category of finite affine algebraic groups is equivalent to the category of finite formal groups.

The category of finite commutative affine algebraic groups is self dual. The category of finite commutative affine algebraic groups is equivalent (and dual) to the category of finite commutative formal groups.