CHAPTER 1

Commutative and Noncommutative Algebraic Geometry

Introduction

Throughout we will fix a base field $\mathbb{K}$. The reader may consider it as real numbers or complex numbers or any other of his most favorite fields.

A fundamental and powerful tool for geometry is to associate with each space X the algebra of functions $\mathcal{O}(X)$ from X to the base field (of coefficients). The dream of geometry is that this construction is bijective, i.e. that two different spaces are mapped to two different function algebras and that each algebra is the function algebra of some space.

Actually the spaces and the algebras will form a category. There are admissible maps. For algebras it is quite clear what these maps will be. For spaces this is less obvious, partly due to the fact that we did not say clearly what spaces exactly are. Then the *dream of geometry* would be that these two categories, the category of (certain) spaces and the category of (certain) algebras, are dual to each other.

Algebraic geometry, noncommutative geometry, and theoretical physics have as a basis this fundamental idea, the duality of two categories, the category of spaces (state spaces in physics) and the category of function algebras (algebras of observables) in physics. We will present this duality in the 1. chapter. Certainly the type of spaces as well as the type of algebras will have to be specified.

Theoretical physics uses the categories of locally compact Hausdorff spaces and of commutative C^*-algebras. A famous theorem of Gelfand-Naimark says that these categories are duals of each other.

(Affine) algebraic geometry uses a duality between the categories of affine algebraic schemes and of (reduced) finitely generated commutative algebras.

To get the whole framework of algebraic geometry one needs to go to more general spaces by patching affine spaces together. On the algebra side this amounts to considering sheaves of commutative algebras. We shall not pursue this more general approach to algebraic geometry, since generalizations to noncommutative geometry are still in the state of development and incomplete.

Noncommutative geometry uses either (imaginary) noncommutative spaces and not necessarily commutative algebras or (imaginary) noncommutative spaces and not necessarily commutative C^*-algebras.

We will take an approach to the duality between geometry and algebra that heavily uses functorial tools, especially representable functors. The affine (algebraic) spaces we use will be given in the form of sets of common zeros of certain polynomials, where the zeros can be taken in arbitrary (commutative) $\mathbb{K}$-algebras B. So an affine space will consist of many different sets of zeros, depending on the choice of the coefficient algebra B.

We first give a short introduction to commutative algebraic geometry in this setup and develop a duality between the category of affine (algebraic) spaces and the category of (finitely generated) commutative algebras.

Then we will transfer it to the noncommutative situation. The functorial approach to algebraic geometry is not too often used but it lends itself particularly well to the study of the noncommutative situation. Even in that situation one obtains space-like objects.

The chapter will close with a first step to construct automorphism "groups" of noncommutative spaces. Since the construction of inverses presents special problems we will only construct endomorphism "monoids" in this chapter and postpone the study of invertible endomorphisms or automorphisms to the next chapter.

At the end of the chapter you should

- know how to construct an affine scheme from a commutative algebra,
- know how to construct the function algebra of an affine scheme,
- know what a noncommutative space is and know examples of such,
- understand and be able to construct endomorphism quantum monoids of certain noncommutative spaces,
- understand, why endomorphism quantum monoids are not made out of endomorphisms of a noncommutative space.

1. The Principles of Commutative Algebraic Geometry

We will begin with simplest form of (commutative) geometric spaces and see a duality between these very simple "spaces" and certain commutative algebras. This example will show how the concept of a function algebra can be used to fulfill the dream of geometry in this situation. It will also show the functorial methods that will be applied throughout this text. It is a particularly simple example of a duality as mentioned in the introduction. This example will not be used later on, so we will only sketch the proofs for some of the statements.

Example 1.1.1. Consider a set of points without any additional geometric structure. So the geometric space is just a set. We introduce the notion of its algebra of functions.

Let X be a set. Then $\mathbb{K}^X := \mathrm{Map}(X, \mathbb{K})$ is a $\mathbb{K}$-algebra with componentwise addition and multiplication: $(f+g)(x) := f(x) + g(x)$ and $(fg)(x) := f(x)g(x)$. We study this fact in more detail.

The set $\mathbb{K}^X$ considered as a vector space with the addition $(f+g)(x) := f(x)+g(x)$ and the scalar multiplication $(\alpha f)(x) := \alpha f(x)$ defines a representable contravariant functor

$$\mathbb{K}^{-} : \mathbf{Set} \longrightarrow \mathbf{Vec}.$$

This functor is a representable functor represented by $\mathbb{K}$. In fact $\mathbb{K}^h : \mathbb{K}^Y \longrightarrow \mathbb{K}^X$ is a linear map for every map $h : X \longrightarrow Y$ since $\mathbb{K}^h(\alpha f + \beta g)(x) = (\alpha f + \beta g)(h(x)) = \alpha f(h(x)) + \beta g(h(x)) = (\alpha f h + \beta g h)(x) = (\alpha \mathbb{K}^h(f) + \beta \mathbb{K}^h(g))(x)$ hence $\mathbb{K}^h(\alpha f + \beta g) = \alpha \mathbb{K}^h(f) + \beta \mathbb{K}^h(g)$.

Consider the homomorphism $\tau : \mathbb{K}^X \otimes \mathbb{K}^Y \longrightarrow \mathbb{K}^{X\times Y}$, defined by $\tau(f \otimes g)(x,y) := f(x)g(y)$. In order to obtain a unique homomorphism τ defined on the tensor product we have to show that $\tau' : \mathbb{K}^X \times \mathbb{K}^Y \longrightarrow \mathbb{K}^{X\times Y}$ is a bilinear map : $\tau'(f+f', g)(x,y) = (f+f')(x)g(y) = (f(x)+f'(x))g(y) = f(x)g(y)+f'(x)g(y) = (\tau'(f,g)+\tau'(f',g))(x,y)$ gives the additivity in the left hand argument. The additivity in the right hand argument and the bilinearity is checked similarly. One can check that τ is always injective. If X or Y are finite then τ is bijective.

As a special example we obtain a multiplication $\nabla : \mathbb{K}^X \otimes \mathbb{K}^X \xrightarrow{\tau} \mathbb{K}^{X\times X} \xrightarrow{\mathbb{K}^\Delta} \mathbb{K}^X$ where $\Delta : X \longrightarrow X \times X$ in **Set** is the diagonal map $\Delta(x) := (x,x)$. Furthermore we get a unit $\eta : \mathbb{K}^{\{*\}} \xrightarrow{\mathbb{K}^\epsilon} \mathbb{K}^X$ where $\epsilon : X \longrightarrow \{*\}$ is the unique map into the one element set. One verifies easily that $(\mathbb{K}^X, \eta, \nabla)$ is a $\mathbb{K}$-algebra. Two properties are essential here, the associativity and the unit of $\mathbb{K}$ and the fact that (X, Δ, ϵ) is a "comonoid" in the category **Set**:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "1\times\Delta"] \\
X \times X \arrow[r, "\Delta\times 1"'] & X \times X \times X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "1_X"] & X \times X \arrow[d, "1\times\epsilon"] \\
X \times X \arrow[r, "\epsilon\times 1"'] & \{*\} \times X \cong X \cong X \times \{*\}.
\end{tikzcd}$$

Since $\mathbb{K}^{-}$ is a functor these two diagrams carry over to the category **Vec** and produce the required diagrams for a $\mathbb{K}$-algebra.

For a map $f : X \longrightarrow Y$ we obtain a homomorphism of algebras $\mathbb{K}^f : \mathbb{K}^Y \longrightarrow \mathbb{K}^X$ because the diagrams

$$\begin{tikzcd}
\mathbb{K}^Y \otimes \mathbb{K}^Y \arrow[r, "\tau"] \arrow[d, "\mathbb{K}^f\otimes\mathbb{K}^f"] & \mathbb{K}^{Y\times Y} \arrow[r, "\mathbb{K}^\Delta"] \arrow[d, "\mathbb{K}^{f\times f}"] & \mathbb{K}^Y \arrow[d, "\mathbb{K}^f"] \\
\mathbb{K}^X \otimes \mathbb{K}^X \arrow[r, "\tau"] & \mathbb{K}^{X\times X} \arrow[r, "\mathbb{K}^\Delta"] & \mathbb{K}^X
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{K}^{\{*\}} \cong \mathbb{K} \arrow[dl, "\eta"] \arrow[dr, "\eta"'] & \\
\mathbb{K}^Y \arrow[rr, "\mathbb{K}^f"] & & \mathbb{K}^X
\end{tikzcd}$$

commute.

Thus

$$\mathbb{K}^{-} : \mathbf{Set} \longrightarrow \mathbb{K}\text{-}\mathbf{cAlg}$$

is a contravariant functor.

By the definition of the set-theoretic (cartesian) product we know that $\mathbb{K}^X = \prod_X \mathbb{K}$. This identity does not only hold on the set level, it holds also for the algebra structures on $\mathbb{K}^X$ resp. $\prod_X \mathbb{K}$.

We now construct an inverse functor

$$\mathrm{Spec} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}.$$

For each point $x \in X$ there is a maximal ideal m_x of $\prod_X \mathbb{K}$ defined by $m_x := \{f \in \mathrm{Map}(X, \mathbb{K}) | f(x) = 0\}$. If X is a finite set then these are exactly all maximal ideals of $\prod_X \mathbb{K}$. To show this we observe the following. The surjective homomorphism $p_x : \prod_X \mathbb{K} \longrightarrow \mathbb{K}$ has kernel m_x hence m_x is a maximal ideal. If $m \subseteq \prod_X \mathbb{K}$ is a maximal ideal and $a = (\alpha_1, \ldots, \alpha_n) \in m$ then for any $\alpha_i \neq 0$ we get $(0, \ldots, 0, 1_i, 0, \ldots, 0) = (0, \ldots, 0, \alpha_i^{-1}, 0, \ldots, 0)(\alpha_1, \ldots, \alpha_n) \in m$ hence the i-th factor $0 \times \ldots \times \mathbb{K} \times \ldots \times 0$ of $\prod_X \mathbb{K}$ is in m. So the elements $a \in m$ must have at least one common component $\alpha_j = 0$ since $m \neq \mathbb{K}$. But more than one such a component is impossible since we would get zero divisors in the residue class algebra. Thus $m = m_x$ where $x \in X$ is the j-th elements of the set.

One can easily show more namely that the ideals m_x are precisely all prime ideals of $\mathrm{Map}(X, \mathbb{K})$.

With each commutative algebra A we can associate the set $\mathrm{Spec}(A)$ of all prime ideals of A. That defines a functor Spec: $\mathbb{K}$-**Alg** $\longrightarrow$ **Set**. Applied to algebras of the form $\mathbb{K}^X = \prod_X \mathbb{K}$ with a finite set X this functor recovers X as $X \cong \mathrm{Spec}(\mathbb{K}^X)$. Thus the dream of geometry is satisfied in this particular example.

The above example shows that we may hope to gain some information on the space (set) X by knowing its algebra of functions $\mathbb{K}^X$ and applying the functor Spec to it. For finite sets and certain algebras the functors $\mathbb{K}^{-}$ and Spec actually define a category duality. We are going to expand this duality to larger categories.

We shall carry some geometric structure into the sets X and will study the connection between these geometric spaces and their algebras of functions. For this purpose we will describe sets of points by their coordinates. Examples are the circle or the parabola. More generally the geometric spaces we are going to consider are so called affine schemes described by polynomial equations. We will see that such geometric spaces are completely described by their algebras of functions. Here the Yoneda Lemma will play a central rôle.

We will, however, take a different approach to functions algebras and geometric spaces, than one does in algebraic geometry. We use the functorial approach, which lends itself to an easier access to the principles of noncommutative geometry. We will define geometric spaces as certain functors from the category of commutative algebras to the category of sets. These sets will have a strong geometrical meaning. The functors will associate with each algebra A the set of points of a "geometric variety", where the points have coordinates in the algebra A.

Definition 1.1.2. The functor $\mathbb{A} = \mathbb{A}^1 : \mathbb{K}$-**cAlg** $\longrightarrow$ **Set** (the underlying functor) that associates with each commutative $\mathbb{K}$-algebra A its space (set) of points (elements) A is called the *affine line*.

Lemma 1.1.3. *The functor "affine line" is a representable functor.*

PROOF. By Lemma 2.3.5 the representing object is $\mathbb{K}[x]$. Observe that it is unique up to isomorphism. $\square$

Definition 1.1.4. The functor $\mathbb{A}^2 : \mathbb{K}$-**cAlg** $\longrightarrow$ **Set** that associates with each commutative algebra A the space (set) of points (elements) of the plane A^2 is called the *affine plane*.

Lemma 1.1.5. *The functor "affine plane" is a representable functor.*

PROOF. Similar to Lemma 2.3.9 the representing object is $\mathbb{K}[x_1, x_2]$. This algebra is unique up to isomorphism. $\square$

Let $p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$ be a family of polynomials. We want to consider the (geometric) variety of zeros of these polynomials. Observe that $\mathbb{K}$ may not contain sufficiently many zeros for these polynomials. Thus we are going to admit zeros in extension fields of $\mathbb{K}$ or more generally in arbitrary commutative $\mathbb{K}$-algebras.

In the following rather simple buildup of commutative algebraic geometry, the reader should carefully verify in which statements and proofs the commutativity is really needed. Most of the following will be verbally generalized to not necessarily commutative algebras.

Definition 1.1.6. Given a set of polynomials $\{p_1, \ldots, p_m\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$. The functor $\mathcal{X}$ that associates with each commutative algebra A the set $\mathcal{X}(A)$ of zeros of the polynomials (p_i) in A^n is called an *affine algebraic variety* or an *affine scheme* (in $\mathbb{A}^n$) with defining polynomials $p_1, \ldots, p_m$. The elements in $\mathcal{X}(A)$ are called the *A-points* of $\mathcal{X}$.

Theorem 1.1.7. *The affine scheme $\mathcal{X}$ in $\mathbb{A}^n$ with defining polynomials $p_1, \ldots, p_m$ is a representable functor with representing algebra*

$$\mathcal{O}(\mathcal{X}) := \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m),$$

called the affine algebra *of the functor $\mathcal{X}$.*

PROOF. First we show that the affine scheme $\mathcal{X} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ with the defining polynomials $p_1, \ldots, p_m$ is a functor. Let $f : A \longrightarrow B$ be a homomorphism of commutative algebras. The induced map $f^n : A^n \longrightarrow B^n$ defined by application of f on the components restricts to $\mathcal{X}(A) \subseteq A^n$ as $\mathcal{X}(f) : \mathcal{X}(A) \longrightarrow \mathcal{X}(B)$. This map is well-defined for let $(a_1, \ldots, a_n) \in \mathcal{X}(A)$ be a zero for all polynomials $p_1, \ldots, p_m$ then $p_i(f(a_1), \ldots, f(a_n)) = f(p_i(a_1, \ldots, a_n)) = f(0) = 0$ for all i hence $f^n(a_1, \ldots, a_n) = (f(a_1), \ldots, f(a_n)) \in B^n$ is a zero for all polynomials. Thus $\mathcal{X}(f) : \mathcal{X}(A) \longrightarrow \mathcal{X}(B)$ is well-defined. Functoriality of $\mathcal{X}$ is clear now.

Now we show that $\mathcal{X}$ is representable by $\mathcal{O}(\mathcal{X}) = \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$. Observe that $(p_1, \ldots, p_m)$ denotes the (two-sided) ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials $p_1, \ldots, p_m$. We know that each n-tupel $(a_1, \ldots, a_n) \in A^n$ uniquely determines an algebra homomorphism $f : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow A$ by $f(x_1) = a_1, \ldots, f(x_n) = a_n$. (The polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ in $\mathbb{K}\text{-}\mathbf{cAlg}$ is free over the set $\{x_1, \ldots, x_n\}$, or $\mathbb{K}[x_1, \ldots, x_n]$ together with the embedding $\iota : \{x_1, \ldots, x_n\} \longrightarrow \mathbb{K}[x_1, \ldots, x_n]$ is a couniversal solution of the problem given by the underlying functor $\mathbb{A} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ and the set $\{x_1, \ldots, x_n\} \in \mathbf{Set}$.) This homomorphism of algebras maps polynomials $p(x_1, \ldots, x_n)$ into $f(p) = p(a_1, \ldots, a_n)$. Hence $(a_1, \ldots, a_n)$ is a common zero of the polynomials $p_1, \ldots, p_m$ if and only if $f(p_i) = p_i(a_1, \ldots, a_n) = 0$, i.e. $p_1, \ldots, p_m$ are in the kernel of f. This happens if and only if f vanishes on the ideal $(p_1, \ldots, p_m)$ or in other word can be factorized through the residue class map

$$\nu : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$$

This induces a bijection

$$\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(\mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m), A) \ni f \mapsto (f(x_1), \ldots, f(x_n)) \in \mathcal{X}(A).$$

Now it is easy to see that this bijection is a natural isomorphism (in A). $\square$

If no polynomials are given for the above construction, then the functor under this construction is the affine space $\mathbb{A}^n$ of dimension n. By giving polynomials the functor $\mathcal{X}$ becomes a subfunctor of $\mathbb{A}^n$, because it defines subsets $\mathcal{X}(A) \subseteq \mathbb{A}^n(A) = A^n$. Both functors are representable functors. The embedding is induced by the homomorphism of algebras $\nu : \mathbb{K}[x_1, \dots, x_n] \longrightarrow \mathbb{K}[x_1, \dots, x_n]/(p_1, \dots, p_m)$.

Problem 1.1.1. 1. Determine the affine algebra of the functor "unit circle" S^1 in $\mathbb{A}^2$.
2. Determine the affine algebra of the functor "unit sphere" S^{n-1} in $\mathbb{A}^n$.
3. Let $\mathcal{X}$ denote the plane curve $y = x^2$. Then $\mathcal{X}$ is isomorphic to the affine line.
4. Let $\mathcal{Y}$ denote the plane curve $xy = 1$. Then $\mathcal{Y}$ is not isomorphic to the affine line. (Hint: An isomorphism $\mathbb{K}[x, x^{-1}] \longrightarrow \mathbb{K}[y]$ sends x to a polynomial $p(y)$ which must be invertible. Consider the highest coefficient of $p(y)$ and show that $p(y) \in \mathbb{K}$. But that means that the map cannot be bijective.)
5. Let $\mathbb{K} = \mathbb{C}$ be the field of complex numbers. Show that the unit functor $U : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ in Lemma 2.3.7 is naturally isomorphic to the unit circle functor S^1. (Hint: There is an algebra isomorphism between the representing algebras $\mathbb{K}[e, e^{-1}]$ and $\mathbb{K}[c, s]/(c^2 + s^2 - 1)$.)
6. * Let $\mathbb{K}$ be an algebraically closed field. Let p be an irreducible square polynomial in $\mathbb{K}[x, y]$. Let $\mathcal{Z}$ be the conic section defined by p with the affine algebra $\mathbb{K}[x, y]/(p)$. Show that $\mathcal{Z}$ is naturally isomorphic either to $\mathcal{X}$ or to $\mathcal{Y}$ from parts 3. resp. 4.

Remark 1.1.8. Affine algebras of affine schemes are finitely generated commutative algebras and any such algebra is an affine algebra of some affine scheme, since $A \cong \mathbb{K}[x_1, \dots, x_n]/(p_1, \dots, p_m)$ (Hilbert basis theorem).

The polynomials $p_1, \dots, p_m$ are not uniquely determined by the affine algebra of an affine scheme. Not even the ideal generated by the polynomials in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ is uniquely determined. Also the number of variables $x_1, \dots, x_n$ is not uniquely determined.

The $\mathbb{K}$-points $(\alpha_1, \dots, \alpha_n) \in \mathcal{X}(\mathbb{K})$ of an affine scheme $\mathcal{X}$ (with coefficients in the base field $\mathbb{K}$) are called *rational points*. They do not suffice to completely describe the affine scheme.

Let for example $\mathbb{K} = \mathbb{R}$ the set of rational numbers. If $\mathcal{X}$ and $\mathcal{Y}$ are affine schemes with affine algebras $\mathcal{O}(\mathcal{X}) := \mathbb{K}[x, y]/(x^2 + y^2 + 1)$ and $\mathcal{O}(\mathcal{Y}) := \mathbb{K}[x]/(x^2 + 1)$ then both schemes have no rational points. The scheme $\mathcal{Y}$, however, has exactly two complex points (with coefficients in the field $\mathbb{C}$ of complex numbers) and the scheme $\mathcal{X}$ has infinitely many complex points, hence $\mathcal{X}(\mathbb{C}) \not\cong \mathcal{Y}(\mathbb{C})$. This does not result from the embeddings into different spaces $\mathbb{A}^2$ resp. $\mathbb{A}^1$. In fact we also have $\mathcal{O}(\mathcal{Y}) = \mathbb{K}[x]/(x^2 + 1) \cong \mathbb{K}[x, y]/(x^2 + 1, y)$, so $\mathcal{Y}$ can be considered as an affine scheme in $\mathbb{A}^2$.

Since each affine scheme $\mathcal{X}$ is isomorphic to the functor $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(\mathcal{O}(\mathcal{X}),\text{-})$ we will henceforth identify these two functors, thus removing annoying isomorphisms.

Definition 1.1.9. Let $\mathbb{K}$-**Aff** denote the category of all commutative finitely generated (or affine cf. 1.1.8) $\mathbb{K}$-algebras. An *affine algebraic variety* is a representable functor $\mathbb{K}\text{-}\mathbf{Aff}(A,\text{-}) : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathbf{Set}$. The affine algebraic varieties together with the natural transformations form the *category of affine algebraic varieties* $\mathrm{Var}(\mathbb{K})$ over $\mathbb{K}$. The functor that associates with each affine algebra A its affine algebraic variety represented by A is denoted by $\mathrm{Spec} : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathrm{Var}(\mathbb{K})$, $\mathrm{Spec}(A) = \mathbb{K}\text{-}\text{-}\mathbf{Aff}(A,\text{-})$.

By the Yoneda Lemma the functor

$$\mathrm{Spec} : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathrm{Var}(\mathbb{K})$$

is an antiequivalence (or duality) of categories with inverse functor

$$\mathcal{O} : \mathrm{Var}(\mathbb{K}) \longrightarrow \mathbb{K}\text{-}\mathbf{Aff}\,.$$

An affine algebraic variety is completely described by its affine algebra $\mathcal{O}(\mathcal{X})$. Thus the dream of geometry is realized.

Arbitrary (not necessarily finitely generated) commutative algebras also define representable functors (defined on the category of all commutative algebras). Thus we also have "infinite dimensional" varieties which we will call *geometric spaces* or *affine varieties*. We denote their category by $\mathrm{Geom}(\mathbb{K})$ and get a commutative diagram

$$\begin{tikzcd} \mathbb{K}\text{-}\mathbf{Aff} \arrow[r, "\mathrm{Spec}"] \arrow[d] & \mathrm{Var}(\mathbb{K}) \arrow[d] \\ \mathbb{K}\text{-}\mathbf{cAlg} \arrow[r, "\cong^{o}"'] & \mathrm{Geom}(\mathbb{K}) \end{tikzcd}$$

We call the representable functors $\mathcal{X} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ *geometric spaces* or *affine varieties*, and the representable functors $\mathcal{X} : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathbf{Set}$ *affine schemes* or *affine algebraic varieties*. This is another realization of the dream of geometry.

The geometric spaces can be viewed as sets of zeros in arbitrary commutative $\mathbb{K}$-algebras B of arbitrarily many polynomials with arbitrarily many variables. The function algebra of $\mathcal{X}$ will be called the *affine algebra* of $\mathcal{X}$ in both cases.

Example 1.1.10. A somewhat less trivial example is the state space of a circular pendulum (of length 1). The location is in $L = \{(a,b) \in A^2 | a^2 + b^2 = 1\}$, the momentum is in $M = \{p \in A\}$ which is a straight line. So the whole geometric space for the pendulum is $(L \times M)(A) = \{(a,b,p) | a,b,p \in A; a^2 + b^2 = 1\}$. This geometric space is represented by $\mathbb{K}[x,y,z]/(x^2 + y^2 - 1)$ since

$$(L \times M)(A) = \{(a,b,p) | a,b,p \in A; a^2 + b^2 = 1\} \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x,y,z]/(x^2 + y^2 - 1), A).$$

The two antiequivalences of categories above give rise to the question for the function algebra. If a representable functor $\mathcal{X} = \mathbb{K}\text{-}\mathbf{cAlg}(A,\text{-})$ is viewed as geometric sets of zeros of certain polynomials, i.e. as spaces with coordinates in arbitrary commutative algebras B, (plus functorial behavior), then it is not clear why the representing algebra A should be anything like an algebra of functions on these geometric sets. It is not even clear where these functions should assume their values. Only if we can show that A can be viewed as a reasonable algebra of functions, we should talk about a realization of the dream of geometry. But this will be done in the following theorem. We will consider functions as maps (coordinate functions) from the geometric set $\mathcal{X}(B)$ to the set of coordinates B, maps that are natural in B. Such coordinate functions are just natural transformations from $\mathcal{X}$ to the underlying functor $\mathbb{A}$.

Theorem 1.1.11. *Let $\mathcal{X}$ be a geometric space with the affine algebra $A = \mathcal{O}(\mathcal{X})$. Then $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ as $\mathbb{K}$-algebras, where $\mathbb{A} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ is the underlying functor or affine line. The isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ induces a natural transformation $A \times \mathcal{X}(B) \longrightarrow B$ (natural in B).*

Proof. First we define an isomorphism between the sets A and $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$. Because of $\mathcal{X} = \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(A, \text{-}) =: \mathbb{K}\text{-}\mathbf{cAlg}(A, \text{-})$ and $\mathbb{A} = \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(\mathbb{K}[x], \text{-}) =: \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \text{-})$ the Yoneda Lemma gives us

$$\mathrm{Nat}(\mathcal{X}, \mathbb{A}) = \mathrm{Nat}(\mathbb{K}\text{-}\mathbf{cAlg}(A, \text{-}), \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \text{-})) \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], A) = \mathbb{A}(A) \cong A$$

on the set level. Let $\phi : A \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ denote the given isomorphism. ϕ is defined by $\phi(a)(B)(p)(x) := p(a)$. By the Yoneda Lemma its inverse is given by $\phi^{-1}(\alpha := \alpha((A)(1)(x)$.

$\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ carries an algebra structure given by the algebra structure of the coefficients. For a coefficient algebra B, a B-point $p : A \longrightarrow B$ in $\mathcal{X}(B) = \mathbb{K}\text{-}\mathbf{Alg}(A, B)$, and $\alpha, \beta \in \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ we have $\alpha(B)(p) \in \mathbb{A}(B) = B$. Hence $(\alpha + \beta)(B)(p) := (\alpha(B) + \beta(B))(p) = \alpha(B)(p) + \beta(B)(p)$ and $(\alpha \cdot \beta)(B)(p) := (\alpha(B) \cdot \beta(B))(p) = \alpha(B)(p) \cdot \beta(B)(p)$ make $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ an algebra.

Let a be an arbitrary element in A. By the isomorphism given above this element induces an algebra homomorphism $g_a : \mathbb{K}[x] \longrightarrow A$ mapping x onto a. This algebra homomorphism induces the natural transformation $\phi(a) : \mathcal{X} \longrightarrow \mathbb{A}$. On the B-level it is just the composition with g_a, i.e. $\phi(a)(B)(p) = (\mathbb{K}[x] \xrightarrow{g_a} A \xrightarrow{p} B)$. Since such a homomorphism is completely described by the image of x we get $\phi(a)(B)(p)(x) = p(a)$. To compare the algebra structures of A and $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ let $a, a' \in A$. We have $\phi(a)(B)(p)(x) = p(a)$ and $\phi(a')(B)(p)(x) = p(a')$, hence $\phi(a + a')(B)(p)(x) = p(a + a') = p(a) + p(a') = \phi(a)(B)(p)(x) + \phi(a')(B)(p)(x) = (\phi(a)(B)(p) + \phi(a')(B)(p))(x) = (\phi(a)(B) + \phi(a')(B))(p)(x) = (\phi(a) + \phi(a'))(B)(p)(x)$. Analogously we get $\phi(aa')(B)(p)(x) = p(aa') = p(a)p(a') = (\phi(a) \cdot \phi(a'))(B)(p)(x)$, and thus $\phi(a + a') = \phi(a) + \phi(a')$ and $\phi(aa') = \phi(a) \cdot \phi(a')$. Hence addition and multiplication in $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ are defined by the addition and the multiplication of the values $p(a) + p(a')$ resp. $p(a)p(a')$.

We describe the action $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ of A on $\mathcal{X}(B)$. Let $p : A \longrightarrow B$ be a B-point in $\mathbb{K}\text{-}\mathbf{cAlg}(A, B) = \mathcal{X}(B)$. For each $a \in A$ the image $\phi(a) : \mathcal{X} \longrightarrow \mathbb{A}$

is a natural transformation hence we have maps $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ such that $\psi(B)(a, p) = p(a)$. Finally each homomorphism of algebras $f : B \longrightarrow B'$ induces a commutative diagram

$$\begin{tikzcd} A \times \mathcal{X}(B) \arrow[r, "\psi(B)"] \arrow[d, "A \times \mathcal{X}(f)"'] & B \arrow[d, "f"] \\ A \times \mathcal{X}(B') \arrow[r, "\psi(B')"'] & B' \end{tikzcd}$$

Thus $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ is a natural transformation. □

Remark 1.1.12. Observe that the isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ induces a natural transformation $A \times \mathcal{X}(B) \longrightarrow B$ (natural in B). In particular the affine algebra A can be viewed as the set of functions from the set of B-points $\mathcal{X}(B)$ into the "base" ring B (functions which are natural in B). In this sense the algebra A may be considered as function algebra of the geometric space $\mathcal{X}$. Thus we will call A the *function algebra* of $\mathcal{X}$.

One can show that the algebra A is universal with respect to the property, that for each commutative algebra D and each natural transformation $\rho : D \times \mathcal{X}(\text{-}) \longrightarrow \text{-}$ there is a unique homomorphism of algebras $f : D \longrightarrow A$, such that the triangle

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f \times 1_{\mathcal{X}(B)}"'] \arrow[dr, "\rho(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes. We will show this result later on for noncommutative algebras. The universal property implies that the function algebra A of an geometric space $\mathcal{X}$ is unique up to isomorphism.

Let $\mathcal{X}$ be an geometric space with function algebra $A = \mathcal{O}(\mathcal{X})$. If $p : A \longrightarrow \mathbb{K}$ is a rational point of $\mathcal{X}$, i.e. a homomorphism of algebras, then $\mathrm{Im}(p) = \mathbb{K}$ hence $\mathrm{Ker}(p)$ is a maximal ideal of A of codimension 1. Conversely let $\mathfrak{m}$ be a maximal ideal of A of codimension 1 then this defines a rational point $p : A \longrightarrow A/\mathfrak{m} \cong \mathbb{K}$. If $\mathbb{K}$ is algebraicly closed and $\mathfrak{m}$ an arbitrary maximal ideal of A, then $A/\mathfrak{m}$ is a finitely generated $\mathbb{K}$-algebra and a field extension of $\mathbb{K}$, hence it coincides with $\mathbb{K}$. Thus the codimension of $\mathfrak{m}$ is 1. The set of maximal ideals of A is called the *maximal spectrum* $\mathrm{Spec}_m(A)$. This is the approach of algebraic geometry to recover the geometric space of (rational) points from the function algebra A. We will not follow this approach since it does not easily extend to noncommutative geometry.

Problem 1.1.2. Let $\mathcal{X}$ be an affine scheme with affine algebra

$$A = \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m).$$

Define "coordinate functions" $q_i : \mathcal{X}(B) \longrightarrow B$ which describe the coordinates of B-points and identify these coordinate functions with elements of A.

Now we will study morphisms between geometric spaces.

Theorem 1.1.13. *Let $\mathcal{X} \subseteq \mathbb{A}^r$ and $\mathcal{Y} \subseteq \mathbb{A}^s$ be affine algebraic varieties and let $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a natural transformation. Then there are polynomials*

$$p_1(x_1, \ldots, x_r), \ldots, p_s(x_1, \ldots, x_r) \in \mathbb{K}[x_1, \ldots, x_r],$$

such that

$$\phi(A)(a_1, \ldots, a_r) = (p_1(a_1, \ldots, a_r), \ldots, p_s(a_1, \ldots, a_r)),$$

*for all $A \in \mathbb{K}$-**Aff** and all $(a_1, \ldots, a_r) \in \mathcal{X}(A)$, i.e. the morphisms between affine algebraic varieties are of polynomial type.*

Proof. Let $\mathcal{O}(\mathcal{X}) = \mathbb{K}[x_1, \ldots, x_r]/I$ and $\mathcal{O}(\mathcal{Y}) = \mathbb{K}[y_1, \ldots, y_s]/J$. For $A \in \mathbb{K}$-**Alg** and $(a_1, \ldots, a_r) \in \mathcal{X}(A)$ let $f : \mathbb{K}[x_1, \ldots, x_r]/I \longrightarrow A$ with $f(x_i) = a_i$ be the homomorphism obtained from $\mathcal{X}(A) \cong \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x_1, \ldots, x_r]/I, A)$. The natural transformation ϕ is given by composition with a homomorphism $g : \mathbb{K}[y_1, \ldots, y_s]/J \longrightarrow \mathbb{K}[x_1, \ldots, x_r]/I$ hence we get

$$\phi(A) : \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x_1, \ldots, x_r]/I, A) \ni f \mapsto fg \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[y_1, \ldots, y_s]/J, A).$$

Since g is described by $g(y_i) = p_i(x_1, \ldots, x_r) \in \mathbb{K}[x_1, \ldots, x_r]$ we get

$$\begin{aligned}&\phi(A)(a_1, \ldots, a_s) = (fg(y_1), \ldots, fg(y_s))\\ &= (f(p_1(x_1, \ldots, x_r)), \ldots, f(p_s(x_1, \ldots, x_r)))\\ &= (p_1(a_1, \ldots, a_r), \ldots, p_s(a_1, \ldots, a_r)).\end{aligned}$$

□

An analogous statement holds for geometric spaces.

Example 1.1.14. The isomorphism between the affine line (1.1.2) and the parabola is given by the isomorphism $f : \mathbb{K}[x, y]/(y - x^2) \longrightarrow \mathbb{K}[z]$, $f(x) = z$, $f(y) = z^2$ that has the inverse function $f^{-1}(z) = x$. On the affine schemes $\mathbb{A}$, the affine line, and $\mathbb{P}$, the parabola, the induced map is $f : \mathbb{A}(A) \ni a \mapsto (a, a^2) \in \mathbb{P}(A)$ resp. $f^{-1} : \mathbb{P}(A) \ni (a, b) \mapsto a \in \mathbb{A}(A)$.