CHAPTER 1

Commutative and Noncommutative Algebraic Geometry

Introduction

Throughout we will fix a base field $\mathbb{K}$. The reader may consider it as real numbers or complex numbers or any other of his most favorite fields.

A fundamental and powerful tool for geometry is to associate with each space X the algebra of functions $\mathcal{O}(X)$ from X to the base field (of coefficients). The dream of geometry is that this construction is bijective, i.e. that two different spaces are mapped to two different function algebras and that each algebra is the function algebra of some space.

Actually the spaces and the algebras will form a category. There are admissible maps. For algebras it is quite clear what these maps will be. For spaces this is less obvious, partly due to the fact that we did not say clearly what spaces exactly are. Then the *dream of geometry* would be that these two categories, the category of (certain) spaces and the category of (certain) algebras, are dual to each other.

Algebraic geometry, noncommutative geometry, and theoretical physics have as a basis this fundamental idea, the duality of two categories, the category of spaces (state spaces in physics) and the category of function algebras (algebras of observables) in physics. We will present this duality in the 1. chapter. Certainly the type of spaces as well as the type of algebras will have to be specified.

Theoretical physics uses the categories of locally compact Hausdorff spaces and of commutative C^*-algebras. A famous theorem of Gelfand-Naimark says that these categories are duals of each other.

(Affine) algebraic geometry uses a duality between the categories of affine algebraic schemes and of (reduced) finitely generated commutative algebras.

To get the whole framework of algebraic geometry one needs to go to more general spaces by patching affine spaces together. On the algebra side this amounts to considering sheaves of commutative algebras. We shall not pursue this more general approach to algebraic geometry, since generalizations to noncommutative geometry are still in the state of development and incomplete.

Noncommutative geometry uses either (imaginary) noncommutative spaces and not necessarily commutative algebras or (imaginary) noncommutative spaces and not necessarily commutative C^*-algebras.

We will take an approach to the duality between geometry and algebra that heavily uses functorial tools, especially representable functors. The affine (algebraic) spaces we use will be given in the form of sets of common zeros of certain polynomials, where the zeros can be taken in arbitrary (commutative) $\mathbb{K}$-algebras B. So an affine space will consist of many different sets of zeros, depending on the choice of the coefficient algebra B.

We first give a short introduction to commutative algebraic geometry in this setup and develop a duality between the category of affine (algebraic) spaces and the category of (finitely generated) commutative algebras.

Then we will transfer it to the noncommutative situation. The functorial approach to algebraic geometry is not too often used but it lends itself particularly well to the study of the noncommutative situation. Even in that situation one obtains space-like objects.

The chapter will close with a first step to construct automorphism "groups" of noncommutative spaces. Since the construction of inverses presents special problems we will only construct endomorphism "monoids" in this chapter and postpone the study of invertible endomorphisms or automorphisms to the next chapter.

At the end of the chapter you should

- know how to construct an affine scheme from a commutative algebra,
- know how to construct the function algebra of an affine scheme,
- know what a noncommutative space is and know examples of such,
- understand and be able to construct endomorphism quantum monoids of certain noncommutative spaces,
- understand, why endomorphism quantum monoids are not made out of endomorphisms of a noncommutative space.

1. The Principles of Commutative Algebraic Geometry

We will begin with simplest form of (commutative) geometric spaces and see a duality between these very simple "spaces" and certain commutative algebras. This example will show how the concept of a function algebra can be used to fulfill the dream of geometry in this situation. It will also show the functorial methods that will be applied throughout this text. It is a particularly simple example of a duality as mentioned in the introduction. This example will not be used later on, so we will only sketch the proofs for some of the statements.

Example 1.1.1. Consider a set of points without any additional geometric structure. So the geometric space is just a set. We introduce the notion of its algebra of functions.

Let X be a set. Then $\mathbb{K}^X := \mathrm{Map}(X, \mathbb{K})$ is a $\mathbb{K}$-algebra with componentwise addition and multiplication: $(f+g)(x) := f(x) + g(x)$ and $(fg)(x) := f(x)g(x)$. We study this fact in more detail.

The set $\mathbb{K}^X$ considered as a vector space with the addition $(f+g)(x) := f(x)+g(x)$ and the scalar multiplication $(\alpha f)(x) := \alpha f(x)$ defines a representable contravariant functor

$$\mathbb{K}^{-} : \mathbf{Set} \longrightarrow \mathbf{Vec}.$$

This functor is a representable functor represented by $\mathbb{K}$. In fact $\mathbb{K}^h : \mathbb{K}^Y \longrightarrow \mathbb{K}^X$ is a linear map for every map $h : X \longrightarrow Y$ since $\mathbb{K}^h(\alpha f + \beta g)(x) = (\alpha f + \beta g)(h(x)) = \alpha f(h(x)) + \beta g(h(x)) = (\alpha fh + \beta gh)(x) = (\alpha \mathbb{K}^h(f) + \beta \mathbb{K}^h(g))(x)$ hence $\mathbb{K}^h(\alpha f + \beta g) = \alpha \mathbb{K}^h(f) + \beta \mathbb{K}^h(g)$.

Consider the homomorphism $\tau : \mathbb{K}^X \otimes \mathbb{K}^Y \longrightarrow \mathbb{K}^{X \times Y}$, defined by $\tau(f \otimes g)(x, y) := f(x)g(y)$. In order to obtain a unique homomorphism τ defined on the tensor product we have to show that $\tau' : \mathbb{K}^X \times \mathbb{K}^Y \longrightarrow \mathbb{K}^{X \times Y}$ is a bilinear map : $\tau'(f + f', g)(x, y) = (f + f')(x)g(y) = (f(x) + f'(x))g(y) = f(x)g(y) + f'(x)g(y) = (\tau'(f, g) + \tau'(f', g))(x, y)$ gives the additivity in the left hand argument. The additivity in the right hand argument and the bilinearity is checked similarly. One can check that τ is always injective. If X or Y are finite then τ is bijective.

As a special example we obtain a multiplication $\nabla : \mathbb{K}^X \otimes \mathbb{K}^X \xrightarrow{\tau} \mathbb{K}^{X \times X} \xrightarrow{\mathbb{K}^{\Delta}} \mathbb{K}^X$ where $\Delta : X \longrightarrow X \times X$ in **Set** is the diagonal map $\Delta(x) := (x, x)$. Furthermore we get a unit $\eta : \mathbb{K}^{\{*\}} \xrightarrow{\mathbb{K}^{\epsilon}} \mathbb{K}^X$ where $\epsilon : X \longrightarrow \{*\}$ is the unique map into the one element set. One verifies easily that $(\mathbb{K}^X, \eta, \nabla)$ is a $\mathbb{K}$-algebra. Two properties are essential here, the associativity and the unit of $\mathbb{K}$ and the fact that (X, Δ, ϵ) is a "comonoid" in the category **Set**:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "1 \times \Delta"] \\
X \times X \arrow[r, "\Delta \times 1"'] & X \times X \times X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "1_X"] & X \times X \arrow[d, "1\times\epsilon"] \\
X \times X \arrow[r, "\epsilon\times 1"'] & \{*\} \times X \cong X \cong X \times \{*\}.
\end{tikzcd}$$

Since $\mathbb{K}^{-}$ is a functor these two diagrams carry over to the category **Vec** and produce the required diagrams for a $\mathbb{K}$-algebra.

For a map $f : X \longrightarrow Y$ we obtain a homomorphism of algebras $\mathbb{K}^f : \mathbb{K}^Y \longrightarrow \mathbb{K}^X$ because the diagrams

$$\begin{tikzcd}
\mathbb{K}^Y \otimes \mathbb{K}^Y \arrow[r, "\tau"] \arrow[d, "\mathbb{K}^f\otimes\mathbb{K}^f"] & \mathbb{K}^{Y\times Y} \arrow[r, "\mathbb{K}^\Delta"] \arrow[d, "\mathbb{K}^{f\times f}"] & \mathbb{K}^Y \arrow[d, "\mathbb{K}^f"] \\
\mathbb{K}^X \otimes \mathbb{K}^X \arrow[r, "\tau"] & \mathbb{K}^{X\times X} \arrow[r, "\mathbb{K}^\Delta"] & \mathbb{K}^X
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{K}^{\{*\}} \cong \mathbb{K} \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\
\mathbb{K}^Y \arrow[rr, "\mathbb{K}^f"] & & \mathbb{K}^X
\end{tikzcd}$$

commute.

Thus

$$\mathbb{K}^{-} : \mathbf{Set} \longrightarrow \mathbb{K}\text{-}\mathbf{cAlg}$$

is a contravariant functor.

By the definition of the set-theoretic (cartesian) product we know that $\mathbb{K}^X = \prod_X \mathbb{K}$. This identity does not only hold on the set level, it holds also for the algebra structures on $\mathbb{K}^X$ resp. $\prod_X \mathbb{K}$.

We now construct an inverse functor

$$\mathrm{Spec} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}.$$

For each point $x \in X$ there is a maximal ideal m_x of $\prod_X \mathbb{K}$ defined by $m_x := \{f \in \mathrm{Map}(X, \mathbb{K}) | f(x) = 0\}$. If X is a finite set then these are exactly all maximal ideals of $\prod_X \mathbb{K}$. To show this we observe the following. The surjective homomorphism $p_x : \prod_X \mathbb{K} \longrightarrow \mathbb{K}$ has kernel m_x hence m_x is a maximal ideal. If $m \subseteq \prod_X \mathbb{K}$ is a maximal ideal and $a = (\alpha_1, \ldots, \alpha_n) \in m$ then for any $\alpha_i \neq 0$ we get $(0, \ldots, 0, 1_i, 0, \ldots, 0) = (0, \ldots, 0, \alpha_i^{-1}, 0, \ldots, 0)(\alpha_1, \ldots, \alpha_n) \in m$ hence the i-th factor $0 \times \ldots \times \mathbb{K} \times \ldots \times 0$ of $\prod_X \mathbb{K}$ is in m. So the elements $a \in m$ must have at least one common component $\alpha_j = 0$ since $m \neq \mathbb{K}$. But more than one such a component is impossible since we would get zero divisors in the residue class algebra. Thus $m = m_x$ where $x \in X$ is the j-th elements of the set.

One can easily show more namely that the ideals m_x are precisely all prime ideals of $\mathrm{Map}(X, \mathbb{K})$.

With each commutative algebra A we can associate the set $\mathrm{Spec}(A)$ of all prime ideals of A. That defines a functor Spec: $\mathbb{K}$-**Alg** $\longrightarrow$ **Set**. Applied to algebras of the form $\mathbb{K}^X = \prod_X \mathbb{K}$ with a finite set X this functor recovers X as $X \cong \mathrm{Spec}(\mathbb{K}^X)$. Thus the dream of geometry is satisfied in this particular example.

The above example shows that we may hope to gain some information on the space (set) X by knowing its algebra of functions $\mathbb{K}^X$ and applying the functor Spec to it. For finite sets and certain algebras the functors $\mathbb{K}^{-}$ and Spec actually define a category duality. We are going to expand this duality to larger categories.

We shall carry some geometric structure into the sets X and will study the connection between these geometric spaces and their algebras of functions. For this purpose we will describe sets of points by their coordinates. Examples are the circle or the parabola. More generally the geometric spaces we are going to consider are so called affine schemes described by polynomial equations. We will see that such geometric spaces are completely described by their algebras of functions. Here the Yoneda Lemma will play a central rôle.

We will, however, take a different approach to functions algebras and geometric spaces, than one does in algebraic geometry. We use the functorial approach, which lends itself to an easier access to the principles of noncommutative geometry. We will define geometric spaces as certain functors from the category of commutative algebras to the category of sets. These sets will have a strong geometrical meaning. The functors will associate with each algebra A the set of points of a "geometric variety", where the points have coordinates in the algebra A.

Definition 1.1.2. The functor $\mathbb{A} = \mathbb{A}^1 : \mathbb{K}$-**cAlg** $\longrightarrow$ **Set** (the underlying functor) that associates with each commutative $\mathbb{K}$-algebra A its space (set) of points (elements) A is called the *affine line*.

Lemma 1.1.3. *The functor "affine line" is a representable functor.*

Proof. By Lemma 2.3.5 the representing object is $\mathbb{K}[x]$. Observe that it is unique up to isomorphism. $\square$

Definition 1.1.4. The functor $\mathbb{A}^2 : \mathbb{K}$-**cAlg** $\longrightarrow$ **Set** that associates with each commutative algebra A the space (set) of points (elements) of the plane A^2 is called the *affine plane*.

Lemma 1.1.5. *The functor "affine plane" is a representable functor.*

Proof. Similar to Lemma 2.3.9 the representing object is $\mathbb{K}[x_1, x_2]$. This algebra is unique up to isomorphism. $\square$

Let $p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$ be a family of polynomials. We want to consider the (geometric) variety of zeros of these polynomials. Observe that $\mathbb{K}$ may not contain sufficiently many zeros for these polynomials. Thus we are going to admit zeros in extension fields of $\mathbb{K}$ or more generally in arbitrary commutative $\mathbb{K}$-algebras.

In the following rather simple buildup of commutative algebraic geometry, the reader should carefully verify in which statements and proofs the commutativity is really needed. Most of the following will be verbally generalized to not necessarily commutative algebras.

Definition 1.1.6. Given a set of polynomials $\{p_1, \ldots, p_m\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$. The functor $\mathcal{X}$ that associates with each commutative algebra A the set $\mathcal{X}(A)$ of zeros of the polynomials (p_i) in A^n is called an *affine algebraic variety* or an *affine scheme* (in $\mathbb{A}^n$) with defining polynomials $p_1, \ldots, p_m$. The elements in $\mathcal{X}(A)$ are called the *A-points* of $\mathcal{X}$.

Theorem 1.1.7. *The affine scheme $\mathcal{X}$ in $\mathbb{A}^n$ with defining polynomials $p_1, \ldots, p_m$ is a representable functor with representing algebra*

$$\mathcal{O}(\mathcal{X}) := \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m),$$

called the affine algebra *of the functor $\mathcal{X}$.*

Proof. First we show that the affine scheme $\mathcal{X} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ with the defining polynomials $p_1, \ldots, p_m$ is a functor. Let $f : A \longrightarrow B$ be a homomorphism of commutative algebras. The induced map $f^n : A^n \longrightarrow B^n$ defined by application of f on the components restricts to $\mathcal{X}(A) \subseteq A^n$ as $\mathcal{X}(f) : \mathcal{X}(A) \longrightarrow \mathcal{X}(B)$. This map is well-defined for let $(a_1, \ldots, a_n) \in \mathcal{X}(A)$ be a zero for all polynomials $p_1, \ldots, p_m$ then $p_i(f(a_1), \ldots, f(a_n)) = f(p_i(a_1, \ldots, a_n)) = f(0) = 0$ for all i hence $f^n(a_1, \ldots, a_n) = (f(a_1), \ldots, f(a_n)) \in B^n$ is a zero for all polynomials. Thus $\mathcal{X}(f) : \mathcal{X}(A) \longrightarrow \mathcal{X}(B)$ is well-defined. Functoriality of $\mathcal{X}$ is clear now.

Now we show that $\mathcal{X}$ is representable by $\mathcal{O}(\mathcal{X}) = \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$. Observe that $(p_1, \ldots, p_m)$ denotes the (two-sided) ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials $p_1, \ldots, p_m$. We know that each n-tupel $(a_1, \ldots, a_n) \in A^n$ uniquely determines an algebra homomorphism $f : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow A$ by $f(x_1) = a_1, \ldots, f(x_n) = a_n$. (The polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ in $\mathbb{K}\text{-}\mathbf{cAlg}$ is free over the set $\{x_1, \ldots, x_n\}$, or $\mathbb{K}[x_1, \ldots, x_n]$ together with the embedding $\iota : \{x_1, \ldots, x_n\} \longrightarrow \mathbb{K}[x_1, \ldots, x_n]$ is a couniversal solution of the problem given by the underlying functor $\mathbb{A} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ and the set $\{x_1, \ldots, x_n\} \in \mathbf{Set}$.) This homomorphism of algebras maps polynomials $p(x_1, \ldots, x_n)$ into $f(p) = p(a_1, \ldots, a_n)$. Hence $(a_1, \ldots, a_n)$ is a common zero of the polynomials $p_1, \ldots, p_m$ if and only if $f(p_i) = p_i(a_1, \ldots, a_n) = 0$, i.e. $p_1, \ldots, p_m$ are in the kernel of f. This happens if and only if f vanishes on the ideal $(p_1, \ldots, p_m)$ or in other word can be factorized through the residue class map

$$\nu : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$$

This induces a bijection

$$\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(\mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m), A) \ni f \mapsto (f(x_1), \ldots, f(x_n)) \in \mathcal{X}(A).$$

Now it is easy to see that this bijection is a natural isomorphism (in A). □

If no polynomials are given for the above construction, then the functor under this construction is the affine space $\mathbb{A}^n$ of dimension n. By giving polynomials the functor $\mathcal{X}$ becomes a subfunctor of $\mathbb{A}^n$, because it defines subsets $\mathcal{X}(A) \subseteq \mathbb{A}^n(A) = A^n$. Both functors are representable functors. The embedding is induced by the homomorphism of algebras $\nu : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$.

Problem 1.1.1. 1. Determine the affine algebra of the functor "unit circle" S^1 in $\mathbb{A}^2$.
2. Determine the affine algebra of the functor "unit sphere" S^{n-1} in $\mathbb{A}^n$.
3. Let $\mathcal{X}$ denote the plane curve $y = x^2$. Then $\mathcal{X}$ is isomorphic to the affine line.
4. Let $\mathcal{Y}$ denote the plane curve $xy = 1$. Then $\mathcal{Y}$ is not isomorphic to the affine line. (Hint: An isomorphism $\mathbb{K}[x, x^{-1}] \longrightarrow \mathbb{K}[y]$ sends x to a polynomial $p(y)$ which must be invertible. Consider the highest coefficient of $p(y)$ and show that $p(y) \in \mathbb{K}$. But that means that the map cannot be bijective.)
5. Let $\mathbb{K} = \mathbb{C}$ be the field of complex numbers. Show that the unit functor $U : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ in Lemma 2.3.7 is naturally isomorphic to the unit circle functor S^1. (Hint: There is an algebra isomorphism between the representing algebras $\mathbb{K}[e, e^{-1}]$ and $\mathbb{K}[c, s]/(c^2 + s^2 - 1)$.)
6. * Let $\mathbb{K}$ be an algebraically closed field. Let p be an irreducible square polynomial in $\mathbb{K}[x, y]$. Let $\mathcal{Z}$ be the conic section defined by p with the affine algebra $\mathbb{K}[x, y]/(p)$. Show that $\mathcal{Z}$ is naturally isomorphic either to $\mathcal{X}$ or to $\mathcal{Y}$ from parts 3. resp. 4.

Remark 1.1.8. Affine algebras of affine schemes are finitely generated commutative algebras and any such algebra is an affine algebra of some affine scheme, since $A \cong \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$ (Hilbert basis theorem).

The polynomials $p_1, \ldots, p_m$ are not uniquely determined by the affine algebra of an affine scheme. Not even the ideal generated by the polynomials in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ is uniquely determined. Also the number of variables $x_1, \ldots, x_n$ is not uniquely determined.

The $\mathbb{K}$-points $(\alpha_1, \ldots, \alpha_n) \in \mathcal{X}(\mathbb{K})$ of an affine scheme $\mathcal{X}$ (with coefficients in the base field $\mathbb{K}$) are called *rational points*. They do not suffice to completely describe the affine scheme.

Let for example $\mathbb{K} = \mathbb{R}$ the set of rational numbers. If $\mathcal{X}$ and $\mathcal{Y}$ are affine schemes with affine algebras $\mathcal{O}(\mathcal{X}) := \mathbb{K}[x, y]/(x^2 + y^2 + 1)$ and $\mathcal{O}(\mathcal{Y}) := \mathbb{K}[x]/(x^2 + 1)$ then both schemes have no rational points. The scheme $\mathcal{Y}$, however, has exactly two complex points (with coefficients in the field $\mathbb{C}$ of complex numbers) and the scheme $\mathcal{X}$ has infinitely many complex points, hence $\mathcal{X}(\mathbb{C}) \not\cong \mathcal{Y}(\mathbb{C})$. This does not result from the embeddings into different spaces $\mathbb{A}^2$ resp. $\mathbb{A}^1$. In fact we also have $\mathcal{O}(\mathcal{Y}) = \mathbb{K}[x]/(x^2 + 1) \cong \mathbb{K}[x, y]/(x^2 + 1, y)$, so $\mathcal{Y}$ can be considered as an affine scheme in $\mathbb{A}^2$.

Since each affine scheme $\mathcal{X}$ is isomorphic to the functor $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(\mathcal{O}(\mathcal{X}), \text{-})$ we will henceforth identify these two functors, thus removing annoying isomorphisms.

Definition 1.1.9. Let $\mathbb{K}$-**Aff** denote the category of all commutative finitely generated (or affine cf. 1.1.8) $\mathbb{K}$-algebras. An *affine algebraic variety* is a representable functor $\mathbb{K}\text{-}\mathbf{Aff}(A, \text{-}) : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathbf{Set}$. The affine algebraic varieties together with the natural transformations form the *category of affine algebraic varieties* $\mathrm{Var}(\mathbb{K})$ over $\mathbb{K}$. The functor that associates with each affine algebra A its affine algebraic variety represented by A is denoted by $\mathrm{Spec} : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathrm{Var}(\mathbb{K})$, $\mathrm{Spec}(A) = \mathbb{K}\text{-}\text{-}\mathbf{Aff}(A, \text{-})$.

By the Yoneda Lemma the functor

$$\mathrm{Spec} : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathrm{Var}(\mathbb{K})$$

is an antiequivalence (or duality) of categories with inverse functor

$$\mathcal{O} : \mathrm{Var}(\mathbb{K}) \longrightarrow \mathbb{K}\text{-}\mathbf{Aff}\,.$$

An affine algebraic variety is completely described by its affine algebra $\mathcal{O}(\mathcal{X})$. Thus the dream of geometry is realized.

Arbitrary (not necessarily finitely generated) commutative algebras also define representable functors (defined on the category of all commutative algebras). Thus we also have "infinite dimensional" varieties which we will call *geometric spaces* or *affine varieties*. We denote their category by $\mathrm{Geom}(\mathbb{K})$ and get a commutative diagram

$$\begin{tikzcd}
\mathbb{K}\text{-}\mathbf{Aff} \arrow[r, "\mathrm{Spec}"] \arrow[d] & \mathrm{Var}(\mathbb{K}) \arrow[d] \\
\mathbb{K}\text{-}\mathbf{cAlg} \arrow[r, "\cong^{o}"'] & \mathrm{Geom}(\mathbb{K})
\end{tikzcd}$$

We call the representable functors $\mathcal{X} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ *geometric spaces* or *affine varieties*, and the representable functors $\mathcal{X} : \mathbb{K}\text{-}\mathbf{Aff} \longrightarrow \mathbf{Set}$ *affine schemes* or *affine algebraic varieties*. This is another realization of the dream of geometry.

The geometric spaces can be viewed as sets of zeros in arbitrary commutative $\mathbb{K}$-algebras B of arbitrarily many polynomials with arbitrarily many variables. The function algebra of $\mathcal{X}$ will be called the *affine algebra* of $\mathcal{X}$ in both cases.

Example 1.1.10. A somewhat less trivial example is the state space of a circular pendulum (of length 1). The location is in $L = \{(a,b) \in A^2 | a^2 + b^2 = 1\}$, the momentum is in $M = \{p \in A\}$ which is a straight line. So the whole geometric space for the pendulum is $(L \times M)(A) = \{(a,b,p) | a,b,p \in A; a^2 + b^2 = 1\}$. This geometric space is represented by $\mathbb{K}[x,y,z]/(x^2 + y^2 - 1)$ since

$$(L \times M)(A) = \{(a,b,p) | a,b,p \in A; a^2 + b^2 = 1\} \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x,y,z]/(x^2 + y^2 - 1), A).$$

The two antiequivalences of categories above give rise to the question for the function algebra. If a representable functor $\mathcal{X} = \mathbb{K}\text{-}\mathbf{cAlg}(A, \text{-})$ is viewed as geometric sets of zeros of certain polynomials, i.e. as spaces with coordinates in arbitrary commutative algebras B, (plus functorial behavior), then it is not clear why the representing algebra A should be anything like an algebra of functions on these geometric sets. It is not even clear where these functions should assume their values. Only if we can show that A can be viewed as a reasonable algebra of functions, we should talk about a realization of the dream of geometry. But this will be done in the following theorem. We will consider functions as maps (coordinate functions) from the geometric set $\mathcal{X}(B)$ to the set of coordinates B, maps that are natural in B. Such coordinate functions are just natural transformations from $\mathcal{X}$ to the underlying functor $\mathbb{A}$.

Theorem 1.1.11. *Let $\mathcal{X}$ be a geometric space with the affine algebra $A = \mathcal{O}(\mathcal{X})$. Then $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ as $\mathbb{K}$-algebras, where $\mathbb{A} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$ is the underlying functor or affine line. The isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ induces a natural transformation $A \times \mathcal{X}(B) \longrightarrow B$ (natural in B).*

Proof. First we define an isomorphism between the sets A and $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$. Because of $\mathcal{X} = \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(A, \text{-}) =: \mathbb{K}\text{-}\mathbf{cAlg}(A, \text{-})$ and $\mathbb{A} = \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{cAlg}}(\mathbb{K}[x], \text{-}) =: \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \text{-})$ the Yoneda Lemma gives us

$$\mathrm{Nat}(\mathcal{X}, \mathbb{A}) = \mathrm{Nat}(\mathbb{K}\text{-}\mathbf{cAlg}(A, \text{-}), \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], \text{-})) \cong \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x], A) = \mathbb{A}(A) \cong A$$

on the set level. Let $\phi : A \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ denote the given isomorphism. ϕ is defined by $\phi(a)(B)(p)(x) := p(a)$. By the Yoneda Lemma its inverse is given by $\phi^{-1}(\alpha := \alpha((A)(1)(x)$.

$\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ carries an algebra structure given by the algebra structure of the coefficients. For a coefficient algebra B, a B-point $p : A \longrightarrow B$ in $\mathcal{X}(B) = \mathbb{K}\text{-}\mathbf{Alg}(A, B)$, and $\alpha, \beta \in \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ we have $\alpha(B)(p) \in \mathbb{A}(B) = B$. Hence $(\alpha + \beta)(B)(p) := (\alpha(B) + \beta(B))(p) = \alpha(B)(p) + \beta(B)(p)$ and $(\alpha \cdot \beta)(B)(p) := (\alpha(B) \cdot \beta(B))(p) = \alpha(B)(p) \cdot \beta(B)(p)$ make $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ an algebra.

Let a be an arbitrary element in A. By the isomorphism given above this element induces an algebra homomorphism $g_a : \mathbb{K}[x] \longrightarrow A$ mapping x onto a. This algebra homomorphism induces the natural transformation $\phi(a) : \mathcal{X} \longrightarrow \mathbb{A}$. On the B-level it is just the composition with g_a, i.e. $\phi(a)(B)(p) = (\mathbb{K}[x] \xrightarrow{g_a} A \xrightarrow{p} B)$. Since such a homomorphism is completely described by the image of x we get $\phi(a)(B)(p)(x) = p(a)$. To compare the algebra structures of A and $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ let $a, a' \in A$. We have $\phi(a)(B)(p)(x) = p(a)$ and $\phi(a')(B)(p)(x) = p(a')$, hence $\phi(a + a')(B)(p)(x) = p(a + a') = p(a) + p(a') = \phi(a)(B)(p)(x) + \phi(a')(B)(p)(x) = (\phi(a)(B)(p) + \phi(a')(B)(p))(x) = (\phi(a)(B) + \phi(a')(B))(p)(x) = (\phi(a) + \phi(a'))(B)(p)(x)$. Analogously we get $\phi(aa')(B)(p)(x) = p(aa') = p(a)p(a') = (\phi(a) \cdot \phi(a'))(B)(p)(x)$, and thus $\phi(a + a') = \phi(a) + \phi(a')$ and $\phi(aa') = \phi(a) \cdot \phi(a')$. Hence addition and multiplication in $\mathrm{Nat}(\mathcal{X}, \mathbb{A})$ are defined by the addition and the multiplication of the values $p(a) + p(a')$ resp. $p(a)p(a')$.

We describe the action $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ of A on $\mathcal{X}(B)$. Let $p : A \longrightarrow B$ be a B-point in $\mathbb{K}\text{-}\mathbf{cAlg}(A, B) = \mathcal{X}(B)$. For each $a \in A$ the image $\phi(a) : \mathcal{X} \longrightarrow \mathbb{A}$

is a natural transformation hence we have maps $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ such that $\psi(B)(a,p) = p(a)$. Finally each homomorphism of algebras $f : B \longrightarrow B'$ induces a commutative diagram

$$\begin{tikzcd} A \times \mathcal{X}(B) \arrow[r, "\psi(B)"] \arrow[d, "A \times \mathcal{X}(f)"'] & B \arrow[d, "f"] \\ A \times \mathcal{X}(B') \arrow[r, "\psi(B')"'] & B' \end{tikzcd}$$

Thus $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ is a natural transformation. $\square$

Remark 1.1.12. Observe that the isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ induces a natural transformation $A \times \mathcal{X}(B) \longrightarrow B$ (natural in B). In particular the affine algebra A can be viewed as the set of functions from the set of B-points $\mathcal{X}(B)$ into the "base" ring B (functions which are natural in B). In this sense the algebra A may be considered as function algebra of the geometric space $\mathcal{X}$. Thus we will call A the *function algebra* of $\mathcal{X}$.

One can show that the algebra A is universal with respect to the property, that for each commutative algebra D and each natural transformation $\rho : D \times \mathcal{X}(\text{-}) \longrightarrow \text{-}$ there is a unique homomorphism of algebras $f : D \longrightarrow A$, such that the triangle

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f \times 1_{\mathcal{X}(B)}"'] \arrow[dr, "\rho(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes. We will show this result later on for noncommutative algebras. The universal property implies that the function algebra A of an geometric space $\mathcal{X}$ is unique up to isomorphism.

Let $\mathcal{X}$ be an geometric space with function algebra $A = \mathcal{O}(\mathcal{X})$. If $p : A \longrightarrow \mathbb{K}$ is a rational point of $\mathcal{X}$, i.e. a homomorphism of algebras, then $\mathrm{Im}(p) = \mathbb{K}$ hence $\mathrm{Ker}(p)$ is a maximal ideal of A of codimension 1. Conversely let $\mathfrak{m}$ be a maximal ideal of A of codimension 1 then this defines a rational point $p : A \longrightarrow A/\mathfrak{m} \cong \mathbb{K}$. If $\mathbb{K}$ is algebraicly closed and $\mathfrak{m}$ an arbitrary maximal ideal of A, then $A/\mathfrak{m}$ is a finitely generated $\mathbb{K}$-algebra and a field extension of $\mathbb{K}$, hence it coincides with $\mathbb{K}$. Thus the codimension of $\mathfrak{m}$ is 1. The set of maximal ideals of A is called the *maximal spectrum* $\mathrm{Spec}_m(A)$. This is the approach of algebraic geometry to recover the geometric space of (rational) points from the function algebra A. We will not follow this approach since it does not easily extend to noncommutative geometry.

Problem 1.1.2. Let $\mathcal{X}$ be an affine scheme with affine algebra

$$A = \mathbb{K}[x_1, \ldots, x_n]/(p_1, \ldots, p_m).$$

Define "coordinate functions" $q_i : \mathcal{X}(B) \longrightarrow B$ which describe the coordinates of B-points and identify these coordinate functions with elements of A.

Now we will study morphisms between geometric spaces.

Theorem 1.1.13. *Let $\mathcal{X} \subseteq \mathbb{A}^r$ and $\mathcal{Y} \subseteq \mathbb{A}^s$ be affine algebraic varieties and let $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a natural transformation. Then there are polynomials*

$$p_1(x_1, \ldots, x_r), \ldots, p_s(x_1, \ldots, x_r) \in \mathbb{K}[x_1, \ldots, x_r],$$

such that

$$\phi(A)(a_1, \ldots, a_r) = (p_1(a_1, \ldots, a_r), \ldots, p_s(a_1, \ldots, a_r)),$$

*for all $A \in \mathbb{K}$-**Aff** and all $(a_1, \ldots, a_r) \in \mathcal{X}(A)$, i.e. the morphisms between affine algebraic varieties are of polynomial type.*

Proof. Let $\mathcal{O}(\mathcal{X}) = \mathbb{K}[x_1, \ldots, x_r]/I$ and $\mathcal{O}(\mathcal{Y}) = \mathbb{K}[y_1, \ldots, y_s]/J$. For $A \in \mathbb{K}$-**Alg** and $(a_1, \ldots, a_r) \in \mathcal{X}(A)$ let $f : \mathbb{K}[x_1, \ldots, x_r]/I \longrightarrow A$ with $f(x_i) = a_i$ be the homomorphism obtained from $\mathcal{X}(A) \cong \mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}[x_1, \ldots, x_r]/I, A)$. The natural transformation ϕ is given by composition with a homomorphism $g : \mathbb{K}[y_1, \ldots, y_s]/J \longrightarrow \mathbb{K}[x_1, \ldots, x_r]/I$ hence we get

$$\phi(A) : \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[x_1, \ldots, x_r]/I, A) \ni f \mapsto fg \in \mathbb{K}\text{-}\mathbf{cAlg}(\mathbb{K}[y_1, \ldots, y_s]/J, A).$$

Since g is described by $g(y_i) = p_i(x_1, \ldots, x_r) \in \mathbb{K}[x_1, \ldots, x_r]$ we get

$$\begin{aligned} &\phi(A)(a_1, \ldots, a_s) = (fg(y_1), \ldots, fg(y_s)) \\ &= (f(p_1(x_1, \ldots, x_r)), \ldots, f(p_s(x_1, \ldots, x_r))) \\ &= (p_1(a_1, \ldots, a_r), \ldots, p_s(a_1, \ldots, a_r)). \end{aligned}$$

$\square$

An analogous statement holds for geometric spaces.

Example 1.1.14. The isomorphism between the affine line (1.1.2) and the parabola is given by the isomorphism $f : \mathbb{K}[x, y]/(y - x^2) \longrightarrow \mathbb{K}[z]$, $f(x) = z$, $f(y) = z^2$ that has the inverse function $f^{-1}(z) = x$. On the affine schemes $\mathbb{A}$, the affine line, and $\mathbb{P}$, the parabola, the induced map is $f : \mathbb{A}(A) \ni a \mapsto (a, a^2) \in \mathbb{P}(A)$ resp. $f^{-1} : \mathbb{P}(A) \ni (a, b) \mapsto a \in \mathbb{A}(A)$.

2. Quantum Spaces and Noncommutative Geometry

Now we come to noncommutative geometric spaces and their function algebras. Many of the basic principles of commutative algebraic geometry as introduced in 1.1 carry over to noncommutative geometry. Our main aim, however, is to study the symmetries (automorphisms) of noncommutative spaces which lead to the notion of a quantum group.

Since the construction of noncommutative geometric spaces has deep applications in theoretical physics we will also call these spaces quantum spaces.

Definition 1.2.1. Let A be a (not necessarily commutative) $\mathbb{K}$-algebra. Then the functor $\mathcal{X} := \mathbb{K}\text{-}\mathbf{Alg}(A,\text{-}) : \mathbb{K}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ represented by A is called *(affine) noncommutative (geometric) space* or *quantum space*. The elements of $\mathbb{K}\text{-}\mathbf{Alg}(A,B)$ are called *B-points* of $\mathcal{X}$. A *morphism of noncommutative spaces* $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a natural transformation.

This definition implies immediately

Corollary 1.2.2. *The noncommutative spaces form a category* **QS** *that is dual to the category of* $\mathbb{K}$*-algebras.*

Remark 1.2.3. Thus one often calls the dual category $\mathbb{K}\text{-}\mathbf{Alg}^{op}$ category of noncommutative spaces.

If A is a finitely generated algebra then it may be considered as a residue class algebra $A \cong \mathbb{K}\langle x_1, \ldots, x_n\rangle/I$ of a polynomial algebra in noncommuting variables (cf. A.6). If $I = (p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n))$ is the two-sided ideal generated by the polynomials $p_1, \ldots, p_m$ then the sets $\mathbb{K}\text{-}\mathbf{Alg}(A,B)$ can be considered as sets of zeros of these polynomials in B^n. In fact, we have $\mathbb{K}\text{-}\mathbf{Alg}(\mathbb{K}\langle x_1, \ldots, x_n\rangle, B) \cong \mathrm{Map}(\{x_1, \ldots, x_n\}, B) = B^n$. Thus $\mathbb{K}\text{-}\mathbf{Alg}(A,B)$ can be considered as the set of those homomorphisms of algebras from $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ to B that vanish on the ideal I or as the set of zeros of these polynomials in B^n.

Similar to Theorem 1.1.13 one shows also in the noncommutative case that morphisms between noncommutative spaces are described by polynomials.

The Theorem 1.1.11 on the operation of the affine algebra $A = \mathcal{O}(\mathcal{X})$ on $\mathcal{X}$ as function algebra can be carried over to the noncommutative case as well: the natural transformation $\psi(B) : A \times \mathcal{X}(B) \longrightarrow B$ (natural in B) is given by $\psi(B)(a,p) := p(a)$ and comes from the isomorphism $A \cong \mathrm{Nat}(\mathcal{X}, \mathbb{A})$.

Now we come to a claim on the function algebra A that we did not prove in the commutative case, but that holds in the commutative as well as in the noncommutative situation.

Lemma 1.2.4. *Let D be a set and $\phi : D \times \mathcal{X}(\text{-}) \longrightarrow \mathbb{A}(\text{-})$ be a natural transformation. Then there exists a unique map $f : D \longrightarrow A$ such that the diagram*

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f\times 1"'] \arrow[dr, "\phi(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes.

Proof. Let $\phi : D \times \mathcal{X} \longrightarrow \mathbb{A}$ be given. We first define a map $f' : D \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$ by $f'(d)(B)(p) := \phi(B)(d, p)$.

We claim that $f'(d) : \mathcal{X} \longrightarrow \mathbb{A}$ is a natural transformation. Observe that the diagram

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[r, "\phi(B)"] \arrow[d, "D \times \mathcal{X}(g)"'] & \mathbb{A}(B) = B \arrow[d, "g"] \\ D \times \mathcal{X}(B') \arrow[r, "\phi(B')"'] & \mathbb{A}(B') = B' \end{tikzcd}$$

commutes for any $g : B \longrightarrow B'$, since ϕ is a natural transformation. Thus the diagram

$$\begin{tikzcd} \mathcal{X}(B) \arrow[r, "f'(d)(B)"] \arrow[d, "\mathcal{X}(g)"'] & \mathbb{A}(B) = B \arrow[d, "g"] \\ \mathcal{X}(B') \arrow[r, "f'(d)(B')"'] & \mathbb{A}(B') = B' \end{tikzcd}$$

commutes since

$$\begin{aligned} (g \circ f'(d)(B))(p) &= (g \circ \phi(B))(d, p) \\ &= \phi(B') \circ (1 \times \mathcal{X}(g))(d, p) \\ &= \phi(B')(d, \mathcal{X}(g)(p)) \\ &= f'(d)(B')(\mathcal{X}(g)(p)). \end{aligned}$$

Hence $f'(d) \in \mathrm{Nat}(X, \mathbb{A})$ and $f' : D \longrightarrow \mathrm{Nat}(\mathcal{X}, \mathbb{A})$.

Now we define $f : D \longrightarrow A$ as $D \xrightarrow{f'} \mathrm{Nat}(\mathcal{X}, \mathbb{A}) \cong A$. By using the isomorphism from 1.1.11 we get $f(d) = f'(d)(A)(1)$. (Actually we get $f(d) = f'(d)(A)(1)(x)$ but we identify $\mathbb{A}(B)$ and B by $\mathbb{A}(B) \ni p \mapsto p(x) \in B$.)

Then we get

$$\begin{aligned}
\psi(B)(f\times 1)(d,p) &= \psi(B)(f(d),p)\\
&= \psi(B)(f'(d)(A)(1)(x),p) \text{ (by definition of } f)\\
&= p\circ f'(d)(A)(1) \text{ (since we may omit } x)\\
&= p\circ \phi(A)(d,1) \text{ (by definition of } f')\\
&= \phi(B)(D\times \mathcal{X}(p))(d,1) \text{ (since } \phi \text{ is a natural transformation)}\\
&= \phi(B)(d,p).
\end{aligned}$$

Hence the diagram in the Lemma commutes.

To show the uniqueness of f let $g : D \longrightarrow A$ be a homomorphism such that $\psi(B)(g\times 1) = \phi(B)$. Then we have

$$f(d) = f'(d)(A)(1) = \phi(A)(d,1) = \psi(A)(g\times 1)(d,1) = \psi(A)(f(d),1) = 1\circ g(d) = g(d)$$

hence $f = g$. □

Problem 1.2.3. ***Definition:*** Let D be an algebra. A natural transformation $\phi : D\times\mathcal{X} \longrightarrow \mathbb{A}$ is called an *algebra action* if $\phi(B)(\text{-},p) : D \longrightarrow \mathbb{A}(B) = B$ is an algebra homomorphism for all B and all $p\in\mathcal{X}(B)$.

Lemma: The natural transformation $\psi : A\times\mathcal{X} \longrightarrow \mathbb{A}$ is an algebra action.

Theorem: Let D be an algebra and $\phi : D\times\mathcal{X}(\text{-}) \longrightarrow \mathbb{A}(\text{-})$ be an algebra action. Then there exists a unique algebra homomorphism $f : D \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
D\times\mathcal{X}(B) \arrow[d, "f\times 1"'] \arrow[dr, "\phi(B)"] & \\
A\times\mathcal{X}(B) \arrow[r, "\psi(B)"'] & B
\end{tikzcd}$$

commutes.

Definition 1.2.5. The noncommutative space $\mathbb{A}_q^{2|0}$ with the function algebra

$$\mathcal{O}(\mathbb{A}_q^{2|0}) := \mathbb{K}\langle x,y\rangle/(xy - q^{-1}yx)$$

with $q\in\mathbb{K}\setminus\{0\}$ is called the *(deformed) quantum plane*. The noncommutative space $\mathbb{A}_q^{0|2}$ with the function algebra

$$\mathcal{O}(\mathbb{A}_q^{0|2}) := \mathbb{K}\langle \xi,\eta\rangle/(\xi^2,\eta^2,\xi\eta + q\eta\xi)$$

is called the *dual (deformed) quantum plane*. We have

$$\mathbb{A}_q^{2|0}(A) = \left\{ \begin{pmatrix} x\\ y\end{pmatrix} \middle| x,y\in A; xy = q^{-1}yx \right\}$$

and

$$\mathbb{A}_q^{0|2}(A) = \left\{ \begin{pmatrix} \xi, & \eta\end{pmatrix} \middle| \xi,\eta\in A; \xi^2 = 0, \eta^2 = 0, \xi\eta = -q\eta\xi \right\}.$$

Definition 1.2.6. Let $\mathcal{X}$ be a noncommutative space with function algebra A and let $\mathcal{X}_c$ be the restriction of the functor $\mathcal{X} : \mathbb{K}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ to the category of commutative algebras: $\mathcal{X}_c : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbf{Set}$. Then we call $\mathcal{X}_c$ the *commutative part* of the noncommutative space $\mathcal{X}$.

Lemma 1.2.7. *The commutative part $\mathcal{X}_c$ of a noncommutative space $\mathcal{X}$ is an affine variety.*

PROOF. The underlying functor $\mathbb{A} : \mathbb{K}\text{-}\mathbf{cAlg} \longrightarrow \mathbb{K}\text{-}\mathbf{Alg}$ has a left adjoint functor $\mathbb{K}\text{-}\mathbf{Alg} \ni A \mapsto A/[A,A] \in \mathbb{K}\text{-}\mathbf{cAlg}$ where $[A,A]$ denotes the two-sided ideal of A generated by the elements $ab - ba$. In fact for each homomorphism of algebras $f : A \longrightarrow B$ with a commutative algebra B there is a factorization through $A/[A,A]$ since f vanishes on the elements $ab - ba$.

Hence if $A = \mathcal{O}(\mathcal{X})$ is the function algebra of $\mathcal{X}$ then $A/[A,A]$ is the representing algebra for $\mathcal{X}_c$. $\square$

Remark 1.2.8. For any commutative algebra (of coefficients) B the spaces $\mathcal{X}$ and $\mathcal{X}_c$ have the same B-points: $\mathcal{X}(B) = \mathcal{X}_c(B)$. The two spaces differ only for noncommutative algebras of coefficients. In particular for commutative fields B as algebras of coefficients the quantum plane $\mathbb{A}_q^{2|0}$ has only B-points on the two axes since the function algebra $\mathbb{K}\langle x, y\rangle/(xy - q^{-1}yx, xy - yx) \cong K[x,y]/(xy)$ defines only B-points (b_1, b_2) where at least one of the coefficients is zero.

Problem 1.2.4. Let S_3 be the symmetric group and $A := \mathbb{K}[S_3]$ be the group algebra on S_3. Describe the points of $\mathcal{X}(B) = \mathbb{K}\text{-}\mathbf{Alg}(A, B)$ as a subspace of $\mathbb{A}^2(B)$. What is $\mathcal{X}_c(B)$ and what is the affine algebra of $\mathcal{X}_c$?

To understand how Hopf algebras fit into the context of noncommutative spaces we have to better understand the tensor product in $\mathbb{K}\text{-}\mathbf{Alg}$.

Definition 1.2.9. Let $A = \mathcal{O}(\mathcal{X})$ and $A' = \mathcal{O}(\mathcal{Y})$ be the function algebras of the noncommutative spaces $\mathcal{X}$ resp. $\mathcal{Y}$. Two B-points $p : A \longrightarrow B$ in $\mathcal{X}(B)$ and $p' : A' \longrightarrow B$ in $\mathcal{Y}(B)$ are called *commuting points* if we have for all $a \in A$ and all $a' \in A'$

$$p(a)p'(a') = p'(a')p(a),$$

i.e. if the images of the two homomorphisms p and p' commute.

Remark 1.2.10. To show that the points p and p' commute, it is sufficient to check that the images of the algebra generators $p(x_1), \dots, p(x_m)$ commute with the images of the algebra generators $p'(y_1), \dots, p'(y_n)$ under the multiplication. This means that we have

$$b_i b'_j = b'_j b_i$$

for the B-points $(b_1, \dots, b_m) \in \mathcal{X}(B)$ and $(b'_1, \dots, b'_n) \in \mathcal{Y}(B)$.

Definition 1.2.11. The functor

$$(\mathcal{X} \perp \mathcal{Y})(B) := \{(p,p') \in \mathcal{X}(B) \times \mathcal{Y}(B) | p, p' \text{ commute}\}$$

is called the *orthogonal product* of the noncommutative spaces $\mathcal{X}$ and $\mathcal{Y}$.

Remark 1.2.12. Together with $\mathcal{X}$ and $\mathcal{Y}$ the orthogonal product $\mathcal{X} \perp \mathcal{Y}$ is again a functor, since homomorphisms $f : B \longrightarrow B'$ are compatible with the multiplication and thus preserve commuting points. Hence $\mathcal{X} \perp \mathcal{Y}$ is a subfunctor of $\mathcal{X} \times \mathcal{Y}$.

Lemma 1.2.13. *If $\mathcal{X}$ and $\mathcal{Y}$ are noncommutative spaces, then $\mathcal{X} \perp \mathcal{Y}$ is a noncommutative space with function algebra $\mathcal{O}(\mathcal{X} \perp \mathcal{Y}) = \mathcal{O}(\mathcal{X}) \otimes \mathcal{O}(\mathcal{Y})$.*

If $\mathcal{X}$ and $\mathcal{Y}$ have finitely generated function algebras then the function algebra of $\mathcal{X} \perp \mathcal{Y}$ is also finitely generated.

Proof. Let $A := \mathcal{O}(\mathcal{X})$ and $A' := \mathcal{O}(\mathcal{Y})$. Let $(p,p') \in (\mathcal{X} \perp \mathcal{Y})(B)$ be a pair of commuting points. Then there is a unique homomorphism of algebras $h : A \otimes A' \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "p"'] & A \otimes A' \arrow[d, "h"] & A' \arrow[l, "\iota'"'] \arrow[dl, "p'"] \\ & B. & \end{tikzcd}$$

Define $h(a \otimes a') := p(a)p'(a')$ and check the necessary properties. Observe that for an arbitrary homomorphism of algebras $h : A \otimes A' \longrightarrow B$ the images of elements of the form $a \otimes 1$ and $1 \otimes a'$ commute since these elements already commute in $A \otimes A'$. Thus we have

$$(\mathcal{X} \perp \mathcal{Y})(B) \cong \mathbb{K}\text{-}\mathbf{Alg}(A \otimes A', B).$$

If the algebra A is generated by the elements $a_1, \ldots, a_m$ and the algebra A' is generated by the elements $a'_1, \ldots, a'_n$ then the algebra $A \otimes A'$ is generated by the elements $a_i \otimes 1$ and $1 \otimes a'_j$. □

Proposition 1.2.14. *The orthogonal product of noncommutative spaces is associative, i.e. for noncommutative spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ we have*

$$(\mathcal{X} \perp \mathcal{Y}) \perp \mathcal{Z} \cong \mathcal{X} \perp (\mathcal{Y} \perp \mathcal{Z}).$$

Proof. Let B be a coefficient algebra and let $p_x \in \mathcal{X}(B)$, $p_y \in \mathcal{Y}(B)$, and $p_z \in \mathcal{Z}(B)$ be points such that $((p_x, p_y), p_z)$ is a pair of commuting points in $((\mathcal{X} \perp \mathcal{Y}) \perp \mathcal{Z})(B)$. In particular (p_x, p_y) is also a pair of commuting points. Thus we have for all $a \in A := \mathcal{O}(\mathcal{X})$, $a' \in A' := \mathcal{O}(\mathcal{Y})$, and $a'' \in A'' := \mathcal{O}(\mathcal{Z})$

$$p_x(a)p_y(a')p_z(a'') = (p_x, p_y)(a \otimes a')p_z(a'') = p_z(a'')(p_x, p_y)(a \otimes a') = p_z(a'')p_x(a)p_y(a')$$

and

$$p_x(a)p_y(a') = p_y(a')p_x(a).$$

If we choose $a = 1$ then we get $p_y(a')p_z(a'') = p_z(a'')p_y(a')$. For arbitrary a, a', a'' we then get

$$p_x(a)p_y(a')p_z(a'') = p_z(a'')p_x(a)p_y(a') = p_z(a'')p_y(a')p_x(a) = p_y(a')p_z(a'')p_x(a)$$

hence (p_y, p_z) and $(p_x, (p_y, p_z))$ are also pairs of commuting points. $\square$

Problem 1.2.5. Show that the orthogonal product of quantum spaces $\mathcal{X} \perp \mathcal{Y}$ is a tensor product for the category **QS** (in the sense of monoidal categories – if you know already what that is).

3. Quantum Monoids and their Actions on Quantum Spaces

We use the orthogonal product introduced in the previous section as "product" to define the notion of a monoid (some may call it an algebra w.r.t. the orthogonal product). Observe that on the geometric level the orthogonal product consists only of commuting points. So whenever we define a morphism on the geometric side with domain an orthogonal product of quantum spaces $f : \mathcal{X} \perp \mathcal{Y} \longrightarrow \mathcal{Z}$ then we only have to define what happens to *commuting* pairs of points. That makes it much easier to define such morphisms for noncommutative coordinate algebras.

We are going to define monoids in this sense and study their actions on quantum spaces.

Let E be the functor represented by $\mathbb{K}$. It maps each algebra H to the one-element set $\{\iota : \mathbb{K} \longrightarrow H\}$.

Definition 1.3.1. Let $\mathcal{M}$ be a noncommutative space and let

$$m : \mathcal{M} \perp \mathcal{M} \longrightarrow \mathcal{M} \text{ and } e : E \longrightarrow \mathcal{M}$$

be morphisms in **QS** such that the diagrams

$$\begin{tikzcd}
\mathcal{M} \perp \mathcal{M} \perp \mathcal{M} \arrow[r, "m \perp 1"] \arrow[d, "1 \perp m"'] & \mathcal{M} \perp \mathcal{M} \arrow[d, "m"] \\
\mathcal{M} \perp \mathcal{M} \arrow[r, "m"'] & \mathcal{M}
\end{tikzcd}$$

and

$$\begin{tikzcd}
E \perp \mathcal{M} \cong \mathcal{M} \cong \mathcal{M} \perp E \arrow[r, "\mathrm{id}\perp\eta"] \arrow[d, "\eta\perp\mathrm{id}"'] \arrow[dr, "1_{\mathcal{M}}"] & \mathcal{M} \perp \mathcal{M} \arrow[d, "\nabla"] \\
\mathcal{M} \perp \mathcal{M} \arrow[r, "\nabla"'] & \mathcal{M}
\end{tikzcd}$$

commute. Then $(\mathcal{M}, m, e)$ is called a *quantum monoid.*

Proposition 1.3.2. *Let $\mathcal{M}$ be a noncommutative space with function algebra H. Then H is a bialgebra if and only if $\mathcal{M}$ is a quantum monoid.*

Proof. Since the functors $\mathcal{M} \perp \mathcal{M}$, $\mathcal{M} \perp E$ and $E \perp \mathcal{M}$ are represented by $H \otimes H$ resp. $H \otimes \mathbb{K} \cong H$ resp. $\mathbb{K} \otimes H \cong H$ the Yoneda Lemma defines a bijection between the morphisms $m : \mathcal{M} \perp \mathcal{M} \longrightarrow \mathcal{M}$ and the algebra homomorphisms $\Delta : H \longrightarrow H \otimes H$ and similarly a bijection between the morphisms $e : E \longrightarrow \mathcal{M}$ and the algebra homomorphisms $\varepsilon : H \longrightarrow \mathbb{K}$. Again by the Yoneda Lemma the bialgebra diagrams in $\mathbb{K}$-**Alg** commute if and only if the corresponding diagrams for a quantum monoid commute. $\square$

Observe that a similar result cannot be formulated for Hopf algebras H since neither the antipode S nor the multiplication $\nabla : H \otimes H \longrightarrow H$ are algebra homomorphisms. In contrast to affine algebraic groups (2.3.2) Hopf algebras in the category $\mathbb{K}\text{-}\mathbf{Alg}^{op} \cong QR$ are not groups. Nevertheless, one defines

Definition 1.3.3. A functor defined on the category of $\mathbb{K}$-algebras and represented by a Hopf algebra H is called a *quantum group*.

Definition 1.3.4. Let $\mathcal{X}$ be a noncommutative space and let $\mathcal{M}$ be a quantum monoid. A morphism (a natural transformation) of quantum spaces $\rho : \mathcal{M} \perp \mathcal{X} \longrightarrow \mathcal{X}$ is called an *operation* of $\mathcal{M}$ on $\mathcal{X}$ if the diagrams

$$\begin{tikzcd}
\mathcal{M} \perp \mathcal{M} \perp \mathcal{X} \arrow[r, "m \perp 1"] \arrow[d, "1 \perp \rho"'] & \mathcal{M} \perp \mathcal{X} \arrow[d, "\rho"] \\
\mathcal{M} \perp \mathcal{X} \arrow[r, "\rho"'] & \mathcal{X}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{X} \cong E \perp \mathcal{X} \arrow[r, "\eta \perp \mathrm{id}"] \arrow[dr, "\mathrm{id}_{\mathcal{X}}"] & \mathcal{M} \perp \mathcal{X} \arrow[d, "\rho"] \\
 & \mathcal{X}
\end{tikzcd}$$

commute. We call $\mathcal{X}$ a noncommutative $\mathcal{M}$-*space*.

Proposition 1.3.5. *Let $\mathcal{X}$ be a noncommutative space with function algebra $A = \mathcal{O}(\mathcal{X})$. Let $\mathcal{M}$ be a quantum monoid with function algebra $B = \mathcal{O}(\mathcal{M})$. Let $\rho : \mathcal{M} \perp \mathcal{X} \longrightarrow \mathcal{X}$ be a morphism in* **QS** *and let $f : A \longrightarrow B \otimes A$ be the associated homomorphism of algebras. Then the following are equivalent*

1. $(\mathcal{X}, \mathcal{M}, \rho)$ is an operation of the quantum monoid $\mathcal{M}$ on the noncommutative space $\mathcal{X}$;

2. (A, H, f) define an H-comodule algebra.

Proof. The homomorphisms of algebras $\Delta \otimes 1_A$, $1_B \otimes f$, $\epsilon \otimes 1_A$ etc. represent the morphisms of quantum spaces $m \perp \mathrm{id}$, $\mathrm{id} \perp \rho$, $\eta \perp \mathrm{id}$ etc. Hence the required diagrams are transferred by the Yoneda Lemma. $\square$

Example 1.3.6. 1. The quantum monoid of "quantum matrices":
We consider the algebra

$$M_q(2) := \mathbb{K}\langle a, b, c, d\rangle / I = \mathbb{K}\left\langle \begin{matrix} a & b \\ c & d \end{matrix} \right\rangle / I$$

where the two-sided ideal I is generated by the elements

$$ab - q^{-1}ba, ac - q^{-1}ca, bd - q^{-1}db, cd - q^{-1}dc, ad - da - (q^{-1} - q)bc, bc - cb.$$

The quantum space $\mathcal{M}_q(2)$ associated with the algebra $M_q(2)$ is given by

$$\begin{aligned}\mathcal{M}_q(2)(A) &= \mathbb{K}\text{-}\mathbf{Alg}(M_q(2), A)\\ &= \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} | a', b', c', d' \in A; a'b' = q^{-1}b'a', \dots, b'c' = c'b' \right\}\end{aligned}$$

where each homomorphism of algebras $f : M_q(2) \longrightarrow A$ is described by the quadruple (a', b', c', d') of images of the algebra generators a, b, c, d. The images must satisfy the same relations that generate the two-sided ideal I hence

$$\begin{gathered} a'b' = q^{-1}b'a', a'c' = q^{-1}c'a', b'd' = q^{-1}d'b', c'd' = q^{-1}d'c', \\ b'c' = c'b', a'd' - q^{-1}b'c' = d'a' - qc'b'. \end{gathered}$$

We write these quadruples as 2×2-matrices and call them *quantum matrices.* The unusual commutation relations are chosen so that the following examples work.

The quantum space of quantum matrices turns out to be a quantum monoid. We give both the algebraic (with function algebras) and the geometric (with quantum spaces) approach to define the multiplication.

a) *The algebraic approach:*

The algebra $M_q(2)$ is a bialgebra with the diagonal

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

i.e. by $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = c \otimes a + d \otimes c$ and $\Delta(d) = c \otimes b + d \otimes d$, and with the counit

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. $\varepsilon(a) = 1$, $\varepsilon(b) = 0$, $\varepsilon(c) = 0$, and $\varepsilon(d) = 1$. We have to prove that Δ and ε are homomorphisms of algebras and that the coalgebra laws are satisfied. To obtain a homomorphism of algebras $\Delta : M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$ we define $\Delta : \mathbb{K}\langle a, b, c, d\rangle \longrightarrow M_q(2) \otimes M_q(2)$ on the free algebra (the polynomial ring in noncommuting variables) $\mathbb{K}\langle a, b, c, d\rangle$ generated by the set $\{a, b, c, d\}$ and show that it vanishes on the ideal I or more simply on the generators of the ideal. Then it factors through a unique homomorphism of algebras $\Delta : M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$. We check this only for one generator of the ideal I:

$$\begin{aligned} &\Delta(ab - q^{-1}ba) = \Delta(a)\Delta(b) - q^{-1}\Delta(b)\Delta(a) = \\ &= (a \otimes a + b \otimes c)(a \otimes b + b \otimes d) - q^{-1}(a \otimes b + b \otimes d)(a \otimes a + b \otimes c) \\ &= aa \otimes ab + ab \otimes ad + ba \otimes cb + bb \otimes cd - q^{-1}(aa \otimes ba + ab \otimes bc + ba \otimes da + bb \otimes dc) \\ &= aa \otimes (ab - q^{-1}ba) + ab \otimes (ad - q^{-1}bc) + ba \otimes (cb - q^{-1}da) + bb \otimes (cd - q^{-1}dc) \\ &= ba \otimes (q^{-1}ad - q^{-2}bc + cb - q^{-1}da) \equiv 0 \mod (I). \end{aligned}$$

The reader should check the other identities.

The coassociativity follows from

$$(\Delta\otimes 1)\Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix} = (\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix})\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix} =$$
$$= \begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix}) = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix} = (1\otimes\Delta)\Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix}.$$

The reader should check the properties of the counit.

b) *The geometric approach:*

$M_q(2)$ has a rather remarkable (and actually well known) comultiplication that is better understood by using the induced multiplication of commuting points. Given two commuting quantum matrices $\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$ in $\mathcal{M}_q(2)(A)$. Then their matrix product

$$\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2\\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2\end{pmatrix}$$

is again a quantum matrix. To prove this we only check one of the relations

$$\begin{aligned}(a_1a_2+b_1c_2)(a_1b_2+b_1d_2) &= a_1a_2a_1b_2+a_1a_2b_1d_2+b_1c_2a_1b_2+b_1c_2b_1d_2\\ &= a_1a_1a_2b_2+a_1b_1a_2d_2+b_1a_1c_2b_2+b_1b_1c_2d_2\\ &= q^{-1}a_1a_1b_2a_2+q^{-1}b_1a_1(d_2a_2+(q^{-1}-q)b_2c_2)+b_1a_1b_2c_2+q^{-1}b_1b_1d_2c_2\\ &= q^{-1}(a_1a_1b_2a_2+a_1b_1b_2c_2+b_1a_1d_2a_2+b_1b_1d_2c_2)\\ &= q^{-1}(a_1b_2a_1a_2+a_1b_2b_1c_2+b_1d_2a_1a_2+b_1d_2b_1c_2)\\ &= q^{-1}(a_1b_2+b_1d_2)(a_1a_2+b_1c_2)\end{aligned}$$

We have used that the two points are commuting points. This multiplication obviously is a natural transformation $\mathcal{M}_q(2)\perp\mathcal{M}_q(2)(A)\longrightarrow\mathcal{M}_q(2)(A)$ (natural in A). It is associative and has unit $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$. For the associativity observe that by 1.2.14

$$((\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}),\begin{pmatrix} a_3 & b_3\\ c_3 & d_3\end{pmatrix})$$

is a pair of commuting points if and only if

$$(\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},(\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix},\begin{pmatrix} a_3 & b_3\\ c_3 & d_3\end{pmatrix}))$$

is a pair of commuting points.

Since $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$ for all quantum matrices $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathcal{M}_q(2)(B)$ we see that $\mathcal{M}_q(2)$ is a quantum monoid.

It remains to show that the multiplication of $\mathcal{M}_q(2)$ and the comultiplication of $M_q(2)$ correspond to each other by the Yoneda Lemma. The identity morphism of

$M_q(2) \otimes M_q(2)$ is given by the pair of commuting points
$(\iota_1, \iota_2) \in \mathcal{M}_q(2) \perp \mathcal{M}_q(2)(M_q(2) \otimes M_q(2)) = \mathbb{K}\text{-}\mathbf{Alg}(M_q(2) \otimes M_q(2), M_q(2) \otimes M_q(2))$.
Since $\iota_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1 = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{pmatrix}$ and $\iota_2 = 1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \otimes a & 1 \otimes b \\ 1 \otimes c & 1 \otimes d \end{pmatrix}$ we have $\mathrm{id} = (\iota_1, \iota_2) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1, 1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix})$. The Yoneda Lemma defines the diagonal as the image of the identity under $\mathbb{K}\text{-}\mathbf{Alg}(M_q(2) \otimes M_q(2), M_q(2) \otimes M_q(2)) \longrightarrow \mathbb{K}\text{-}\mathbf{Alg}(M_q(2), M_q(2) \otimes M_q(2))$ by the multiplication. So $\Delta(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \Delta = \iota_1 * \iota_2 = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Thus $M_q(2)$ defines a quantum monoid $\mathcal{M}_q(2)$ with

$$\mathcal{M}_q(2)(B) = \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \middle| a', b', c', d' \in B; a'b' = q^{-1}b'a', \ldots, b'c' = c'b' \right\}.$$

This is the deformed version of $\mathcal{M}_2^\times$ the multiplicative monoid of the 2×2-matrices of commutative algebras.

2. Let $A_q^{2|0} = \mathbb{K}\langle x, y \rangle / (xy - q^{-1}yx)$ be the function algebra of the quantum plane $\mathbb{A}_q^{2|0}$. By the definition 1.2.5 we have

$$\mathbb{A}_q^{2|0}(A') = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in A'; xy = q^{-1}yx \right\}.$$

The set

$$\mathcal{M}_q(2)(A') = \left\{ \begin{pmatrix} u & x \\ y & z \end{pmatrix} \middle| u, x, y, z \in A'; ux = q^{-1}xu, \ldots, xy = yx \right\}$$

operates on this quantum plane by matrix multiplication

$$\mathcal{M}_q(2)(A') \perp \mathbb{A}_q^{2|0}(A') \ni (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}_q^{2|0}(A').$$

Again one should check that the required equations are preserved. Since we have a matrix multiplication we get an operation as in the preceding proposition. In particular $A_q^{2|0}$ is a $M_q(2)$-comodule algebra.

As in example 1. we get the comultiplication as $\delta(\begin{pmatrix} x \\ y \end{pmatrix}) = \delta = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$.

3. Let $A_q^{0|2} = \mathbb{K}\langle \xi, \eta \rangle / (\xi^2, \eta^2, \xi\eta + q\eta\xi)$ be the function algebra of the dual quantum plane $\mathbb{A}_q^{0|2}$. By the definition 1.2.5 we have

$$\mathbb{A}_q^{0|2}(A') = \left\{ \begin{pmatrix} a' & b' \end{pmatrix} \middle| a', b' \in A'; a'^2 = 0, b'^2 = 0, a'b' = -qb'a' \right\}.$$

The quantum monoid $M_q(2)$ also operates on the dual quantum plane by matrix multiplication

$$\mathbb{A}_q^{0|2}(A') \perp \mathcal{M}_q(2)(A') \ni \left(\begin{pmatrix}\xi & \eta\end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto \begin{pmatrix}\xi & \eta\end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{A}_q^{0|2}(A').$$

This gives another example of a $M_q(2)$-comodule algebra $A_q^{0|2} \longrightarrow A_q^{0|2} \otimes M_q(2)$ with $\delta(\begin{pmatrix}\xi & \eta\end{pmatrix}) = \delta = (\begin{pmatrix}\xi & \eta\end{pmatrix} \otimes 1) * (1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix}\xi & \eta\end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

What is now the reason for the remarkable relations on $\mathcal{M}_q(2)$? It is based on a fact that we will show later namely that $M_q(2)$ is the universal quantum monoid acting on the quantum plane $\mathbb{A}_q^{2|0}$ from the left and on the dual quantum plane $\mathbb{A}_q^{0|2}$ from the right. This however happens in the category of quantum planes represented by quadratic algebras. Here we will show a simpler theorem for finite dimensional algebras.

Problem 1.3.6. Determine the $\mathbb{H}$-points of the quantum plane $\mathbb{A}_q^{2|0}$ where $\mathbb{H}$ is the $\mathbb{R}$-algebra of the quaternions.

Definition 1.3.7. 1. Let $\mathcal{X}$ be a quantum space. A quantum space $\mathcal{M}(\mathcal{X})$ together with a morphism of quantum spaces $\mu : \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \longrightarrow \mathcal{X}$ is called a *quantum space acting universally* on $\mathcal{X}$ (or simply a *universal quantum space for* $\mathcal{X}$) if for every quantum space $\mathcal{Y}$ and every morphism of quantum spaces $f : \mathcal{Y} \perp \mathcal{X} \longrightarrow \mathcal{X}$ there is a unique morphism of quantum spaces $g : \mathcal{Y} \longrightarrow \mathcal{M}(\mathcal{X})$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{Y} \perp \mathcal{X} \arrow[d, "g \perp 1_{\mathcal{X}}"'] \arrow[dr, "f"] & \\ \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \arrow[r, "\mu"'] & \mathcal{X}. \end{tikzcd}$$

2. Let A be a $\mathbb{K}$-algebra. A $\mathbb{K}$-algebra $M(A)$ together with a homomorphism of algebras $\delta : A \longrightarrow M(A) \otimes A$ is called an *algebra coacting universally on* A (or simply a *universal algebra for* A) if for every $\mathbb{K}$-algebra B and every homomorphism of $\mathbb{K}$-algebras $f : A \longrightarrow B \otimes A$ there exists a unique homomorphism of algebras $g : M(A) \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g \otimes 1_A"] \\ & B \otimes A \end{tikzcd}$$

By the universal properties the universal algebra $M(A)$ for A and the universal quantum space $\mathcal{M}(\mathcal{X})$ for $\mathcal{X}$ are unique up to isomorphism.

Proposition 1.3.8. 1. *Let A be a $\mathbb{K}$-algebra with universal algebra $M(A)$ and $\delta : A \longrightarrow M(A) \otimes A$. Then $M(A)$ is a bialgebra and A is an $M(A)$-comodule algebra by δ.*

2. *If B is a bialgebra and if $f : A \longrightarrow B \otimes A$ defines the structure of a B-comodule algebra on A then there is a unique homomorphism $g : M(A) \longrightarrow B$ of bialgebras such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g \otimes 1_A"] \\ & B \otimes A \end{tikzcd}$$

The corresponding statement for quantum spaces and quantum monoids is the following.

Proposition 1.3.9. 1. *Let $\mathcal{X}$ be a quantum space with universal quantum space $\mathcal{M}(\mathcal{X})$ and $\mu : \mathcal{M}(\mathcal{X}) \perp A \longrightarrow A$. Then $\mathcal{M}(\mathcal{X})$ is a quantum monoid and $\mathcal{X}$ is an $\mathcal{M}(\mathcal{X})$-space by μ.*

2. *If $\mathcal{Y}$ is another quantum monoid and if $f : \mathcal{Y} \perp \mathcal{X} \longrightarrow \mathcal{X}$ defines the structure of a $\mathcal{Y}$-space on $\mathcal{X}$ then there is a unique morphism of quantum monoids $g : \mathcal{Y} \longrightarrow \mathcal{M}(\mathcal{X})$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{Y} \perp \mathcal{X} \arrow[d, "g \perp 1_{\mathcal{X}}"'] \arrow[dr, "f"] & \\ \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \arrow[r, "\mu"'] & \mathcal{X}. \end{tikzcd}$$

Proof. We give the proof for the algebra version of the proposition. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"'] & M(A) \otimes A \arrow[d, "\Delta \otimes 1_A"] \\ M(A) \otimes A \arrow[r, "1_{M(A)} \otimes \delta"'] & M(A) \otimes M(A) \otimes A \end{tikzcd}$$

where the morphism of algebras Δ is defined by the universal property of $M(A)$ with respect to the algebra morphism $(1_{M(A)} \otimes \delta)\delta$. Furthermore there is a unique morphism of algebras $\epsilon : M(A) \longrightarrow \mathbb{K}$ such that

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "1_A"'] & M(A) \otimes A \arrow[d, "\epsilon \otimes 1_A"] \\ & A \cong \mathbb{K} \otimes A \end{tikzcd}$$

commutes.

The coalgebra axioms arise from the following commutative diagrams

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"] & M(A) \otimes A \arrow[d, "\Delta\otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)}\otimes\delta"] \arrow[d, shift right, "\Delta\otimes 1_A"'] \arrow[d, shift left, "1_{M(A)}\otimes\delta"] & M(A) \otimes M(A) \otimes A \arrow[d, shift right, "\Delta\otimes 1_{M(A)}\otimes 1_A"'] \arrow[d, shift left, "1_{M(A)}\otimes\Delta\otimes 1_A"] \\
M(A) \otimes M(A) \otimes A \arrow[r, "1_{M(A)}\otimes 1_{M(A)}\otimes\delta"] & M(A) \otimes M(A) \otimes M(A) \otimes A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[d, "\delta"] & & M(A) \otimes A \arrow[d, "\Delta\otimes 1_A"'] \arrow[dd, bend left, "1_{M(A)}\otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)}\otimes\delta"] \arrow[drr, "1_{M(A)}\otimes 1_A"'] & M(A) \otimes M(A) \otimes A \arrow[dr, phantom] & M(A) \otimes M(A) \otimes A \arrow[d, "1_{M(A)}\otimes\epsilon\otimes 1_A"'] \\
 & & M(A) \otimes A \cong M(A) \otimes \mathbb{K} \otimes A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dd, bend right, "1_A"'] & M(A) \otimes A \arrow[d, "\Delta\otimes 1_A"] \\
M(A) \otimes A \arrow[r, "1_{M(A)}\otimes\delta"] \arrow[d, "\epsilon\otimes 1_A"] & M(A) \otimes M(A) \otimes A \arrow[d, "\epsilon\otimes 1_{M(A)}\otimes 1_A"] \\
A \arrow[r, "\delta"] & M(A) \otimes A \cong \mathbb{K} \otimes M(A) \otimes A.
\end{tikzcd}$$

In fact these diagrams imply by the uniqueness of the induced homomorphisms of algebras $(\Delta \otimes 1_{M(A)})\Delta = (1_{M(A)} \otimes \Delta)\Delta$, $(1_{M(A)} \otimes \epsilon)\Delta = 1_{M(A)}$ and $\epsilon \otimes (1_{M(A)})\Delta = 1_{M(A)}$. Finally A is an $M(A)$-comodule algebra by the definition of Δ and ϵ.

Now assume that a structure of a B-comodule algebra on A is given by a bialgebra B and $f : A \longrightarrow B \otimes A$. Then there is a unique homomorphism of algebras $g : M(A) \longrightarrow B$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[d, "g\otimes 1_A"] \\
 & B \otimes A
\end{tikzcd}$$

commutes. Then the following diagram

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"'] & M(A) \otimes A \arrow[r, shift left, "\Delta\otimes 1_A"] \arrow[r, shift right, "1_{M(A)}\otimes\delta"'] \arrow[d, "g\otimes 1_A"] & M(A) \otimes M(A) \otimes A \arrow[d, "g\otimes g\otimes 1_A"] \\
& B \otimes A \arrow[r, shift left, "\Delta_B\otimes 1_A"] \arrow[r, shift right, "1_B\otimes f"'] & B \otimes B \otimes A
\end{tikzcd}$$

implies $((g \otimes g)\Delta \otimes 1_A)\delta = (g \otimes g \otimes 1_A)(\Delta \otimes 1_A)\delta = (g \otimes g \otimes 1_A)(1_{M(A)} \otimes \delta)\delta = (g \otimes (g \otimes 1_A)\delta)\delta = (1_B \otimes (g \otimes 1_A)\delta)(g \otimes 1_A)\delta = (1_B \otimes f)f = (\Delta_B \otimes 1_A)f = (\Delta_B \otimes 1_A)(g \otimes 1_A)\delta = (\Delta_B g \otimes 1_A)\delta$ hence $(g \otimes g)\Delta = \Delta_B g$. Furthermore the diagram

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[dr, "f"] \arrow[ddr, "1_A"'] & M(A) \otimes A \arrow[d, "g\otimes 1_A"'] \arrow[dd, bend left=60, "\epsilon\otimes 1_A"] \\
& B \otimes A \arrow[d, "\epsilon_B\otimes 1_A"'] \\
& A \cong \mathbb{K} \otimes A
\end{tikzcd}$$

implies $\epsilon_B g = \epsilon$. Thus g is a homomorphism of bialgebras. $\square$

Since universal algebras for algebras A tend to become very big they do not exist in general. But a theorem of Tambara's says that they exist for finite dimensional algebras (over a field $\mathbb{K}$).

Definition 1.3.10. If $\mathcal{X}$ is a quantum space with finite dimensional function algebra then we call $\mathcal{X}$ a *finite quantum space*.

The following theorem is the quantum space version and equivalent to a theorem of Tambara.

Theorem 1.3.11. *Let $\mathcal{X}$ be a finite quantum space. Then there exists a (universal) quantum space $\mathcal{M}(\mathcal{X})$ with morphism of quantum spaces $\mu : \mathcal{M}(\mathcal{X}) \perp \mathcal{X} \longrightarrow \mathcal{X}$.*

The algebra version of this theorem is

Theorem 1.3.12. (Tambara) *Let A be a finite dimensional $\mathbb{K}$-algebra. Then there exists a (universal) $\mathbb{K}$-algebra $M(A)$ with homomorphism of algebras $\delta : A \longrightarrow M(A) \otimes A$.*

Proof. We are going to construct the $\mathbb{K}$-algebra $M(A)$ quite explicitly. First we observe that $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a coalgebra (cf. problem A.6.8) with the structural morphism $\Delta : A^* \longrightarrow (A \otimes A)^* \cong A^* \otimes A^*$. Denote the dual basis by $\sum_{i=1}^n a_i \otimes \bar{a}^i \in A \otimes A^*$. Now let $T(A \otimes A^*)$ be the tensor algebra of the vector space

$A \otimes A^*$. Consider elements of the tensor algebra

$$\begin{aligned}
&xy \otimes \zeta \in A \otimes A^*,\\
&x \otimes y \otimes \Delta(\zeta) \in A \otimes A \otimes A^* \otimes A^* \cong A \otimes A^* \otimes A \otimes A^*,\\
&1 \otimes \zeta \in A \otimes A^*,\\
&\zeta(1) \in \mathbb{K}.
\end{aligned}$$

The following elements

$$xy \otimes \zeta - x \otimes y \otimes \Delta(\zeta) \tag{1}$$

and

$$1 \otimes \zeta - \zeta(1) \tag{2}$$

generate a two-sided ideal $I \subseteq T(A \otimes A^*)$. Now we define

$$M(A) := T(A \otimes A^*)/I$$

and the cooperation $\delta : A \ni a \longrightarrow \sum_{i=1}^n (a \otimes \bar{a}^i) \otimes a_i \in T(A \otimes A^*)/I \otimes A$. This is a well-defined linear map.

To show that this map is a homomorphism of algebras we first describe the multiplication of A by $a_i a_j = \sum_k \alpha_{ij}^k a_k$. Then the comultiplication of A^* is given by $\Delta(\bar{a}^k) = \sum_{ij} \alpha_{ij}^k \bar{a}^i \otimes \bar{a}^j$ since $(\Delta(\bar{a}^k), a_l \otimes a_m) = (\bar{a}^k, a_l a_m) = \sum_r \alpha_{lm}^r (\bar{a}^k, a_r) = \alpha_{lm}^k = \sum_{ij} \alpha_{ij}^k (\bar{a}^i, a_l)(\bar{a}^j, a_m) = (\sum_{ij} \alpha_{ij}^k \bar{a}^i \otimes \bar{a}^j, a_l \otimes a_m)$. Now write $1 = \sum \beta^k a_k$. Then we get $\epsilon(\bar{a}^i) = \beta^i$ since $\epsilon(\bar{a}^i) = (\bar{a}^i, 1) = \sum_j \beta^j (\bar{a}^i, a_j) = \beta^i$. So we have $\delta(a)\delta(b) = (\sum_{i=1}^n (a \otimes \bar{a}^i) \otimes a_i) \cdot (\sum_{j=1}^n (b \otimes \bar{a}^j) \otimes a_j) = \sum_{ij} (a \otimes b \otimes \bar{a}^i \otimes \bar{a}^j) \otimes a_i a_j = \sum_{ijk} \alpha_{ij}^k (a \otimes b \otimes \bar{a}^i \otimes \bar{a}^j) \otimes a_k = \sum_k (a \otimes b \otimes \Delta(\bar{a}^k)) \otimes a_k = \sum_k (ab \otimes \bar{a}^k) \otimes a_k = \delta(ab)$. Furthermore we have $\delta(1) = \sum_i (1 \otimes \bar{a}^i) \otimes a_i = \sum_i \bar{a}^i(1) \otimes a_i = 1 \otimes \sum_i \bar{a}^i(1) a_i = 1 \otimes 1$. Hence δ is a homomorphism of algebras.

Now we have to show that there is a unique g for each f. First of all $f : A \longrightarrow B \otimes A$ induces uniquely determined linear maps $f_i : A \longrightarrow B$ with $f(a) = \sum_i f_i(a) \otimes a_i$ since the a_i form a basis. Since f is a homomorphism of algebras we get from $\sum_k f_k(a) \otimes a_k = f(ab) = f(a)f(b) = \sum_{ij} (f_i(a) \otimes a_i)(f_j(b) \otimes a_j) = \sum_{ij} f_i(a) f_j(b) \otimes a_i a_j = \sum_{ijk} \alpha_{ij}^k f_i(a) f_j(b) \otimes a_k$ by comparison of coefficients

$$f_k(ab) = \sum_{ij} \alpha_{ij}^k f_i(a) f_j(b).$$

Furthermore we define $g(a \otimes \bar{a}) := (1 \otimes \bar{a}) f(a) \in B$. Then we get in particular $g(a \otimes \bar{a}^i) = (1 \otimes \bar{a}^i)(\sum_j f_j(a) \otimes a_j) = f_i(a)$. We can extend the map g to a homomorphism of algebras $g : T(A \otimes A^*) \longrightarrow B$. Applied to the generators of the ideal we get $g(ab \otimes \bar{a}^k - a \otimes b \otimes \Delta(\bar{a}^k)) = (1 \otimes \bar{a}^k) \sum_l f_l(ab) \otimes a_l - \sum_{rsij} \alpha_{ij}^k (1 \otimes \bar{a}^i)(f_r(a) \otimes a_r) \cdot (1 \otimes \bar{a}^j)(f_s(b) \otimes a_s) = f_k(ab) - \sum_{ij} \alpha_{ij}^k f_i(a) f_j(b) = 0$. We have furthermore $g(1 \otimes \zeta - \zeta(1)) = (1 \otimes \zeta) f(1) - \zeta(1) = (1 \otimes \zeta)(1 \otimes 1) - \zeta(1) = 1\zeta(1) - \zeta(1) = 0$. Thus the homomorphism of algebras g vanishes on the ideal I so it may be factored through $M(A) = T(A)/I$. Denote this factorization also by g. Then the diagram commutes since $(g \otimes 1_A)\delta(a) = (g \otimes 1_A)(\sum_i (a \otimes \bar{a}^i) \otimes a_i) = \sum_i (1 \otimes \bar{a}^i) f(a) \otimes a_i = \sum_{ij} f_j(a)(\bar{a}^i, a_j) \otimes a_i = \sum_i f_i(a) \otimes a_i = f(a)$.

We still have to show that g is uniquely determined. Assume that we also have $(h \otimes 1_A)\delta = f$ then $\sum_i h(a \otimes \bar{a}^i) \otimes a_i = (h \otimes 1_A)\delta(a) = f(a) = \sum_i f_i(a) \otimes a_i$ hence $h(a \otimes \bar{a}^i) = f_i(a) = g(a \otimes \bar{a}^i)$, i.e. $g = h$. $\square$

Definition 1.3.13. Let A be a $\mathbb{K}$-algebra. The universal algebra $M(A)$ for A that is a bialgebra is also called the *coendomorphism bialgebra* of A.

Problem 1.3.7.

1. Determine explicitly the dual coalgebra A^* of the algebra $A := \mathbb{K}\langle x \rangle/(x^2)$. (Hint: Find a basis for A.)
2. Determine and describe the coendomorphism bialgebra of A from problem 1.1. (Hint: Determine first a set of algebra generators of $M(A)$. Then describe the relations.)
3. Determine explicitly the dual coalgebra A^* of $A := \mathbb{K}\langle x \rangle/(x^3)$.
4. Determine and describe the coendomorphism bialgebra of A from problem 1.3.
5. (*) Determine explicitly the dual coalgebra A^* of $A := \mathbb{K}\langle x, y \rangle/I$ where the ideal I is generated as a two-sided ideal by the polynomials
$$xy - q^{-1}yx, x^2, y^2.$$
6. (*) Determine the coendomorphism bialgebra of A from problem 1.5.
7. Let A be a finite dimensional $\mathbb{K}$-algebra with universal bialgebra $A \longrightarrow B \otimes A$. Show
   i) that $A^{op} \longrightarrow B^{op} \otimes A^{op}$ is universal (where A^{op} has the multiplication $\nabla\tau : A \otimes A \longrightarrow A \otimes A \longrightarrow A$);
   ii) that $A \cong A^{op}$ implies $B \cong B^{op}$ (as bialgebras);
   iii) that for commutative algebras A the algebra B satisfies $B \cong B^{op}$ but that B need not be commutative.
   iv) Find an isomorphism $B \cong B^{op}$ for the bialgebra $B = \mathbb{K}\langle a, b \rangle/(a^2, ab+ba)$. (compare problem 1.7 2).

Sketch of solution:

1/2. The bialgebra has the form $B = \mathbb{K}\langle a, b\rangle/(a^2, ab+ba)$ with $\Delta(a) = a\otimes 1+b\otimes a$, $\Delta(b) = b \otimes b$ and $\varepsilon(a) = 0$, $\varepsilon(b) = 1$. The coaction is $\delta(x) = a \otimes 1 + b \otimes x$.

3/4. A has the basis $1, x, x^2$. The dual coalgebra has the dual basis e, ζ, ζ_2 with $\Delta(e) = e \otimes e$, $\Delta(\zeta) = \zeta \otimes e + e \otimes \zeta$ and $\Delta(\zeta_2) = \zeta_2 \otimes e + \zeta \otimes \zeta + e \otimes \zeta_2$.

The universal bialgebra $B = T(A \otimes A^*)/I$ satisfies $\delta(x) = x \otimes e \otimes 1 + x \otimes \zeta \otimes x + x \otimes \zeta_2 \otimes x^2 = a \otimes 1 + b \otimes x + c \otimes x^2$. Thus it is generated by the elements $a = x \otimes e$, $b = x \otimes \zeta$ and $c = x \otimes \zeta_2$. The multiplication table and the relations arise from

$$\begin{aligned}
&1 \otimes e = 1,\\
&1 \otimes \zeta = 1 \otimes \zeta_2 = 0,\\
&x^2 \otimes e = (x \otimes e)(x \otimes e),\\
&x^2 \otimes \zeta = (x \otimes \zeta)(x \otimes e) + (x \otimes e)(x \otimes \zeta),\\
&x^2 \otimes \zeta_2 = (x \otimes \zeta_2)(x \otimes e) + (x \otimes \zeta)(x \otimes \zeta) + (x \otimes e)(x \otimes \zeta_2),
\end{aligned}$$

$$\begin{aligned}
&0 = x^3 \otimes e = (x^2 \otimes e)(x \otimes e),\\
&0 = x^3 \otimes \zeta = (x^2 \otimes \zeta)(x \otimes e) + (x^2 \otimes e)(x \otimes \zeta),\\
&0 = x^3 \otimes \zeta_2 = (x^2 \otimes \zeta_2)(x \otimes e) + (x^2 \otimes \zeta)(x \otimes \zeta) + (x^2 \otimes e)(x \otimes \zeta_2)
\end{aligned}$$

We use the abbreviation $\{u, v\} := u^2v + uvu + vu^2$ and have

$$\begin{aligned}
&a^3 = 0,\\
&\{a, b\} = 0,\\
&\{a, c\} + \{b, a\} = 0.
\end{aligned}$$

The condition $(1 \otimes \delta)\delta = (\Delta \otimes 1)\delta$ implies

$$\begin{aligned}
&\Delta(a) = a \otimes 1 + b \otimes a + c \otimes a^2,\\
&\Delta(b) = b \otimes b + c \otimes (ba + ab),\\
&\Delta(c) = b \otimes c + c \otimes b^2 + c \otimes (ca + ac),\\
&\epsilon(a) = 0,\\
&\epsilon(b) = 1,\\
&\epsilon(c) = 0.
\end{aligned}$$

5/6. A has the basis $1, x, y, xy$. The dual basis of A^* is denoted by e, ξ, η, θ. The diagonal is

$$\begin{aligned}
&\Delta(e) = e \otimes e,\\
&\Delta(\xi) = \xi \otimes e + e \otimes \xi,\\
&\Delta(\eta) = \eta \otimes e + e \otimes \eta,\\
&\Delta(\theta) = \theta \otimes e + e \otimes \theta + \xi \otimes \eta + q\eta \otimes \xi.
\end{aligned}$$

Thus the coendomorphism bialgebra has the algebra generators $a \otimes \zeta$ with $a \in \{1, x, y, xy\}$ and $\zeta \in \{e, \xi, \eta, \theta\}$. The generators of the relations (of I) are given by the equations 1.1 and 1.2. They imply that $1 \otimes e$ is the unit element, that

$1 \otimes \xi = 1 \otimes \eta = 1 \otimes \theta = 0$ and that

$$\begin{aligned}
&ab \otimes e = (a \otimes e)(b \otimes e),\\
&ab \otimes \xi = (a \otimes \xi)(b \otimes e) + (a \otimes 1)(b \otimes \xi),\\
&ab \otimes \eta = (a \otimes \eta)(b \otimes e) + (a \otimes 1)(b \otimes \eta),\\
&ab \otimes \theta = (a \otimes \theta)(b \otimes e) + (a \otimes 1)(b \otimes \theta) + (a \otimes \xi)(b \otimes \eta) + q(a \otimes \eta)(b \otimes \xi).
\end{aligned}$$

Furthermore for ab we have to take into account the relations in A.

We define

$$\begin{aligned}
a &:= x \otimes e, \quad b := x \otimes \xi, \quad c := x \otimes \eta, \quad d := x \otimes \theta,\\
e &:= y \otimes e, \quad f := y \otimes \xi, \quad g := y \otimes \eta, \quad h := x \otimes \theta,
\end{aligned}$$

and get $\delta(x) = a \otimes 1 + b \otimes x + c \otimes y + d \otimes xy$ and $\delta(y) = e \otimes 1 + f \otimes x + g \otimes y + h \otimes xy$. Hence B is generated by $a, \ldots, h$ as an algebra. The relations are

$$\begin{aligned}
&a^2 = e^2 = 0,\\
&ab + ba = ac + ca = ef + fe = eg + ge = 0,\\
&ad + da + bc + qcb = eh + he + fg + qgf = 0,\\
&ae = qea,\\
&af + be = q(fa + eb),\\
&ag + ce = q(ga + ec),\\
&ah - qha + de - qed + bg - q^2 gb + qcf - qfc = 0.
\end{aligned}$$

The diagonal is

$$\begin{aligned}
\Delta(a) &= a \otimes 1 + b \otimes a + c \otimes e + d \otimes ae,\\
\Delta(b) &= b \otimes b + c \otimes f + d \otimes (af + be),\\
\Delta(c) &= b \otimes c + c \otimes g + d \otimes (ag + ce),\\
\Delta(d) &= b \otimes d + c \otimes h + d \otimes (ah + de + bg + q^{-1} cf) \quad \text{etc.}
\end{aligned}$$