ADVANCED ALGEBRA

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1. Tensor Products and Free Modules

1.1. Modules.

Definition 1.1. Let R be a ring (always associative with unit element). A *left R-module* ${}_RM$ is an Abelian group M (with composition written as addition) together with an operation

$$R \times M \ni (r, m) \mapsto rm \in M$$

such that

(1) $(rs)m = r(sm)$,
(2) $(r + s)m = rm + sm$,
(3) $r(m + m') = rm + rm'$,
(4) $1m = m$

for all $r, s \in R$, $m, m' \in M$.
If R is a field then a (left) R-module is a (called a) vector space over R.
A *homomorphism of left R-modules* or simply an *R-module homomorphism* $f : {}_RM \longrightarrow {}_RN$ is a homomorphism of groups with $f(rm) = rf(m)$.
Right R-modules and *homomorphisms of right R-modules* are defined analogously.
We define

$$\mathrm{Hom}_R(.M, .N) := \{f : {}_RM \longrightarrow {}_RN | f \text{ is a homomorphism of left } R\text{-modules}\}.$$

Similarly $\mathrm{Hom}_R(M., N.)$ denotes the set of homomorphisms of right R-modules M_R and N_R.
An R-module homomorphism $f : {}_RM \longrightarrow {}_RN$ is
a *monomorphism* if f is injective,
an *epimorphism* if f is surjective,
an *isomorphism* if f is bijective,
an *endomorphism* if $M = N$,
an *automorphism* if f is an endomorphism and an isomorphism.

Problem 1.1. Let R be a ring and M be an Abelian group. Show that there is a one-to-one correspondence between maps $f : R \times M \longrightarrow M$ that make M into a left R-module and ring homomorphisms (always preserving the unit element) $g : R \longrightarrow \mathrm{End}(M)$.

Lemma 1.2. $\mathrm{Hom}_R(M, N)$ *is an Abelian group by* $(f + g)(m) := f(m) + g(m)$.

Proof. Since N is an Abelian group the set of maps $\mathrm{Map}(M, N)$ is also an Abelian group. The set of group homomorphisms $\mathrm{Hom}(M, N)$ is a subgroup of $\mathrm{Map}(M, N)$ (observe that this holds only for Abelian groups). We show that $\mathrm{Hom}_R(M, N)$ is a subgroup of $\mathrm{Hom}(M, N)$. We must only show that $f - g$ is an R-module homomorphism if f and g are. Obviously $f - g$ is a group homomorphism. Furthermore we have $(f - g)(rm) = f(rm) - g(rm) = rf(m) - rg(m) = r(f(m) - g(m)) = r(f - g)(m)$. □

Problem 1.2. Let $f : M \longrightarrow N$ be an R-module homomorphism.

(1) f is an isomorphism if and only if (iff) there exists an R-module homomorphism $g : N \longrightarrow M$ such that

$$fg = \mathrm{id}_N \text{ and } gf = \mathrm{id}_M .$$

Furthermore g is uniquely determined by f.
(2) The following are equivalent:
(a) f is a monomorphism,
(b) for all R-modules P and all homomorphisms $g, h : P \longrightarrow M$

$$fg = fh \Longrightarrow g = h,$$

(c) for all R-modules P the homomorphism of Abelian groups

$$\mathrm{Hom}_R(P, f): \ \mathrm{Hom}_R(P, M) \ni g \mapsto fg \in \mathrm{Hom}_R(P, N)$$

is a monomorphism.

(3) The following are equivalent:

(a) f is an epimorphism,

(b) for all R-modules P and all homomorphisms $g, h : N \longrightarrow P$

$$gf = hf \Longrightarrow g = h,$$

(c) for all R-modules P the homomorphism of Abelian groups

$$\mathrm{Hom}_R(f, P): \ \mathrm{Hom}_R(N, P) \ni g \mapsto gf \in \mathrm{Hom}_R(M, P)$$

is a monomorphism.

Remark 1.3. Each Abelian group is a $\mathbb{Z}$-module in a unique way. Each homomorphism of Abelian groups is a $\mathbb{Z}$-module homomorphism.

Proof. By exercise 1.1 we have to find a unique ring homomorphism $g : \mathbb{Z} \longrightarrow \mathrm{End}(M)$. This holds more generally. If S is a ring then there is a unique ring homomorphism $g : \mathbb{Z} \longrightarrow S$. Since a ring homomorphism must preserve the unit we have $g(1) = 1$. Define $g(n) := 1 + \ldots + 1$ (n-times) for $n \geq 0$ and $g(-n) := -(1 + \ldots + 1)$ (n-times) for $n > 0$. Then it is easy to check that g is a ring homomorphism and it is obviously unique. This means that M is a $\mathbb{Z}$-module by $nm = m + \ldots + m$ (n-times) for $n \geq 0$ and $(-n)m = -(m + \ldots + m)$ (n-times) for $n > 0$.
If $f : M \longrightarrow N$ is a homomorphism of (Abelian) groups then $f(nm) = f(m + \ldots + m) = f(m) + \ldots + f(m) = nf(m)$ for $n \geq 0$ and $f((-n)m) = f(-(m + \ldots + m)) = -(f(m) + \ldots + f(m)) = (-n)f(m)$ for $n > 0$. Hence f is a $\mathbb{Z}$-module homomorphism. □

Problem 1.3. (1) Let R be a ring. Then ${}_RR$ is a left R-module.

(2) Let M be a Abelian group and $\mathrm{End}(M)$ be the endomorphism ring of M. Then M is an $\mathrm{End}(M)$-module.

(3) $\{(\bar{1}, \bar{0}), (\bar{0}, \bar{1})\}$ is a generating set for the $\mathbb{Z}$-module $\mathbb{Z}/(2) \times \mathbb{Z}/(3)$.

(4) $\{(\bar{1}, \bar{1})\}$ is a generating set for the $\mathbb{Z}$-module $\mathbb{Z}/(2) \times \mathbb{Z}/(3)$.

(5) ${}_{\mathbb{Z}}\mathbb{Z}/(n)$ has no basis as a module, i.e. this module is not free.

(6) Let $V = \bigoplus_{i=0}^{\infty} Kb_i$ be a countably infinite dimensional vector space over the field K. Let $p, q, a, b \in \mathrm{Hom}(V, V)$ be defined by

$$\begin{aligned}
p(b_i) &:= b_{2i}, \\
q(b_i) &:= b_{2i+1}, \\
a(b_i) &:= \begin{cases} b_{i/2}, & \text{if } i \text{ is even, and} \\ 0, & \text{if } i \text{ is odd.} \end{cases} \\
b(b_i) &:= \begin{cases} b_{i-1/2}, & \text{if } i \text{ is odd, and} \\ 0, & \text{if } i \text{ is even.} \end{cases}
\end{aligned}$$

Show $pa + qb = \mathrm{id}_V$, $ap = bq = \mathrm{id}$, $aq = bp = 0$.

Show for $R = \mathrm{End}_K(V)$ that ${}_RR = Ra \oplus Rb$ and $R_R = pR \oplus qR$ holds.

(7) Are $\{(0, \ldots, a, \ldots, 0) | a \in K_n\}$ and $\{(a, 0, \ldots, 0) | a \in K_n\}$ isomorphic as $M_n(K)$-modules?

(8) For each module P there is a module Q such that $P \oplus Q \cong Q$.

(9) Which of the following statements is correct?

(a) $P_1 \oplus Q = P_2 \oplus Q \Longrightarrow P_1 = P_2$?
(b) $P_1 \oplus Q = P_2 \oplus Q \Longrightarrow P_1 \cong P_2$?
(c) $P_1 \oplus Q \cong P_2 \oplus Q \Longrightarrow P_1 \cong P_2$?

(10) $\mathbb{Z}/(2) \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(6) \oplus \ldots \cong \mathbb{Z}/(6) \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(6) \oplus \ldots$.
(11) $\mathbb{Z}/(2) \oplus \mathbb{Z}/(4) \oplus \mathbb{Z}/(4) \oplus \ldots \not\cong \mathbb{Z}/(4) \oplus \mathbb{Z}/(4) \oplus \mathbb{Z}/(4) \oplus \ldots$.
(12) Find two Abelian groups P and Q, such that P is isomorphic to a subgroup of Q and Q is isomorphic to a subgroup of P and $P \not\cong Q$.

1.2. **Tensor products I.**

Definition and Remark 1.4. Let M_R and ${}_RN$ be R-modules, and let A be an Abelian group. A map $f : M \times N \longrightarrow A$ is called *R-bilinear* if

(1) $f(m + m', n) = f(m, n) + f(m', n)$,
(2) $f(m, n + n') = f(m, n) + f(m, n')$,
(3) $f(mr, n) = f(m, rn)$

for all $r \in R,\ m, m' \in M,\ n, n' \in N$.
Let $\mathrm{Bil}_R(M, N; A)$ denote the set of all R-bilinear maps $f : M \times N \longrightarrow A$.
$\mathrm{Bil}_R(M, N; A)$ is an Abelian group with $(f + g)(m, n) := f(m, n) + g(m, n)$.

Definition 1.5. Let M_R and ${}_RN$ be R-modules. An Abelian group $M \otimes_R N$ together with an R-bilinear map

$$\otimes : M \times N \ni (m, n) \mapsto m \otimes n \in M \otimes_R N$$

is called a *tensor product of M and N over R* if for each Abelian group A and for each R-bilinear map $f : M \times N \longrightarrow A$ there exists a unique group homomorphism $g : M \otimes_R N \longrightarrow A$ such that the diagram

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes. The elements of $M \otimes_R N$ are called *tensors*, the elements of the form $m \otimes n$ are called *decomposable tensors*.
Warning: If you want to define a homomorphism $f : M \otimes_R N \longrightarrow A$ with a tensor product as domain you *must* define it by giving an R-bilinear map defined on $M \times N$.

Proposition 1.6. *A tensor product $(M \otimes_R N, \otimes)$ defined by M_R and ${}_RN$ is unique up to a unique isomorphism.*

Proof. Let $(M \otimes_R N, \otimes)$ and $(M \boxtimes_R N, \boxtimes)$ be tensor products. Then

$$\begin{tikzcd} & & M \times N \arrow[dll, "\otimes"'] \arrow[dl, "\boxtimes"] \arrow[dr, "\otimes"] \arrow[drr, "\boxtimes"] & & \\ M \otimes_R N \arrow[r, "h"] & M \boxtimes_R N \arrow[rr, "k"] & & M \otimes_R N \arrow[r, "h"] & M \boxtimes_R N \end{tikzcd}$$

implies $k = h^{-1}$. $\square$

Because of this fact we will henceforth talk about *the* tensor product of M and N over R.

Proposition 1.7. (Rules of computation in a tensor product) *Let $(M \otimes_R N, \otimes)$ be the tensor product. Then we have for all $r \in R$, $m, m' \in M$, $n, n' \in N$*

(1) *$M \otimes_R N = \{\sum_i m_i \otimes n_i \mid m_i \in M, n_i \in N\}$,*
(2) *$(m + m') \otimes n = m \otimes n + m' \otimes n$,*
(3) *$m \otimes (n + n') = m \otimes n + m \otimes n'$,*
(4) *$mr \otimes n = m \otimes rn$ (observe in particular, that $\otimes : M \times N \longrightarrow M \otimes N$ is not injective in general),*
(5) *if $f : M \times N \longrightarrow A$ is an R-bilinear map and $g : M \otimes_R N \longrightarrow A$ is the induced homomorphism, then*

$$g(m \otimes n) = f(m, n).$$

Proof. (1) Let $B := \langle m \otimes n \rangle \subseteq M \otimes_R N$ denote the subgroup of $M \otimes_R N$ generated by the decomposable tensors $m \otimes n$. Let $j : B \longrightarrow M \otimes_R N$ be the embedding homomorphism. We get an induced map $\otimes' : M \times N \longrightarrow B$. The following diagram

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes'"] \arrow[dr, "\otimes'"'] & B \arrow[r, "j"] & M \otimes_R N \arrow[dl, "p"] \arrow[d, "jp"] \\
 & B \arrow[r, "j"] & M \otimes_R N
\end{tikzcd}$$

induces a unique p with $p \circ j \circ \otimes' = p \circ \otimes = \otimes'$ since $\otimes'$ is R-bilinear. Because of $jp \circ \otimes = j \circ \otimes' = \otimes = \mathrm{id}_{M \otimes_R N} \circ \otimes$ we get $jp = \mathrm{id}_{M \otimes_R N}$, hence the embedding j is surjective and thus the identity.
(2) $(m + m') \otimes n = \otimes(m + m', n) = \otimes(m, n) + \otimes(m', n) = m \otimes n + m' \otimes n$.
(3) and (4) analogously.
(5) is precisely the definition of the induced homomorphism. □

To construct tensor products, we need the notion of a free module.

1.3. Free modules.

Definition 1.8. Let X be a set and R be a ring. An R-module RX together with a map $\iota : X \longrightarrow RX$ is called a *free R-module generated by X* (or *an R-module freely generated by X*), if for every R-module M and for every map $f : X \longrightarrow M$ there exists a unique homomorphism of R-modules $g : RX \longrightarrow M$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "f"'] & RX \arrow[d, "g"] \\
 & M
\end{tikzcd}$$

commutes.
An R-module F is a *free R-module* if there is a set X and a map $\iota : X \longrightarrow F$ such that F is freely generated by X. Such a set X (or its image $\iota(X)$) is called a *free generating set for F*.

Warning: If you want to define a homomorphism $g : RX \longrightarrow M$ with a free module as domain you should define it by giving a map $f : X \longrightarrow M$.

Proposition 1.9. *A free R-module $\iota : X \longrightarrow RX$ defined over a set X is unique up to a unique isomorphism of R-modules.*

Proof. follows from the following diagram

$$\begin{tikzcd}
& & X \arrow[dll, "\iota"'] \arrow[dl, "\iota'"] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\
RX \arrow[r, "h"] & RX' \arrow[rr, "k"] & & RX \arrow[r, "h"] & RX'
\end{tikzcd}$$

□

Proposition 1.10. (Rules of computation in a free R-module) *Let $\iota : X \longrightarrow RX$ be a free R-module over X. Let $\widetilde{x} := \iota(x) \in RX$ for all $x \in X$. Then we have*

(1) *$\widetilde{X} = \{\widetilde{x} | \ \exists x \in X : \widetilde{x} = \iota(x)\}$ is a generating set of RX, i.e. each element $m \in RX$ is a linear combination $m = \sum_{i=1}^n r_i \widetilde{x}_i$ of the $\widetilde{x}$.*

(2) *$\widetilde{X} \subseteq RX$ is linearly independent and ι is injective, i.e. if $\sum'_{x \in X} r_x \widetilde{x} = 0$, then we have $\forall x \in X : r_x = 0$.*

Proof. (1) Let $M := \langle \widetilde{x} | x \in X \rangle \subseteq RX$ be the submodule generated by the $\widetilde{x}$. Then the diagram

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "0"'] & RX \arrow[d, shift right, "0"'] \arrow[d, shift left, "\nu"] \\
& RX/M
\end{tikzcd}$$

commutes with both maps 0 and ν. Thus $0 = \nu$ and $RX/M = 0$ and hence $RX = M$.
(2) Let $\sum_{i=0}^n r_i \widetilde{x}_i = 0$ and $r_0 \neq 0$. Let $j : X \longrightarrow R$ be the map given by $j(x_0) = 1, j(x) = 0$ for all $x \neq x_0$. $\Longrightarrow \ \exists g : RX \longrightarrow R$ with

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "j"'] & RX \arrow[d, "g"] \\
& R
\end{tikzcd}$$

commutative and $0 = g(0) = g(\sum_{i=0}^n r_i \widetilde{x}_i) = \sum_{i=0}^n r_i g(\widetilde{x}_i) = \sum_{i=0}^n r_i j(x_i) = r_0$. Contradiction. Hence the second statement. □

Notation 1.11. Since ι is injective we will identify X with it's image in RX and we will write $\sum_{x \in X} r_x x$ for an element $\sum_{x \in X} r_x \iota(x) \in RX$. The coefficients r_x are uniquely determined.

Proposition 1.12. *Let X be a set. Then there exists a free R-module $\iota : X \longrightarrow RX$ over X.*

Proof. Obviously $RX := \{\alpha : X \longrightarrow R | \text{ for almost all } x \in X : \alpha(x) = 0\}$ is a submodule of $\mathrm{Map}(X, R)$ which is an R-module by componentwise addition and multiplication. Define $\iota : X \longrightarrow RX$ by $\iota(x)(y) := \delta_{xy}$.
Let $f : X \longrightarrow M$ be an arbitrary map. Let $\alpha \in RX$. Define $g(\alpha) := \sum_{x \in X} \alpha(x) \cdot f(x)$. Then g is well defined, because we have $\alpha(x) \neq 0$ for only finitely many $x \in X$. Furthermore g is an R-module homomorphism: $rg(\alpha) + sg(\beta) = r \sum \alpha(x) \cdot f(x) + s \sum \beta(x) \cdot f(x) = \sum (r\alpha(x) + s\beta(x)) \cdot f(x) = \sum (r\alpha + s\beta)(x) \cdot f(x) = g(r\alpha + s\beta)$.
Furthermore we have $g\iota = f : g\iota(x) = \sum_{y \in X} \iota(x)(y) \cdot f(y) = \sum \delta_{xy} \cdot f(y) = f(x)$. For $\alpha \in RX$ we have $\alpha = \sum_{x \in X} \alpha(x)\iota(x)$ since $\alpha(y) = \sum \alpha(x)\iota(x)(y)$. In order to show that g is uniquely determined by f, let $h \in \mathrm{Hom}_R(RX, M)$ be given with $h\iota = f$. Then $h(\alpha) = h(\sum \alpha(x)\iota(x)) = \sum \alpha(x) h\iota(x) = \sum \alpha(x) f(x) = g(\alpha)$ hence $h = g$. □

Remark 1.13. If the base ring $\mathbb{K}$ is a field then a $\mathbb{K}$-module is a vector space. Each vector space V has a basis X (proof by Zorn's lemma). V together with the embedding $X \longrightarrow V$ is a free $\mathbb{K}$-module (as one shows in Linear Algebra). Hence every vector space is free. This is why one always defines vector space homomorphisms only on the basis.
For a vector space V any two bases have the same number of elements. This is not true for free modules over an arbitrary ring (see Exercise 1.4).

Problem 1.4. Show that for $R := \mathrm{End}_{\mathbb{K}}(V)$ for a vector space V of infinite countable dimension there is an isomorphism of left R-modules ${}_RR \cong {}_RR \oplus {}_RR$. Conclude that R is a free module on a generating set $\{1\}$ with one element and also free on a generating set with two elements.

Problem 1.5. Let $\iota : X \longrightarrow RX$ be a free module. Let $f : X \longrightarrow M$ be a map and $g : RX \longrightarrow M$ be the induced R-module homomorphism. Then

$$g(\sum_X r_x x) = \sum_X r_x f(x).$$

1.4. **Tensor products II.**

Proposition 1.14. *Given R-modules M_R and ${}_RN$. Then there exists a tensor product $(M \otimes_R N, \otimes)$.*

Proof. Define $M \otimes_R N := \mathbb{Z}(M \times N)/U$ where $\mathbb{Z}(M \times N)$ is a free $\mathbb{Z}$-module over $M \times N$ (the free Abelian group) and U is generated by

$$\begin{array}{l} \iota(m+m', n) - \iota(m,n) - \iota(m',n) \\ \iota(m, m+n') - \iota(m,n) - \iota(m,n') \\ \iota(mr, n) - \iota(m, rn) \end{array}$$

for all $r \in R$, $m, m' \in M$, $n, n' \in N$. Consider

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[drr, "\psi"'] & \mathbb{Z}(M \times N) \arrow[r, "\nu"] \arrow[dr, "\rho"] & M \otimes_R N = \mathbb{Z}(M \times N)/U \arrow[d, "g"] \\ & & A \end{tikzcd}$$

Let ψ be given. Then there is a unique $\rho \in \mathrm{Hom}(\mathbb{Z}(M \times N), A)$ such that $\rho\iota = \psi$. Since ψ is R-bilinear we get $\rho(\iota(m+m', n) - \iota(m,n) - \iota(m'n)) = \psi(m+m', n) - \psi(m,n) - \psi(m', n) = 0$ and similarly $\rho(\iota(m, n+n') - \iota(m,n) - \iota(m,n')) = 0$ and $\rho(\iota(mr,n) - \iota(m,rn)) = 0$. So we get $\rho(U) = 0$. This implies that there is a unique $g \in \mathrm{Hom}(M \otimes_R N, A)$ such that $g\nu = \rho$ (homomorphism theorem). Let $\otimes := \nu \circ \iota$. Then $\otimes$ is bilinear since $(m+m') \otimes n = \nu \circ \iota(m+m', n) = \nu(\iota(m+m', n)) = \nu(\iota(m+m', n) - \iota(m,n) - \iota(m', n) + \iota(m,n) + \iota(m', n)) = \nu(\iota(m,n) + \iota(m',n)) = \nu \circ \iota(m,n) + \nu \circ \iota(m', n) = m \otimes n + m' \otimes n$. The other two properties are obtained in an analogous way.
We have to show that $(M \otimes_R N, \otimes)$ is a tensor product. The above diagram shows that for each Abelian group A and for each R-bilinear map $\psi : M \times N \longrightarrow A$ there is a $g \in \mathrm{Hom}(M \otimes_R N, A)$ such that $g \circ \otimes = \psi$. Given $h \in \mathrm{Hom}(M \otimes_R N, A)$ with $h \circ \otimes = \psi$. Then $h \circ \nu \circ \iota = \psi$. This implies $h \circ \nu = \rho = g \circ \nu$ hence $g = h$. □

Proposition and Definition 1.15. *Given two homomorphisms*

$$f \in \mathrm{Hom}_R(M., M'.) \text{ and } g \in \mathrm{Hom}_R(.N, .N').$$

Then there is a unique homomorphism

$$f \otimes_R g \in \mathrm{Hom}(M \otimes_R N, M' \otimes_R N')$$

such that $f \otimes_R g(m \otimes n) = f(m) \otimes g(n)$, *i.e. the following diagram commutes*

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[d, "f \times g"'] & M \otimes_R N \arrow[d, "f \otimes_R g"] \\ M' \times N' \arrow[r, "\otimes"'] & M' \otimes_R N' \end{tikzcd}$$

Proof. $\otimes \circ (f \times g)$ is bilinear. $\square$

Notation 1.16. We often write $f \otimes_R N := f \otimes_R 1_N$ and $M \otimes_R g := 1_M \otimes_R g$.
We have the following rule of computation:

$$f \otimes_R g = (f \otimes_R N') \circ (M \otimes_R g) = (M' \otimes_R g) \circ (f \otimes_R N)$$

since $f \times g = (f \times N') \circ (M \times g) = (M' \times g) \circ (f \times N)$.

1.5. **Bimodules.**

Definition 1.17. Let R, S be rings and let M be a left R-module and a right S-module. M is called an *R-S-bimodule* if $(rm)s = r(ms)$. We define $\mathrm{Hom}_{R\text{-}S}(.M.,.N.) := \mathrm{Hom}_R(.M,.N) \cap \mathrm{Hom}_S(M.,N.)$.

Remark 1.18. Let M_S be a right S-module and let $R \times M \longrightarrow M$ be a map. M is an R-S-bimodule if and only if

(1) $\forall r \in R : (M \ni m \mapsto rm \in M) \in \mathrm{Hom}_S(M.,M.)$,
(2) $\forall r, r' \in R, m \in M : (r + r')m = rm + r'm$,
(3) $\forall r, r' \in R, m \in M : (rr')m = r(r'm)$,
(4) $\forall m \in M : 1m = m$.

Lemma 1.19. *Let ${}_R M_S$ and ${}_S N_T$ be bimodules. Then ${}_R(M \otimes_S N)_T$ is a bimodule by $r(m \otimes n) := rm \otimes n$ and $(m \otimes n)t := m \otimes nt$.*

Proof. Clearly we have that $(r \otimes_S \mathrm{id})(m \otimes n) = rm \otimes n = r(m \otimes n)$ is a homomorphism. Then (2)-(4) hold. Thus $M \otimes_S N$ is a left R-module. Similarly it is a right T-module. Finally we have $r((m \otimes n)t) = r(m \otimes nt) = rm \otimes nt = (rm \otimes n)t = (r(m \otimes n))t$. $\square$

Corollary 1.20. *Given bimodules ${}_R M_S$, ${}_S N_T$, ${}_R M'_S$, ${}_S N'_T$ and homomorphisms $f \in \mathrm{Hom}_{R\text{-}S}(.M.,.M'.)$ and $g \in \mathrm{Hom}_{S\text{-}T}(.N.,.N'.)$. Then we have $f \otimes_S g \in \mathrm{Hom}_{R\text{-}T}(.M \otimes_S N.,.M' \otimes_S N'.)$.*

Proof. $f \otimes_S g(rm \otimes nt) = f(rm) \otimes g(nt) = r(f \otimes_S g)(m \otimes n)t$. $\square$

Remark 1.21. Unless otherwise defined $\mathbb{K}$ will always be a commutative ring.
Every module M over the commutative ring $\mathbb{K}$ and in particular every vector space over a field $\mathbb{K}$ is a $\mathbb{K}$-$\mathbb{K}$-bimodule by $\lambda m = m\lambda$. Observe that there are $\mathbb{K}$-$\mathbb{K}$-bimodules that do not satisfy $\lambda m = m\lambda$. Take for example an automorphism $\alpha : \mathbb{K} \longrightarrow \mathbb{K}$ and a left $\mathbb{K}$-module M and define $m\lambda := \alpha(\lambda)m$. Then M is such a $\mathbb{K}$-$\mathbb{K}$-bimodule.
The tensor product $M \otimes_{\mathbb{K}} N$ of two $\mathbb{K}$-$\mathbb{K}$-bimodules M and N is again a $\mathbb{K}$-$\mathbb{K}$-bimodule. If we have, however, $\mathbb{K}$-$\mathbb{K}$-bimodules M and N arising from $\mathbb{K}$-modules as above, i.e. satisfying $\lambda m = m\lambda$, then their tensor product $M \otimes_{\mathbb{K}} N$ also satisfies this equation, so $M \otimes_{\mathbb{K}} N$ comes from a (left) $\mathbb{K}$-module. Indeed we have $\lambda m \otimes n = m\lambda \otimes n = m \otimes \lambda n = m \otimes n\lambda$. Thus we can also define a tensor product of two left $\mathbb{K}$-modules.
We often write the tensor product of two vector spaces or two left modules M and N over a commutative ring $\mathbb{K}$ as $M \otimes N$ instead of $M \otimes_{\mathbb{K}} N$ and the tensor product over $\mathbb{K}$ of two $\mathbb{K}$-module homomorphisms f and g as $f \otimes g$ instead of $f \otimes_{\mathbb{K}} g$.
(*Warning:* Do not confuse this with a tensor $f \otimes g$. See the following exercise.)

Problem 1.6. (1) Let M_R, ${}_RN$, M'_R, and ${}_RN'$ be R-modules. Show that the following is a homomorphism of Abelian groups:

$$\mu : \mathrm{Hom}_R(M, M') \otimes_{\mathbb{Z}} \mathrm{Hom}_R(N, N') \ni f \otimes g \mapsto f \otimes_R g \in \mathrm{Hom}(M \otimes_R N, M' \otimes_R N').$$

(2) Find examples where μ is not injective and where μ is not surjective.
(3) Explain why $f \otimes g$ is a decomposable tensor whereas $f \otimes_R g$ is not a tensor.

Theorem 1.22. *Let ${}_RM_S$, ${}_SN_T$, and ${}_TP_U$ be bimodules. Then there are canonical isomorphisms of bimodules*

(1) Associativity Law: $\alpha : (M \otimes_S N) \otimes_T P \cong M \otimes_S (N \otimes_T P)$.
(2) Law of the Left Unit: $\lambda : R \otimes_R M \cong M$.
(3) Law of the Right Unit: $\rho : M \otimes_S S \cong M$.
(4) Symmetry Law: *If M, N are $\mathbb{K}$-modules then there is an isomorphism of $\mathbb{K}$-modules* $\tau : M \otimes N \cong N \otimes M$.
(5) Existence of Inner Hom-Functors: *Let ${}_RM_T$, ${}_SN_T$, and ${}_SP_R$ be bimodules. Then there are canonical isomorphisms of bimodules*

$$\mathrm{Hom}_{S\text{-}T}(.P \otimes_R M., .N.) \cong \mathrm{Hom}_{S\text{-}R}(.P., .\mathrm{Hom}_T(M., N.).) \text{ and}$$

$$\mathrm{Hom}_{S\text{-}T}(.P \otimes_R M., .N.) \cong \mathrm{Hom}_{R\text{-}T}(.M., .\mathrm{Hom}_S(.P, .N).).$$

Proof. We only describe the corresponding homomorphisms.
(1) Use 1.7 (5) to define $\alpha((m \otimes n) \otimes p) := m \otimes (n \otimes p)$.
(2) Define $\lambda : R \otimes_R M \longrightarrow M$ by $\lambda(r \otimes m) := rm$.
(3) Define $\rho : M \otimes_S S \longrightarrow M$ by $\rho(m \otimes s) := ms$.
(4) Define $\tau(m \otimes n) := n \otimes m$.
(5) For $f : P \otimes_R M \longrightarrow N$ define $\phi(f) : P \longrightarrow \mathrm{Hom}_T(M, N)$ by $\phi(f)(p)(m) := f(p \otimes m)$ and $\psi(f) : M \longrightarrow \mathrm{Hom}_S(P, N)$ by $\psi(f)(m)(p) := f(p \otimes m)$. $\square$

Usually one identifies threefold tensor products along the map α so that we can use $M \otimes_S N \otimes_T P := (M \otimes_S N) \otimes_T P = M \otimes_S (N \otimes_T P)$. For the notion of a monoidal or tensor category, however, this canonical isomorphism (natural transformation) is of central importance and will be discussed later.

Problem 1.7.
(1) Give a complete proof of Theorem 1.22. In (5) show how $\mathrm{Hom}_T(M., N.)$ becomes an S-R-bimodule.
(2) Give an explicit proof of $M \otimes_R (X \oplus Y) \cong M \otimes_R X \oplus M \otimes_R Y$.
(3) Show that for every finite dimensional vector space V there is a *unique* element $\sum_{i=1}^n v_i \otimes v_i^* \in V \otimes V^*$ such that the following holds

$$\forall v \in V : \quad \sum_i v_i^*(v) v_i = v.$$

(Hint: Use an isomorphism $\mathrm{End}(V) \cong V \otimes V^*$ and dual bases $\{v_i\}$ of V and $\{v_i^*\}$ of V^*.)
(4) Show that the following diagrams (*coherence diagrams or constraints*) of $\mathbb{K}$-modules commute:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\alpha(A\otimes B,C,D)"] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha(B,C,D)"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \mathbb{K}) \otimes B \arrow[rr, "\alpha(A,\mathbb{K},B)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A \otimes (\mathbb{K} \otimes B) \arrow[dl, "1\otimes\lambda(B)"] \\
& A \otimes B &
\end{tikzcd}$$

(5) Write $\tau(A,B) : A \otimes B \longrightarrow B \otimes A$ for $\tau(A,B) : a \otimes b \mapsto b \otimes a$. Show that

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\tau(X,B)\otimes 1"] \arrow[d, "\alpha"'] & (B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[d, "1\otimes\tau(A,C)"] \\
A \otimes (B \otimes C) \arrow[r, "\tau(A,B\otimes C)"] & (B \otimes C) \otimes A \arrow[r, "\alpha"] & B \otimes (C \otimes A)
\end{tikzcd}$$

commutes for all $\mathbb{K}$-modules A, B, C and that

$$\tau(B,A)\tau(A,B) = \mathrm{id}_{A\otimes B}$$

for all $\mathbb{K}$-modules A and B. Let $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ be $\mathbb{K}$-modules homomorphisms. Show that

$$\begin{tikzcd}
A \otimes B \arrow[r, "\tau(A,B)"] \arrow[d, "f\otimes g"'] & B \otimes A \arrow[d, "g\otimes f"] \\
A' \otimes B' \arrow[r, "\tau(A',B')"] & B' \otimes A'
\end{tikzcd}$$

commutes.

(6) Find an example of M, $N \in \mathbb{K}$-Mod-$\mathbb{K}$ such that $M \otimes_{\mathbb{K}} N \not\cong N \otimes_{\mathbb{K}} M$.

Proposition 1.23. *Let (RX, ι) be a free R-module and ${}_SM_R$ be a bimodule. Then every element $u \in M \otimes_R RX$ has a unique representation $u = \sum_{x\in X} m_x \otimes x$.*

Proof. By 1.10 $\sum_{x\in X} r_x x$ is the general element of RX. Hence we have $u = \sum m_i \otimes \alpha_i = \sum m_i \otimes \sum r_{x,i} x = \sum_i \sum_x m_i r_{x,i} \otimes x = \sum_x (\sum_i m_i r_{x,i}) \otimes x$. To show the uniqueness let $\sum_{y\in X} m_y \otimes y = 0$. Let $x \in X$ and $f_x : RX \longrightarrow R$ be defined by $f_x(\iota(y)) = f_x(y) := \delta_{xy}$. Then $(1_M \otimes_R f_x)(\sum m_y \otimes y) = \sum m_y \otimes f_x(y) = m_x \otimes 1 = 0$ for all $x \in X$. Now let

$$\begin{tikzcd}
M \times R \arrow[r, "\otimes"] \arrow[dr, "\mathrm{mult}"'] & M \otimes_R R \arrow[d, "\rho"] \\
& M
\end{tikzcd}$$

be given. Then $\rho(m_x \otimes 1) = m_x \cdot 1 = m_x = 0$ hence we have uniqueness. From 1.22 (3) we know that ρ is an isomorphism. $\square$

Corollary 1.24. *Let ${}_SM_R$, ${}_RN$ be (bi-)modules. Let M be a free S-module over Y, and N be a free R-module over X. Then $M \otimes_R N$ is a free S-module over $Y \times X$.*

Proof. Consider the diagram

$$\begin{tikzcd}
Y \times X \arrow[r, "\iota_Y \times \iota_X"] \arrow[drr, "f"'] & M \times N \arrow[r, "\otimes"] \arrow[dr, "g"] & {}_SM \otimes_R N \arrow[d, "h"] \\
& & {}_SU
\end{tikzcd}$$

Let f be an arbitrary map. For all $x \in X$ we define homomorphisms $g(\text{-}, x) \in \mathrm{Hom}_S(.M, .U)$ by the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\iota_Y"] \arrow[dr, "f(\text{-},x)"'] & {}_SM \arrow[d, "g(\text{-},x)"] \\ & {}_SU \end{tikzcd}$$

Let $\widetilde{g} \in \mathrm{Hom}_R(.N, .\mathrm{Hom}_S(.M_R, .U))$ be defined by

$$\begin{tikzcd} X \arrow[r, "\iota_x"] \arrow[dr, "g(\text{-},\text{-})"'] & {}_RN \arrow[d, "\widetilde{g}"] \\ & {}_R\mathrm{Hom}_S(.M_R, .U) \end{tikzcd}$$

with $x \mapsto g(\text{-}, x)$. Then we define $g(m, n) := \widetilde{g}(n)(m) =: h(m \otimes n)$. Observe that g is additive in m and in n (because $\widetilde{g}$ is additive in m and in n), and g is R-bilinear, because $g(mr, n) = \widetilde{g}(n)(mr) = (r\widetilde{g}(n))(m) = \widetilde{g}(rn)(m) = g(m, rn)$. Obviously $g(y, x) = f(y, x)$, hence $h \circ \otimes \circ \iota_Y \times \iota_X = f$. Furthermore we have $h(sm \otimes n) = \widetilde{g}(n)(sm) = s(\widetilde{g}(n)(m)) = sh(m \otimes n)$, hence h is an S-module homomorphism.
Let k be an S-module homomorphism satisfying $k \circ \otimes \circ \iota_Y \times \iota_X = f$, then $k \circ \otimes(\text{-}, x) = g(\text{-}, x)$, since $k \circ \otimes$ is S-linear in the first argument. Thus $k \circ \otimes(m, n) = \widetilde{g}(n)(m) = h(m \otimes n)$, and hence $h = k$. $\square$

Problem 1.8. (Tensors in physics:) Let V be a finite dimensional vector space over the field $\mathbb{K}$ and let V^* be its dual space. Let t be a tensor in $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^* = V^{\otimes r} \otimes (V^*)^{\otimes s}$.
(1) Show that for each basis $B = (b_1, \ldots, b_n)$ and dual basis $B^* = (b^1, \ldots, b^n)$ there is a uniquely determined scheme (a family or an $(r+s)$-dimensional matrix) of coefficients $(a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s})$ with $a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \in \mathbb{K}$ such that

$$t = \sum_{i_1=1}^{n} \ldots \sum_{i_r=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{j_s=1}^{n} a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} b_{i_1} \otimes \ldots \otimes b_{i_r} \otimes b^{j_1} \otimes \ldots \otimes b^{j_s}. \tag{1}$$

(2) Show that for each change of bases $L : B \longrightarrow C$ with $c_j = \sum \lambda^i_j b_i$ (with inverse matrix (μ^i_j)) the following *transformation formula* holds

$$a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = \sum_{k_1=1}^{n} \ldots \sum_{k_r=1}^{n} \sum_{l_1=1}^{n} \ldots \sum_{l_s=1}^{n} \lambda^{i_1}_{k_1} \ldots \lambda^{i_r}_{k_r} \mu^{l_1}_{j_1} \ldots \mu^{l_s}_{j_s} a(C)^{k_1,\ldots,k_r}_{l_1,\ldots,l_s} \tag{2}$$

(3) Show that every family of schemes of coefficients $(a(B)|B$ basis of $V)$ with $a(B) = (a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s})$ and $a(B)^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \in K$ satisfying the transformation formula (2) defines a unique tensor (independent of the choice of the basis) $t \in V^{\otimes r} \otimes (V^*)^{\otimes s}$ such that (1) holds.
Rule for physicists: A tensor is a collection of schemes of coefficients that transform according to the transformation formula for tensors.

1.6. **Complexes and exact sequences.**

Definition 1.25. A (finite or infinite) sequence of homomorphisms

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \ldots$$

is called a *complex*, if $f_i f_{i-1} = 0$ for all $i \in I$ (or equivalently $\mathrm{Im}(f_{i-1}) \subseteq \mathrm{Ke}(f_i)$).
A complex is called *exact* or an *exact sequence* if $\mathrm{Im}(f_{i-1}) = \mathrm{Ke}(f_i)$ for all $i \in I$.

Lemma 1.26. *A complex*

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \ldots$$

is exact if and only if the sequences

$$0 \longrightarrow \mathrm{Im}(f_{i-1}) \longrightarrow M_i \longrightarrow \mathrm{Im}(f_i) \longrightarrow 0$$

are exact for all $i \in I$, if and only if the sequences

$$0 \longrightarrow \mathrm{Ke}(f_{i-1}) \longrightarrow M_{i-1} \longrightarrow \mathrm{Ke}(f_i) \longrightarrow 0$$

are exact for all $i \in I$.

Proof. The sequences

$$0 \longrightarrow \mathrm{Ke}(f_i) \longrightarrow M_i \longrightarrow \mathrm{Im}(f_i) \longrightarrow 0$$

are obviously exact since $\mathrm{Ke}(f_i) \longrightarrow M_i$ is a monomorphism, $M_i \longrightarrow \mathrm{Im}(f_i)$ is an epimorphism and $\mathrm{Ke}(f_i)$ is the kernel of $M_i \longrightarrow \mathrm{Im}(f_i)$.
The sequence

$$0 \longrightarrow \mathrm{Im}(f_{i-1}) \longrightarrow M_i \longrightarrow \mathrm{Im}(f_i) \longrightarrow 0$$

is exact if and only if $\mathrm{Im}(f_{i-1}) = \mathrm{Ke}(f_i)$.
The sequence

$$0 \longrightarrow \mathrm{Ke}(f_{i-1}) \longrightarrow M_{i-1} \longrightarrow \mathrm{Ke}(f_i) \longrightarrow 0$$

is exact if and only if $M_{i-1} \longrightarrow \mathrm{Ke}(f_i)$ is surjective, if and only if $\mathrm{Im}(f_{i-1}) = \mathrm{Ke}(f_i)$. $\square$

Problem 1.9. (1) In the tensor product $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ we have $1 \otimes i - i \otimes 1 = 0$.
In the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ we have $1 \otimes i - i \otimes 1 \neq 0$.

(2) For each R-module M we have $R \otimes_R M \cong M$.

(3) Given the $\mathbb{Q}$-vector space $V = \mathbb{Q}^n$.
  (a) Determine $\dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Q}} V)$.
  (b) Describe explicitely an isomorphism $\mathbb{R} \otimes_{\mathbb{Q}} V \cong \mathbb{R}^n$.

(4) Let V be a $\mathbb{Q}$-vector space and W be an $\mathbb{R}$-vector space.
  (a) $\mathrm{Hom}_{\mathbb{R}}(.\mathbb{R}_{\mathbb{Q}}, .W) \cong W$ in $\mathbb{Q}$-Mod.
  (b) $\mathrm{Hom}_{\mathbb{Q}}(.V, .W) \cong \mathrm{Hom}_{\mathbb{R}}(.\mathbb{R} \otimes_{\mathbb{Q}} V, .W)$.
  (c) Let $\dim_{\mathbb{Q}} V < \infty$ and $\dim_{\mathbb{R}} W < \infty$. How can one explain that in 4b we have infinite matrices on the left hand side and finite matrices on the right hand side?
  (d) $\mathrm{Hom}_{\mathbb{Q}}(.V, \mathrm{Hom}_{\mathbb{R}}(.\mathbb{R}, .W) \cong \mathrm{Hom}_{\mathbb{R}}(.\mathbb{R} \otimes_{\mathbb{Q}} V, .W)$.

(5) $\mathbb{Z}/(18) \otimes_{\mathbb{Z}} \mathbb{Z}/(30) \neq 0$.

(6) $m : \mathbb{Z}/(18) \otimes_{\mathbb{Z}} \mathbb{Z}/(30) \ni \overline{x} \otimes \overline{y} \mapsto \overline{xy} \in \mathbb{Z}/(6)$ is a homomorphism and m is bijective.

(7) For $\mathbb{Q}$-vector spaces V and W we have $V \otimes_{\mathbb{Z}} W \cong V \otimes_{\mathbb{Q}} W$.

(8) For each finite Abelian group M we have $\mathbb{Q} \otimes_{\mathbb{Z}} M = 0$.

(9) $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathbb{Z}/(\mathrm{ggT}(m, n))$.

(10) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(n) = 0$.

(11) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/(n)) = 0$.

(12) Determine explicitely isomorphisms for

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q},$$
$$3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}.$$

Show that the following diagram commutes

$$\begin{tikzcd} 3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, "\cong"] \\ \mathbb{Q} \arrow[r, "3\cdot"'] & \mathbb{Q} \end{tikzcd}$$

(13) The homomorphism $2\mathbb{Z} \otimes_Z \mathbb{Z}/(2) \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(2)$ is the zero homomorphism, but both modules are different from zero.

2. Algebras and Coalgebras

2.1. **Algebras.** Let $\mathbb{K}$ be a commutative ring. We consider all $\mathbb{K}$-modules as $\mathbb{K}$-$\mathbb{K}$-bimodules as in Remark 1.21. Tensor products of $\mathbb{K}$-modules will be simply written as $M \otimes N := M \otimes_K N$.

Definition 2.1. A $\mathbb{K}$-*algebra* is a $\mathbb{K}$-module A together with a *multiplication* $\nabla : A \otimes A \longrightarrow A$ ($\mathbb{K}$-module homomorphism) that is associative:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \nabla"] \arrow[d, "\nabla \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}$$

and a *unit* $\eta : \mathbb{K} \longrightarrow A$ ($\mathbb{K}$-module homomorphism):

$$\begin{tikzcd}
\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K} \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\eta \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A.
\end{tikzcd}$$

A $\mathbb{K}$-algebra A is *commutative* if the following diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau"] \arrow[dr, "\nabla"'] & & A \otimes A \arrow[dl, "\nabla"] \\
& A. &
\end{tikzcd}$$

Let A and B be $\mathbb{K}$-algebras. A *homomorphism of algebras* $f : A \longrightarrow B$ is a $\mathbb{K}$-module homomorphism such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\nabla_A"'] & B \otimes B \arrow[d, "\nabla_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{K} \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\
A \arrow[rr, "f"] & & B.
\end{tikzcd}$$

Remark 2.2. Every $\mathbb{K}$-algebra A is a ring with the multiplication

$$A \times A \xrightarrow{\otimes} A \otimes A \xrightarrow{\nabla} A.$$

The unit element is $\eta(1)$, where 1 is the unit element of $\mathbb{K}$.
Obviously the composition of two homomorphisms of algebras is again a homomorphism of algebras. Furthermore the identity map is a homomorphism of algebras.

Problem 2.1. (1) Show that $\mathrm{End}_{\mathbb{K}}(V)$ is a $\mathbb{K}$-algebra.
(2) Show that $(A, \nabla : A \otimes A \longrightarrow A, \eta : \mathbb{K} \longrightarrow A)$ is a $\mathbb{K}$-algebra if and only if A with the multiplication $A \times A \xrightarrow{\otimes} A \otimes A \xrightarrow{\nabla} A$ and the unit $\eta(1)$ is a ring and $\eta : \mathbb{K} \longrightarrow \mathrm{Cent}(A)$ is a ring homomorphism into the *center* of A, where $\mathrm{Cent}(A) := \{a \in A | \forall b \in A : ab = ba\}$.
(3) Let V be a $\mathbb{K}$-module. Show that $D(V) := \mathbb{K} \times V$ with the multiplication $(r_1, v_1)(r_2, v_2) := (r_1 r_2, r_1 v_2 + r_2 v_1)$ is a commutative $\mathbb{K}$-algebra.

Lemma 2.3. *Let A and B be algebras. Then $A \otimes B$ is an algebra with the multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$.*

Proof. Certainly the algebra properties can easily be checked by a simple calculation with elements. For later applications we prefer a diagrammatic proof.
Let $\nabla_A : A \otimes A \longrightarrow A$ and $\nabla_B : B \otimes B \longrightarrow B$ denote the multiplications of the two algebras. Then the new multiplication is $\nabla_{A \otimes B} := (\nabla_A \otimes \nabla_B)(1_A \otimes \tau \otimes 1_B) : A \otimes B \otimes A \otimes B \longrightarrow A \otimes B$ where $\tau : B \otimes A \longrightarrow A \otimes B$ is the symmetry map from Theorem 1.22. Now the following diagram commutes

$$\begin{tikzcd}
A\otimes B\otimes A\otimes B\otimes A\otimes B \arrow[rr, "1\otimes\tau\otimes 1^3"] \arrow[dd, "1^3\otimes\tau\otimes 1"'] & & A\otimes A\otimes B\otimes B\otimes A\otimes B \arrow[r, "\nabla\otimes\nabla\otimes 1^2"] \arrow[dl, "1^3\otimes\tau\otimes 1"'] \arrow[dd, "1^2\otimes\tau_{B\otimes B,A}\otimes 1"] & A\otimes B\otimes A\otimes B \arrow[dd, "1\otimes\tau\otimes 1"] \\
 & A\otimes A\otimes B\otimes A\otimes B\otimes B \arrow[dr, "1^2\otimes\tau\otimes 1^2"] & & \\
A\otimes B\otimes A\otimes A\otimes B\otimes B \arrow[ur, "1\otimes\tau\otimes 1^3"] \arrow[rr, "1\otimes\tau_{B,A\otimes A}\otimes 1^2"'] \arrow[d, "1^2\otimes\nabla\otimes\nabla"'] & & A\otimes A\otimes A\otimes B\otimes B\otimes B \arrow[r, "\nabla\otimes 1\otimes\nabla\otimes 1"] \arrow[d, "1\otimes\nabla\otimes 1\otimes\nabla"] & A\otimes A\otimes B\otimes B \arrow[d, "\nabla\otimes\nabla"] \\
A\otimes B\otimes A\otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & A\otimes A\otimes B\otimes B \arrow[r, "\nabla\otimes\nabla"] & A\otimes B
\end{tikzcd}$$

In the left upper rectangle of the diagram the quadrangle commutes by the properties of the tensor product and the two triangles commute by inner properties of τ. The right upper and left lower rectangles commute since τ is a natural transformation (use Exercise 1.7 (5)) and the right lower rectangle commutes by the associativity of the algebras A and B.
Furthermore we use the homomorphism $\eta = \eta_{A\otimes B} : \mathbb{K} \longrightarrow \mathbb{K} \otimes \mathbb{K} \longrightarrow A \otimes B$ in the following commutative diagram

$$\begin{tikzcd}
\mathbb{K}\otimes A\otimes B \cong A\otimes B \cong A\otimes B\otimes\mathbb{K} \arrow[rr] \arrow[d] \arrow[dr] \arrow[ddrrr, "1"] \arrow[drr] & & A\otimes B\otimes\mathbb{K}\otimes\mathbb{K} \arrow[r, "1^2\otimes\eta\otimes\eta"] \arrow[d, "1\otimes\tau\otimes 1"] & A\otimes B\otimes A\otimes B \arrow[d, "1\otimes\tau\otimes 1"] \\
\mathbb{K}\otimes\mathbb{K}\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\eta\otimes\eta\otimes 1^2"'] & \mathbb{K}\otimes A\otimes\mathbb{K}\otimes B \arrow[d, "\eta\otimes 1\otimes\eta\otimes 1"'] & A\otimes\mathbb{K}\otimes B\otimes\mathbb{K} \arrow[r, "1\otimes\eta\otimes 1\otimes\eta"'] & A\otimes A\otimes B\otimes B \arrow[d, "\nabla\otimes\nabla"] \\
A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] & A\otimes A\otimes B\otimes B \arrow[rr, "\nabla\otimes\nabla"] & & A\otimes B.
\end{tikzcd}$$

$\square$

2.2. **Tensor algebras.**

Definition 2.4. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $T(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow T(V)$ is called a *tensor algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ there exists a unique homomorphism of $\mathbb{K}$-algebras $g : T(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.
Note: If you want to define a homomorphism $g : T(V) \longrightarrow A$ with a tensor algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules defined on V.

Lemma 2.5. *A tensor algebra $(T(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(T(V), \iota)$ and $(T'(V), \iota')$ be tensor algebras over V. Then

$$\begin{tikzcd} & & V \arrow[dll, "\iota"'] \arrow[dl, "\iota'"] \arrow[drr, "\iota"'] \arrow[drrr, "\iota'"] & & & \\ T(V) \arrow[r, "h"] & T'(V) \arrow[rrr, "k"] & & & T(V) \arrow[r, "h"] & T'(V) \end{tikzcd}$$

implies $k = h^{-1}$. ◻

Proposition 2.6. (Rules of computation in a tensor algebra) *Let $(T(V), \iota)$ be the tensor algebra over V. Then we have*

(1) *$\iota : V \longrightarrow T(V)$ is injective (so we may identify the elements $\iota(v)$ and v for all $v \in V$),*
(2) *$T(V) = \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$, where $v_{i_j} \in V$,*
(3) *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules, A is a $\mathbb{K}$-algebra, and $g : T(V) \longrightarrow A$ is the induced homomorphism of $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. (1) Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is defined as in 2.1 (3) to construct $g : T(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.
(2) Let $B := \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $T(V)$ generated by the elements of V. Let $j : B \longrightarrow T(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow T(V)$ factors through a $\mathbb{K}$-module homomorphism $\iota' : V \longrightarrow B$. The following diagram

$$\begin{tikzcd} V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] & T(V) \arrow[dl, "p"'] \arrow[d, "jp"] \\ & B \arrow[r, "j"] & T(V) \end{tikzcd}$$

induces a unique p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ since ι' is a homomorphism of $\mathbb{K}$-modules. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{T(V)} \circ \iota$ we get $jp = \mathrm{id}_{T(V)}$, hence the embedding j is surjective and thus j is the identity.
(3) is precisely the definition of the induced homomorphism. ◻

Proposition 2.7. *Given a $\mathbb{K}$-module V. Then there exists a tensor algebra $(T(V), \iota)$.*

Proof. Define $T^n(V) := V \otimes \ldots \otimes V = V^{\otimes n}$ to be the n-fold tensor product of V. Define $T^0(V) := \mathbb{K}$ and $T^1(V) := V$. We define

$$T(V) := \bigoplus_{i \geq 0} T^i(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots .$$

The components $T^n(V)$ of $T(V)$ are called *homogeneous components.*
The canonical isomorphisms $T^m(V) \otimes T^n(V) \cong T^{m+n}(V)$ taken as multiplication

$$\nabla : T^m(V) \otimes T^n(V) \longrightarrow T^{m+n}(V)$$
$$\nabla : T(V) \otimes T(V) \longrightarrow T(V)$$

and the embedding $\eta : \mathbb{K} = T^0(V) \longrightarrow T(V)$ induce the structure of a $\mathbb{K}$-algebra on $T(V)$. Furthermore we have the embedding $\iota : V \longrightarrow T^1(V) \subseteq T(V)$.
We have to show that $(T(V), \iota)$ is a tensor algebra. Let $f : V \longrightarrow A$ be a homomorphism of $\mathbb{K}$-modules. Each element in $T(V)$ is a sum of decomposable tensors $v_1 \otimes \ldots \otimes v_n$. Define $g : T(V) \longrightarrow A$ by $g(v_1 \otimes \ldots \otimes v_n) := f(v_1) \ldots f(v_n)$ (and $(g : T^0(V) \longrightarrow A) = (\eta : \mathbb{K} \longrightarrow A)$). By induction one sees that g is a homomorphism of algebras. Since $(g : T^1(V) \longrightarrow A) = (f : V \longrightarrow A)$ we get $g \circ \iota = f$. If $h : T(V) \longrightarrow A$ is a homomorphism of algebras with $h \circ \iota = f$ we get $h(v_1 \otimes \ldots \otimes v_n) = h(v_1) \ldots h(v_n) = f(v_1) \ldots f(v_n)$ hence $h = g$. $\square$

Problem 2.2. (1) Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \longrightarrow V \longrightarrow T(V)$ defines a *free algebra* over X, i.e. for every $\mathbb{K}$-algebra A and every map $f : X \longrightarrow A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : T(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.
We write $\mathbb{K}\langle X \rangle := T(\mathbb{K}X)$ and call it the *polynomial ring over $\mathbb{K}$ in the non-commuting variables X.*
(2) Let $T(V)$ and $\iota : V \longrightarrow T(V)$ be a tensor algebra. Regard V as a subset of $T(V)$ by ι. Show that there is a unique homomorphism of algebras $\Delta : T(V) \longrightarrow T(V) \otimes T(V)$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.
(3) Show that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : T(V) \longrightarrow T(V) \otimes T(V) \otimes T(V)$.
(4) Show that there is a unique homomorphism of algebras $\varepsilon : T(V) \longrightarrow \mathbb{K}$ with $\varepsilon(v) = 0$ for all $v \in V$.
(5) Show that $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = \mathrm{id}_{T(V)}$.
(6) Show that there is a unique homomorphism of algebras $S : T(V) \longrightarrow T(V)^{op}$ with $S(v) = -v$. ($T(V)^{op}$ is the *opposite algebra* of $T(V)$ with multiplication $s * t := ts$ for all $s, t \in T(V) = T(V)^{op}$ and where st denotes the product in $T(V)$.)
(7) Show that the diagrams

$$\begin{tikzcd} T(V) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & T(V) \\ T(V) \otimes T(V) \arrow[rr, "1\otimes S", "S\otimes 1"'] & & T(V) \otimes T(V) \arrow[u, "\nabla"'] \end{tikzcd}$$

commute.

2.3. **Symmetric algebras.**

Definition 2.8. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $S(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow S(V)$, such that $\iota(v)\cdot\iota(v') = \iota(v')\cdot\iota(v)$ for all $v, v' \in V$, is called a *symmetric algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$, such that $f(v)\cdot f(v') = f(v')\cdot f(v)$ for all $v, v' \in V$, there exists a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.
Note: If you want to define a homomorphism $g : S(V) \longrightarrow A$ with a symmetric algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ satisfying $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$.

Lemma 2.9. *A symmetric algebra $(S(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(S(V), \iota)$ and $(S'(V), \iota')$ be symmetric algebras over V. Then

$$\begin{tikzcd} & & V \arrow[dll, "\iota"'] \arrow[dl, "\iota'"] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\ S(V) \arrow[r, "h"] & S'(V) \arrow[rr, "k"] & & S(V) \arrow[r, "h"] & S'(V) \end{tikzcd}$$

implies $k = h^{-1}$. $\square$

Proposition 2.10. (Rules of computation in a symmetric algebra) *Let $(S(V), \iota)$ be the symmetric algebra over V. Then we have*

(1) *$\iota : V \longrightarrow S(V)$ is injective (we will identify the elements $\iota(v)$ and v for all $v \in V$),*
(2) *$S(V) = \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*
(3) *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules satisfying $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$, A is a $\mathbb{K}$-algebra, and $g : S(V) \longrightarrow A$ is the induced homomorphism $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. (1) Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is the commutative algebra defined in 2.1 (3) to construct $g : S(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.
(2) Let $B := \{\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $S(V)$ generated by the elements of V. Let $j : B \longrightarrow S(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow S(V)$ factors through a $\mathbb{K}$-module homomorphism $\iota' : V$

$\longrightarrow B$. The following diagram

$$\begin{tikzcd} V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] & S(V) \arrow[dl, "p"'] \arrow[d, "jp"] \\ & B \arrow[r, "j"] & S(V) \end{tikzcd}$$

induces a unique p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ since ι' is a homomorphism of $\mathbb{K}$-modules satisfying $\iota'(v) \cdot \iota'(v') = \iota'(v') \cdot \iota'(v)$ for all $v, v' \in V$. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{S(V)} \circ \iota$ we get $jp = \mathrm{id}_{S(V)}$, hence the embedding j is surjective and thus the identity.
(3) is precisely the definition of the induced homomorphism. □

Proposition 2.11. *Let V be a $\mathbb{K}$-module. The symmetric algebra $(S(V), \iota)$ is commutative and satisfies the following universal property:*
for each commutative $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$ there exists a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

Proof. Commutativity follows from the commutativity of the generators: $vv' = v'v$ which carries over to the elements of the form $\sum_{n,\bar{i}} v_{i_1} \cdot \ldots \cdot v_{i_n}$. The universal property follows since the defining condition $f(v) \cdot f(v') = f(v') \cdot f(v)$ for all $v, v' \in V$ is automatically satisfied. □

Proposition 2.12. *Given a $\mathbb{K}$-module V. Then there exists a symmetric algebra $(S(V), \iota)$.*

Proof. Define $S(V) := T(V)/I$ where $I = \langle vv' - v'v | v, v' \in V \rangle$ is the two-sided ideal generated by the elements $vv' - v'v$. Let ι be the canonical map $V \longrightarrow T(V) \longrightarrow S(V)$. Then the universal property is easily verified by the homomorphism theorem for algebras. □

Problem 2.3. (1) Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \longrightarrow V \longrightarrow S(V)$ defines a *free commutative algebra* over X, i.e. for every commutative $\mathbb{K}$-algebra A and every map $f : X \longrightarrow A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.
The algebra $\mathbb{K}[X] := S(\mathbb{K}X)$ is called the *polynomial ring over $\mathbb{K}$ in the (commuting) variables X.*
(2) Let $S(V)$ and $\iota : V \longrightarrow S(V)$ be a symmetric algebra. Show that there is a unique homomorphism of algebras $\Delta : S(V) \longrightarrow S(V) \otimes S(V)$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.
(3) Show that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : S(V) \longrightarrow S(V) \otimes S(V) \otimes S(V)$.

(4) Show that there is a unique homomorphism of algebras $\varepsilon : S(V) \longrightarrow \mathbb{K}$ with $\varepsilon(v) = 0$ for all $v \in V$.
(5) Show that $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = \mathrm{id}_{S(V)}$.
(6) Show that there is a unique homomorphism of algebras $S : S(V) \longrightarrow S(V)$ with $S(v) = -v$.
(7) Show that the diagrams

$$\begin{tikzcd}
S(V) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & S(V) \\
S(V) \otimes S(V) \arrow[rr, "1\otimes S", "S\otimes 1"'] & & S(V) \otimes S(V) \arrow[u, "\nabla"']
\end{tikzcd}$$

commute.

2.4. **Exterior algebras.**

Definition 2.13. Let $\mathbb{K}$ be a commutative ring. Let V be a $\mathbb{K}$-module. A $\mathbb{K}$-algebra $E(V)$ together with a homomorphism of $\mathbb{K}$-modules $\iota : V \longrightarrow E(V)$, such that $\iota(v)^2 = 0$ for all $v \in V$, is called an *exterior algebra or Grassmann algebra over* V if for each $\mathbb{K}$-algebra A and for each homomorphism of $\mathbb{K}$-modules $f : V \longrightarrow A$, such that $f(v)^2 = 0$ for all $v \in V$, there exists a unique homomorphism of $\mathbb{K}$-algebras $g : E(V) \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "f"'] & E(V) \arrow[d, "g"] \\
& A
\end{tikzcd}$$

commutes.
The multiplication in $E(V)$ is usually denoted by $u \wedge v$.
Note: If you want to define a homomorphism $g : E(V) \longrightarrow A$ with an exterior algebra as domain you should define it by giving a homomorphism of $\mathbb{K}$-modules defined on V satisfying $f(v)^2 = 0$ for all $v, v' \in V$.

Problem 2.4. (1) Let $f : V \longrightarrow A$ be a $\mathbb{K}$-module homomorphism satisfying $f(v)^2 = 0$ for all $v \in V$. Then $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$.
(2) Let 2 be invertible in $\mathbb{K}$ (e.g. $\mathbb{K}$ a field of characteristic $\neq 2$). Let $f : V \longrightarrow A$ be a $\mathbb{K}$-module homomorphism satisfying $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$. Then $f(v)^2 = 0$ for all $v \in V$.

Lemma 2.14. *An exterior algebra $(E(V), \iota)$ defined by V is unique up to a unique isomorphism.*

Proof. Let $(E(V), \iota)$ and $(E'(V), \iota')$ be exterior algebras over V. Then

$$\begin{tikzcd}
& V \arrow[dl, "\iota"'] \arrow[d, "\iota'"] \arrow[dr, "\iota"] \arrow[drr, "\iota'"] & & \\
E(V) \arrow[r, "h"] & E'(V) \arrow[r, "k"] & E(V) \arrow[r, "h"] & E'(V)
\end{tikzcd}$$

implies $k = h^{-1}$. □

Proposition 2.15. (Rules of computation in an exterior algebra) *Let $(E(V), \iota)$ be the exterior algebra over V. Then we have*

(1) *$\iota : V \longrightarrow E(V)$ is injective (we will identify the elements $\iota(v)$ and v for all $v \in V$),*

(2) *$E(V) = \{\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$,*

(3) *if $f : V \longrightarrow A$ is a homomorphism of $\mathbb{K}$-modules satisfying $f(v) \cdot f(v') = -f(v') \cdot f(v)$ for all $v, v' \in V$, A is a $\mathbb{K}$-algebra, and $g : E(V) \longrightarrow A$ is the induced homomorphism $\mathbb{K}$-algebras, then*

$$g(\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n}) = \sum_{n,\bar{i}} f(v_{i_1}) \cdot \ldots \cdot f(v_{i_n}).$$

Proof. (1) Use the embedding homomorphism $j : V \longrightarrow D(V)$, where $D(V)$ is the algebra defined in 2.1 (3) to construct $g : E(V) \longrightarrow D(V)$ such that $g \circ \iota = j$. Since j is injective so is ι.

(2) Let $B := \{\sum_{n,\bar{i}} v_{i_1} \wedge \ldots \wedge v_{i_n} | \bar{i} = (i_1, \ldots, i_n)$ multiindex of length $n\}$. Obviously B is the subalgebra of $E(V)$ generated by the elements of V. Let $j : B \longrightarrow E(V)$ be the embedding homomorphism. Then $\iota : V \longrightarrow E(V)$ factors through a $\mathbb{K}$-module homomorphism $\iota' : V \longrightarrow B$. The following diagram

$$\begin{tikzcd} V \arrow[r, "\iota'"] \arrow[dr, "\iota'"'] & B \arrow[r, "j"] & E(V) \arrow[dl, "p"'] \arrow[d, "jp"] \\ & B \arrow[r, "j"] & E(V) \end{tikzcd}$$

induces a unique p with $p \circ j \circ \iota' = p \circ \iota = \iota'$ since ι' is a homomorphism of $\mathbb{K}$-modules satisfying $\iota'(v) \cdot \iota'(v') = -\iota'(v') \cdot \iota'(v)$ for all $v, v' \in V$. Because of $jp \circ \iota = j \circ \iota' = \iota = \mathrm{id}_{E(V)} \circ \iota$ we get $jp = \mathrm{id}_{E(V)}$, hence the embedding j is surjective and thus j is the identity.

(3) is precisely the definition of the induced homomorphism. $\square$

Proposition 2.16. *Given a $\mathbb{K}$-module V. Then there exists an exterior algebra $(E(V), \iota)$.*

Proof. Define $E(V) := T(V)/I$ where $I = \langle v^2 | v \in V \rangle$ is the two-sided ideal generated by the elements v^2. Let ι be the canonical map $V \longrightarrow T(V) \longrightarrow E(V)$. Then the universal property is easily verified by the homomorphism theorem for algebras. $\square$

Problem 2.5. (1) Let V be a finite dimensional vector space of dimension n. Show that $E(V)$ is finite dimensional of dimension 2^n. (Hint: The homogeneous components $E^i(V)$ have dimension $\begin{pmatrix} n \\ i \end{pmatrix}$.

(2) Show that the symmetric group S_n operates (from the left) on $T^n(V)$ by $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$ with $\sigma \in S_n$ and $v_i \in V$.

(3) A tensor $a \in T^n(V)$ is called *a symmetric tensor* if $\sigma(a) = a$ for all $\sigma \in S_n$. Let $\hat{S}^n(V)$ be the subspace of symmetric tensors in $T^n(V)$.

a) Show that $\mathcal{S} : T^n(V) \ni a \mapsto \sum_{\sigma \in S_n} \sigma(a) \in T^n(V)$ is a linear map (symmetrization).

b) Show that $\mathcal{S}$ has its image in $\hat{S}^n(V)$.

c) Show that $\mathrm{Im}(\mathcal{S}) = \hat{S}^n(V)$ if $n!$ is invertible in $\mathbb{K}$.

d) Show that $\hat{S}^n(V) \hookrightarrow T^n(V) \xrightarrow{\nu} S^n(V)$ is an isomorphism if $n!$ is invertible in $\mathbb{K}$ and $\nu : T^n(V) \longrightarrow S^n(V)$ is the restriction of $\nu : T(V) \longrightarrow S(V)$, where $S(V)$ is the symmetric algebra.

(4) A tensor $a \in T^n(V)$ is called *an antisymmetric tensor* if $\sigma(a) = \varepsilon(\sigma)a$ for all $\sigma \in S_n$ where $\varepsilon(\sigma)$ is the sign of the permutation σ. Let $\hat{E}^n(V)$ be the subspace of antisymmetric tensors in $T^n(V)$.

a) Show that $\mathcal{E} : T^n(V) \ni a \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(a) \in T^n(V)$ is a $\mathbb{K}$-module homomorphism (antisymmetrization).

b) Show that $\mathcal{E}$ has its image in $\hat{E}^n(V)$.
c) Show that $\mathrm{Im}(\mathcal{E}) = \hat{E}^n(V)$ if $n!$ is invertible in $\mathbb{K}$.
d) Show that $\hat{E}^n(V) \hookrightarrow T^n(V) \xrightarrow{\nu} E^n(V)$ is an isomorphism if $n!$ is invertible in $\mathbb{K}$ and $\nu : T^n(V) \longrightarrow E^n(V)$ is the restriction of $\nu : T(V) \longrightarrow E(V)$, where $E(V)$ is the exterior algebra.

2.5. **Left A-modules.**

Definition 2.17. Let A be a $\mathbb{K}$-algebra. A *left A-module* is a $\mathbb{K}$-module M together with a homomorphism $\mu_M : A \otimes M \longrightarrow M$, such that the diagrams

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\nabla\otimes\mathrm{id}"'] & A \otimes M \arrow[d, "\mu"] \\
A \otimes M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \cong \mathbb{K} \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & A \otimes M \arrow[d, "\mu"] \\
 & M
\end{tikzcd}$$

commute.
Let ${}_AM$ and ${}_AN$ be left A-modules and let $f : M \longrightarrow N$ be a $\mathbb{K}$-linear map. The map f is called a *homomorphism of left A-modules* if the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mu_M"] \arrow[d, "1\otimes f"'] & M \arrow[d, "f"] \\
A \otimes N \arrow[r, "\mu_N"'] & N
\end{tikzcd}$$

commutes.

Problem 2.6. Show that an Abelian group M is a left module *over the ring A* if and only if M is a $\mathbb{K}$-module and a left A-module in the sense of Definition 2.17.

2.6. **Coalgebras.**

Definition 2.18. A $\mathbb{K}$-*coalgebra* is a $\mathbb{K}$-module C together with a *comultiplication* or *diagonal* $\Delta : C \longrightarrow C \otimes C$ ($\mathbb{K}$-module homomorphism) that is coassociative:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes\mathrm{id}"] \\
C \otimes C \arrow[r, "\mathrm{id}\otimes\Delta"'] & C \otimes C \otimes C
\end{tikzcd}$$

and a *counit* or *augmentation* $\epsilon : C \longrightarrow \mathbb{K}$ ($\mathbb{K}$-module homomorphism):

$$\begin{tikzcd}
C \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "\mathrm{id}"] & & C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\
C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"'] & \mathbb{K} \otimes C \cong & C \cong C \otimes \mathbb{K}.
\end{tikzcd}$$

A $\mathbb{K}$-coalgebra C is *cocommutative* if the following diagram commutes

$$\begin{tikzcd}
& C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[rr, "\tau"] & & C \otimes C
\end{tikzcd}$$

Let C and D be $\mathbb{K}$-coalgebras. A *homomorphism of coalgebras* $f : C \longrightarrow D$ is a $\mathbb{K}$-module homomorphism such that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\
C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D
\end{tikzcd}$$

and

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"'] & & D. \arrow[dl, "\epsilon_D"] \\
& \mathbb{K} &
\end{tikzcd}$$

Remark 2.19. Obviously the composition of two homomorphisms of coalgebras is again a homomorphism of coalgebras. Furthermore the identity map is a homomorphism of coalgebras.

Problem 2.7. (1) Show that $V \otimes V^*$ is a coalgebra for every finite dimensional vector space V over a field $\mathbb{K}$ if the comultiplication is defined by $\Delta(v \otimes v^*) := \sum_{i=1}^n v \otimes v_i^* \otimes v_i \otimes v^*$ where $\{v_i\}$ and $\{v_i^*\}$ are dual bases of V resp. V^*.
(2) Show that the free $\mathbb{K}$-modules $\mathbb{K}X$ with the basis X and the comultiplication $\Delta(x) = x \otimes x$ is a coalgebra. What is the counit? Is the counit unique?
(3) Show that $\mathbb{K} \oplus V$ with $\Delta(1) = 1 \otimes 1$, $\Delta(v) = v \otimes 1 + 1 \otimes v$ defines a coalgebra.
(4) Let C and D be coalgebras. Then $C \otimes D$ is a coalgebra with the comultiplication $\Delta_{C \otimes D} := (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D) : C \otimes D \longrightarrow C \otimes D \otimes C \otimes D$ and counit $\varepsilon = \varepsilon_{C \otimes D} : C \otimes D \longrightarrow \mathbb{K} \otimes K \longrightarrow \mathbb{K}$. (The proof is analogous to the proof of Lemma 2.3.)

To describe the comultiplication of a $\mathbb{K}$-coalgebra in terms of elements we introduce a notation first introduced by Sweedler similar to the notation $\nabla(a \otimes b) = ab$ used for algebras. Instead of $\Delta(c) = \sum c_i \otimes c_i'$ we write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

Observe that only the complete expression on the right hand side makes sense, not the components $c_{(1)}$ or $c_{(2)}$ which are *not* considered as families of elements of C. This notation alone does not help much in the calculations we have to perform later on. So we introduce a more general notation.

Definition 2.20. (Sweedler Notation) Let M be an arbitrary $\mathbb{K}$-module and C be a $\mathbb{K}$-coalgebra. Then there is a bijection between all multilinear maps

$$f : C \times \ldots \times C \longrightarrow M$$

and all linear maps

$$f' : C \otimes \ldots \otimes C \longrightarrow M.$$

These maps are associated to each other by the formula

$$f(c_1, \ldots, c_n) = f'(c_1 \otimes \ldots \otimes c_n).$$

For $c \in C$ we define

$$\sum f(c_{(1)}, \ldots, c_{(n)}) := f'(\Delta^{n-1}(c)),$$

where Δ^{n-1} denotes the $n-1$-fold application of Δ, for example $\Delta^{n-1} = (\Delta \otimes 1 \otimes \ldots \otimes 1) \circ \ldots \circ (\Delta \otimes 1) \circ \Delta$.
In particular we obtain for the bilinear map $\otimes : C \times C \ni (c, d) \mapsto c \otimes d \in C \otimes C$ (with associated identity map)

$$\sum c_{(1)} \otimes c_{(2)} = \Delta(c),$$

and for the multilinear map $\otimes^2 : C \times C \times C \longrightarrow C \otimes C \otimes C$

$$\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = (\Delta \otimes 1)\Delta(c) = (1 \otimes \Delta)\Delta(c).$$

With this notation one verifies easily

$$\sum c_{(1)} \otimes \ldots \otimes \Delta(c_{(i)}) \otimes \ldots \otimes c_{(n)} = \sum c_{(1)} \otimes \ldots \otimes c_{(n+1)}$$

and

$$\begin{aligned} \sum c_{(1)} \otimes \ldots \otimes \epsilon(c_{(i)}) \otimes \ldots \otimes c_{(n)} &= \sum c_{(1)} \otimes \ldots \otimes 1 \otimes \ldots \otimes c_{(n-1)} \\ &= \sum c_{(1)} \otimes \ldots \otimes c_{(n-1)} \end{aligned}$$

This notation and its application to multilinear maps will also be used in more general contexts like comodules.

Proposition 2.21. *Let C be a coalgebra and A an algebra. Then the composition $f * g := \nabla_A(f \otimes g)\Delta_{\mathcal{C}}$ defines a multiplication*

$$\mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \ni f \otimes g \mapsto f * g \in \mathrm{Hom}(C, A),$$

such that $\mathrm{Hom}(C, A)$ *becomes an algebra. The unit element is given by* $\mathbb{K} \ni \alpha \mapsto (c \mapsto \eta(\alpha\epsilon(c))) \in \mathrm{Hom}(C, A)$.

Proof. The multiplication of $\mathrm{Hom}(C, A)$ obviously is a bilinear map. The multiplication is associative since $(f * g) * h = \nabla_A((\nabla_A(f \otimes g)\Delta_C) \otimes h)\Delta_C = \nabla_A(\nabla_A \otimes 1)((f \otimes g) \otimes h)(\Delta_C \otimes 1)\Delta_C = \nabla_A(1 \otimes \nabla_A)(f \otimes (g \otimes h))(1 \otimes \Delta_C)\Delta_C = \nabla_A(f \otimes (\nabla_A(g \otimes h)\Delta_C))\Delta_C = f * (g * h)$.
Furthermore it is unitary with unit $1_{\mathrm{Hom}(C,A)} = \eta_A\epsilon_C$ since $\eta_A\epsilon_C * f = \nabla_A(\eta_A\epsilon_C \otimes f)\Delta_C = \nabla_A(\eta_A \otimes 1_A)(1_{\mathbb{K}} \otimes f)(\epsilon_C \otimes 1_C)\Delta_C = f$ and similarly $f * \eta_A\epsilon_C = f$. □

Definition 2.22. The multiplication $* : \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C, A)$ is called *convolution*.

Corollary 2.23. *Let C be a $\mathbb{K}$-coalgebra. Then $C^* = \mathrm{Hom}_{\mathbb{K}}(C, \mathbb{K})$ is an $\mathbb{K}$-algebra.*

Proof. Use that $\mathbb{K}$ itself is a $\mathbb{K}$-algebra. □

Remark 2.24. If we write the evaluation as $C^* \otimes C \ni a \otimes c \mapsto \langle a, c\rangle \in \mathbb{K}$ then an element $a \in C^*$ is completely determined by the values of $\langle a, c\rangle$ for all $c \in C$. So the product of a and b in C^* is uniquely determined by the formula

$$\langle a * b, c\rangle = \langle a \otimes b, \Delta(c)\rangle = \sum a(c_{(1)})b(c_{(2)}).$$

The unit element of C^* is $\epsilon \in C^*$.

Lemma 2.25. *Let $\mathbb{K}$ be a field and A be a finite dimensional $\mathbb{K}$-algebra. Then $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a $\mathbb{K}$-coalgebra.*

Proof. Define the comultiplication on A^* by

$$\Delta : A^* \xrightarrow{\nabla^*} (A \otimes A)^* \xrightarrow{\mathrm{can}^{-1}} A^* \otimes A^*.$$

The canonical map $\mathrm{can} : A^* \otimes A^* \longrightarrow (A \otimes A)^*$ is invertible, since A is finite dimensional. By a diagrammatic proof or by calculation with elements it is easy to show that A^* becomes a $\mathbb{K}$-coalgebra. $\square$

Remark 2.26. If $\mathbb{K}$ is an arbitrary commutative ring and A is a $\mathbb{K}$-algebra, then $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a $\mathbb{K}$-coalgebra if A is a finitely generated projective $\mathbb{K}$-module.

Problem 2.8. Find sufficient conditions for an algebra A resp. a coalgebra C such that $\mathrm{Hom}(A, C)$ becomes a coalgebra with co-convolution as comultiplication.

2.7. **Comodules.**

Definition 2.27. Let C be a $\mathbb{K}$-coalgebra. A *left C-comodule* is a $\mathbb{K}$-module M together with a $\mathbb{K}$-module homomorphism $\delta_M : M \longrightarrow C \otimes M$, such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes M \arrow[d, "\Delta\otimes\mathrm{id}"] \\ C \otimes M \arrow[r, "\mathrm{id}\otimes\delta"'] & C \otimes C \otimes M \end{tikzcd}$$

and

$$\begin{tikzcd} M \arrow[d, "\delta"'] \arrow[dr, "\mathrm{id}"] & \\ C \otimes M \arrow[r, "\epsilon\otimes\mathrm{id}"'] & \mathbb{K} \otimes M \cong M. \end{tikzcd}$$

commute.

Let ${}^C M$ and ${}^C N$ be C-comodules and let $f : M \longrightarrow N$ be a $\mathbb{K}$-module homomorphism. The map f is called a *homomorphism of comodules* if the diagram

$$\begin{tikzcd} M \arrow[r, "\delta_M"] \arrow[d, "f"'] & C \otimes M \arrow[d, "1\otimes f"] \\ N \arrow[r, "\delta_N"'] & C \otimes N \end{tikzcd}$$

commutes.

Let N be an arbitrary $\mathbb{K}$-module and M be a C-comodule. Then there is a bijection between all multilinear maps

$$f : C \times \ldots \times C \times M \longrightarrow N$$

and all linear maps

$$f' : C \otimes \ldots \otimes C \otimes M \longrightarrow N.$$

These maps are associated to each other by the formula

$$f(c_1, \ldots, c_n, m) = f'(c_1 \otimes \ldots \otimes c_n \otimes m).$$

For $m \in M$ we define

$$\sum f(m_{(1)}, \ldots, m_{(n)}, m_{(M)}) := f'(\delta^n(m)),$$

where δ^n denotes the n-fold application of δ, i.e. $\delta^n = (1 \otimes \ldots \otimes 1 \otimes \delta) \circ \ldots \circ (1 \otimes \delta) \circ \delta$. In particular we obtain for the bilinear map $\otimes : C \times M \longrightarrow C \otimes M$

$$\sum m_{(1)} \otimes m_{(M)} = \delta(m),$$

and for the multilinear map $\otimes^2 : C \times C \times M \longrightarrow C \otimes C \otimes M$

$$\sum m_{(1)} \otimes m_{(2)} \otimes m_{(M)} = (1 \otimes \delta)\delta(c) = (\Delta \otimes 1)\delta(m).$$

Problem 2.9. Show that a finite dimensional vector space V is a comodule over the coalgebra $V \otimes V^*$ as defined in exercise 2.7 (1) with the coaction $\delta(v) := \sum v \otimes v_i^* \otimes v_i \in (V \otimes V^*) \otimes V$ where $\sum v_i^* \otimes v_i$ is the dual basis of V in $V^* \otimes V$.

Theorem 2.28. *(Fundamental Theorem for Comodules) Let $\mathbb{K}$ be a field. Let M be a left C-comodule and let $m \in M$ be given. Then there exists a finite dimensional subcoalgebra $C' \subseteq C$ and a finite dimensional C'-comodule M' with $m \in M' \subseteq M$ where $M' \subseteq M$ is a $\mathbb{K}$-submodule, such that the diagram*

$$\begin{tikzcd}
M' \arrow[r, "\delta'"] \arrow[d] & C' \otimes M' \arrow[d] \\
M \arrow[r, "\delta"'] & C \otimes M
\end{tikzcd}$$

commutes.

Corollary 2.29. *(1) Each element $c \in C$ of a coalgebra is contained in a finite dimensional subcoalgebra of C.*
(2) Each element $m \in M$ of a comodule is contained in a finite dimensional subcomodule of M.

Corollary 2.30. *(1) Each finite dimensional subspace V of a coalgebra C is contained in a finite dimensional subcoalgebra C' of C.*
(2) Each finite dimensional subspace V of a comodule M is contained in a finite dimensional subcomodule M' of M.

Corollary 2.31. *(1) Each coalgebra is a union of finite dimensional subcoalgebras.*
(2) Each comodule is a union of finite dimensional subcomodules.

Proof. (*of the Theorem*) We can assume that $m \neq 0$ for else we can use $M' = 0$ and $C' = 0$. Under the representations of $\delta(m) \in C \otimes M$ as finite sums of decomposable tensors pick one

$$\delta(m) = \sum_{i=1}^{s} c_i \otimes m_i$$

of shortest length s. Then the families $(c_i | i = 1, \ldots, s)$ and $(m_i | i = 1, \ldots, s)$ are linearly independent. Choose coefficients $c_{ij} \in C$ such that

$$\Delta(c_j) = \sum_{i=1}^{t} c_i \otimes c_{ij}, \quad \forall j = 1, \ldots, s,$$

by suitably extending the linearly independent family $(c_i | i = 1, \ldots, s)$ to a linearly independent family $(c_i | i = 1, \ldots, t)$ and $t \geq s$.
We first show that we can choose $t = s$. By coassociativity we have $\sum_{i=1}^{s} c_i \otimes \delta(m_i) = \sum_{j=1}^{s} \Delta(c_j) \otimes m_j = \sum_{j=1}^{s} \sum_{i=1}^{t} c_i \otimes c_{ij} \otimes m_j$. Since the c_i and the m_j are linearly independent we can compare coefficients and get

$$\delta(m_i) = \sum_{j=1}^{s} c_{ij} \otimes m_j, \quad \forall i = 1, \ldots, s \tag{3}$$

and $0 = \sum_{j=1}^{s} c_{ij} \otimes m_j$ for $i > s$. The last statement implies

$$c_{ij} = 0, \quad \forall i > s, j = 1, \ldots, s.$$

Hence we get $t = s$ and

$$\Delta(c_j) = \sum_{i=1}^{s} c_i \otimes c_{ij}, \quad \forall j = 1, \ldots, s.$$

Define finite dimensional subspaces $C' = \langle c_{ij} | i, j = 1, \ldots, s \rangle \subseteq C$ and $M' = \langle m_i | i = 1, \ldots, s \rangle \subseteq M$. Then by (3) we get $\delta : M' \longrightarrow C' \otimes M'$. We show that $m \in M'$ and that the restriction of Δ to C' gives a $\mathbb{K}$-module homomorphism $\Delta : C' \longrightarrow C' \otimes C'$ so that the required properties of the theorem are satisfied. First observe that $m = \sum \varepsilon(c_i) m_i \in M'$ and $c_j = \sum \varepsilon(c_i) c_{ij} \in C'$. Using coassociativity we get

$$\begin{aligned} \textstyle\sum_{i,j=1}^{s} c_i \otimes \Delta(c_{ij}) \otimes m_j &= \textstyle\sum_{k,j=1}^{s} \Delta(c_k) \otimes c_{kj} \otimes m_j \\ &= \textstyle\sum_{i,j,k=1}^{s} c_i \otimes c_{ik} \otimes c_{kj} \otimes m_j \end{aligned}$$

hence

$$\Delta(c_{ij}) = \sum_{k=1}^{s} c_{ik} \otimes c_{kj}.$$

□

Remark 2.32. We give a sketch of a second proof of Theorem 2.28 which is somewhat more technical. Since C is a $\mathbb{K}$-coalgebra, the dual C^* is an algebra. The comodule structure $\delta : M \longrightarrow C \otimes M$ leads to a module structure by $\rho = (\mathrm{ev} \otimes 1)(1 \otimes \delta) : C^* \otimes M \longrightarrow C^* \otimes C \otimes M \longrightarrow M$. Consider the submodule $N := C^* m$. Then N is finite dimensional, since $c^* m = \sum_{i=1}^{n} \langle c^*, c_i \rangle m_i$ for all $c^* \in C^*$ where $\sum_{i=1}^{n} c_i \otimes m_i = \delta(m)$. Observe that $C^* m$ is a subspace of the space generated by the m_i. But it does not depend on the choice of the m_i. Furthermore if we take $\delta(m) = \sum c_i \otimes m_i$ with a shortest representation then the m_i are in $C^* m$ since $c^* m = \sum \langle c^*, c_i \rangle m_i = m_i$ for c^* an element of a dual basis of the c_i.
N is a C-comodule since $\delta(c^* m) = \sum \langle c^*, c_i \rangle \delta(m_i) = \sum \langle c^*, c_{i(1)} \rangle c_{i(2)} \otimes m_i \in C \otimes C^* m$.
Now we construct a subcoalgebra D of C such that N is a D-comodule with the induced coaction. Let $D := N \otimes N^*$. By 2.9 N is a comodule over the coalgebra $N \otimes N^*$. Construct a $\mathbb{K}$-module homomorphism $\phi : D \longrightarrow C$ by $n \otimes n^* \mapsto \sum n_{(1)} \langle n^*, n_{(N)} \rangle$. By definition of the

dual basis we have $n = \sum n_i \langle n_i^*, n \rangle$. Thus we get

$$\begin{aligned}
(\phi \otimes \phi)\Delta_D(n \otimes n^*) &= (\phi \otimes \phi)(\sum n \otimes n_i^* \otimes n_i \otimes n^*) \\
&= \sum n_{(1)} \langle n_i^*, n_{(N)} \rangle \otimes n_{i(1)} \langle n^*, n_{i(N)} \rangle \\
&= \sum n_{(1)} \otimes n_{i(1)} \langle n^*, n_{i(N)} \rangle \langle n_i^*, n_{(N)} \rangle \\
&= \sum n_{(1)} \otimes n_{(2)} \langle n^*, n_{(N)} \rangle = \sum \Delta_C(n_{(1)}) \langle n^*, n_{(N)} \rangle \\
&= \Delta_C \phi(n \otimes n^*).
\end{aligned}$$

Furthermore $\varepsilon_C \phi(n \otimes n^*) = \varepsilon(\sum n_{(1)} \langle n^*, n_{(N)} \rangle) = \langle n^*, \sum \varepsilon(n_{(1)}) n_{(N)} \rangle = \langle n^*, n \rangle = \varepsilon(n \otimes n^*)$. Hence $\phi : D \longrightarrow C$ is a homomorphism of coalgebras, D is finite dimensional and the image $C' := \phi(D)$ is a finite dimensional subcoalgebra of C. Clearly N is also a C'-comodule, since it is a D-comodule.

Finally we show that the D-comodule structure on N if lifted to the C-comodule structure coincides with the one defined on M. We have

$$\begin{aligned}
\delta_C(c^*m) &= \delta_C(\sum \langle c^*, m_{(1)} \rangle m_{(M)}) = \sum \langle c^*, m_{(1)} \rangle m_{(2)} \otimes m_{(M)} \\
&= \sum \langle c^*, m_{(1)} \rangle m_{(2)} \otimes m_i \langle m_i^*, m_{(M)} \rangle = \sum \langle c^*, m_{(1)} \rangle m_{(2)} \langle m_i^*, m_{(M)} \rangle \otimes m_i \\
&= (\phi \otimes 1)(\sum \langle c^*, m_{(1)} \rangle m_{(M)} \otimes m_i^* \otimes m_i) = (\phi \otimes 1)(\sum c^*m \otimes m_i^* \otimes m_i) \\
&= (\phi \otimes 1)\delta_D(c^*m).
\end{aligned}$$

3. Projective Modules and Generators

3.1. **Products and coproducts.**

Definition 3.1.
(1) Let $(M_i|i \in I)$ be a family of R-modules. An R-module $\prod M_i$ together with a family of homomorphisms $(p_j : \prod M_i \longrightarrow M_j|j \in I)$ is called a *(direct) product* of the M_i and the homomorphisms $p_j : \prod M_i \longrightarrow M_j$ are called *projections*, if for each R-module N and for each family of homomorphisms $(f_j : N \longrightarrow M_j|j \in I)$ there is a unique homomorphism $f : N \longrightarrow \prod M_i$ such that

$$\begin{tikzcd} N \arrow[d, "f"'] \arrow[dr, "f_j"] & \\ \prod M_i \arrow[r, "p_j"'] & M_j \end{tikzcd}$$

commute for all $j \in I$.
(2) "The dual notion is called coproduct": Let $(M_i|i \in I)$ be a family of R-modules. An R-module $\coprod M_i$ together with a family of homomorphisms $(\iota_j : M_j \longrightarrow \coprod M_i|j \in I)$ is called a *coproduct* or *direct sum* of the M_i and the homomorphisms $\iota_j : M_j \longrightarrow \coprod M_i$ are called *injections*, if for each R-module N and for each family of homomorphisms $(f_j : M_j \longrightarrow N|j \in I)$ there is a unique homomorphism $f : \coprod M_j \longrightarrow N$ such that

$$\begin{tikzcd} M_j \arrow[r, "\iota_j"] \arrow[dr, "f_j"'] & \coprod M_i \arrow[d, "f"] \\ & N \end{tikzcd}$$

commute for all $j \in I$.

Remark 3.2. An analogous definition can be given for algebras, coalgebras, comodules, groups, Abelian groups etc.

Note: If you want to define a homomorphism $f : N \longrightarrow \prod M_i$ with a product as range (codomain) you should define it by giving homomorphisms $f_i : N \longrightarrow M_i$.
If you want to define a homomorphism $f : \coprod M_i \longrightarrow N$ with a coproduct as domain you should define it by giving homomorphisms $f_i : M_i \longrightarrow N$.

Lemma 3.3. *Products and coproducts are unique up to a unique isomorphism.*

Proof. analogous to Proposition1.6. □

Proposition 3.4. (Rules of computation in a product of R-modules) *Let $(\prod M_i, (p_j))$ be a product of the family of R-modules $(M_i)_{i\in I}$.*

(1) *There is a bijection of sets*

$$\prod M_i \ni a \mapsto (a_i) := (p_i(a)) \in \{(a_i)|\forall i \in I : a_i \in M_i\}$$

such that $a + b \mapsto (a_i + b_i)$ and $ra \mapsto (ra_i)$.

(2) *If $(f_i : N \longrightarrow M_i)$ is a family of homomorphisms and $f : N \longrightarrow \prod M_i$ is the induced homomorphism then the family associated to $f(n) \in \prod M_i$ is $(f_i(n))$, i.e. $(p_i(f(n))) = (f_i(n))$.*

Proof. Let a family $(a_i|i \in I)$ be given. Form $\varphi_i : \{1\} \longrightarrow M_i$ with $\varphi_i(1) = a_i$ for all $i \in I$. Construct $g_i \in \mathrm{Hom}_R(R, M_i)$ such that the diagrams

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[dr, "\varphi_i"'] & R \arrow[d, "g_i"] \\
& M_i
\end{tikzcd}$$

commute (R is the free R-module over the set $\{1\}$). Then there is a unique $g : R \longrightarrow \prod M_i$ with

$$\begin{tikzcd}
R \arrow[d, "g"'] \arrow[dr, "g_j"] & \\
\prod M_i \arrow[r, "p_j"'] & M_j
\end{tikzcd}$$

for all $j \in I$. The homomorphism g is completely and uniquely determined by $g(1) =: a$ and by the commutative diagram

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[dr] \arrow[ddr, "\varphi_j"'] & R \arrow[d, "g"] \arrow[dd, bend left, "g_j"] \\
& \prod M_i \arrow[d, "p_j"] \\
& M_j
\end{tikzcd}$$

where $p_j(a) = \varphi_j(1) = a_j$. So we have found $a \in \prod M_i$ with $(p_i(a)) = (a_i)$. Hence the map given in the proposition is surjective. Given a and b in $\prod M_i$ with $(p_i(a)) = (p_i(b))$ then $\varphi_j(1) := p_j(a)$ and $\psi_j(1) := p_j(b)$ define equal maps $\varphi_j = \psi_j$, hence the induced maps $g_j : R \longrightarrow M_j$ and $h_j : R \longrightarrow M_j$ are equal so that $g = h$ and hence $a = g(1) = h(1) = b$. Hence the map given in the proposition is bijective.
Since a is uniquely determined by the $p_j(a) = a_j$ we have $p_j(a+b) = p_j(a) + p_j(b) = a_j + b_j$ and $p_j(ra) = rp_j(a) = ra_j$
The last statement is $p_i f = f_i$. $\square$

Remark 3.5. Observe that this construction can always be performed if there is a free object (algebra, coalgebra, comodule, group, Abelian group, etc.) R over the set $\{1\}$ i.e. if

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[dr] & R \arrow[d] \\
& T
\end{tikzcd}$$

has a universal solution.

Proposition 3.6. (Rules of computation in a coproduct of R-modules) *Let $(\coprod M_i, (\iota_j))$ be a coproduct of the family of R-modules $(M_i)_{i \in I}$.*

(1) *The homomorphisms $\iota_j : M_j \longrightarrow \coprod M_i$ are injective.*
(2) *For each element $a \in \coprod M_i$ there are finitely many $a_i \in M_i$ with $a = \sum_{i=1}^n \iota_i(a_i)$. The $a_i \in M_i$ are uniquely determined by a.*

Proof. (1) To show the injectivity of ι_i define $f_i : M_i \longrightarrow M_j$ by

$$f_i := \begin{cases} \mathrm{id}, & i = j, \\ 0, & \text{else.} \end{cases}$$

Then the diagram

$$\begin{tikzcd} M_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & \coprod M_j \arrow[d, "f"] \\ & M_j \end{tikzcd}$$

defines a uniquely determined homomorphism f. For $i = j$ this implies $f\iota_i = \mathrm{id}_{M_i}$, hence ι_i is injective.

(2) Define $\widetilde{M} := \sum \iota_j(M_j) \subseteq \coprod M_j$. Then the following diagram commutes with both 0 and ν

$$\begin{tikzcd} M_i \arrow[r, "\iota_i"] \arrow[dr, "0"'] & \coprod M_j \arrow[d, shift right, "0"'] \arrow[d, shift left, "\nu"] \\ & \coprod M_j/\widetilde{M} \end{tikzcd}$$

Hence $\nu = 0$ and $\coprod M_j = \widetilde{M}$. Let $a = \sum \iota_j(a_j)$. Define f as in (1). Then we have $f(a) = f(\sum \iota_j(a_j)) = \sum f\iota_j(a_j) = \sum f_j(a_j) = a_i$, hence the a_i are uniquely determined by a. $\square$

Propositions 3.4 and 3.6 give already an indication of how to construct products and coproducts.

Proposition 3.7. *Let $(M_i|i \in I)$ be a family of R-modules. Then there exist a product $(\prod M_i, (p_j : \prod M_i \longrightarrow M_j|j \in I))$ and a coproduct $(\coprod M_i, (\iota_j : M_j \longrightarrow \prod M_i|j \in I))$.*

Proof. 1. Define

$$\prod M_i := \{a : I \longrightarrow \cup_{i\in I} M_i |\ \forall j \in I : a(j) = a_j \in M_j\}$$

and $p_j : \prod M_i \longrightarrow M_j$, $p_j(a) := a(j) = a_j \in M_j$. It is easy to check that $\prod M_i$ is an R-module with componentwise operations and that the p_j are homomorphisms. If $(f_j : N \longrightarrow M_j)$ is a family of homomorphisms then there is a unique map $f : N \longrightarrow \prod M_i$, $f(n) = (f_i(n)|i \in I)$ such that $p_j f = f_j$ for all $j \in I$. The following families are equal: $(p_j f(n + n')) = (f_j(n + n')) = (f_j(n) + f_j(n')) = (p_j f(n) + p_j f(n')) = (p_j(f(n) + f(n')))$, hence $f(n + n') = f(n) + f(n')$. Analogously one shows $f(rn) = rf(n)$. Thus f is a homomorphism and $\prod M_i$ is a product.

2. Define

$$\coprod M_i := \{a : I \longrightarrow \cup_{i\in I} M_i |\forall j \in I : a(j) \in M_j ; a \text{ with finite support}\}$$

(the notion *with finite support* means that all but a finite number of the $a(j)$'s are zero) and $\iota_j : M_j \longrightarrow \coprod M_i$, $\iota_j(a_j)(i) := \delta_{ij} a_i$. Then $\coprod M_i \subseteq \prod M_i$ is a submodule and the ι_j are homomorphisms. Given $(f_j : M_j \longrightarrow N|j \in I)$. Define $f(a) = f(\sum \iota_i a_i) = \sum f\iota_i(a_i) = \sum f_i(a_i)$. Then f is an R-module homomorphism and we have $f\iota_i(a_i) = f_i(a_i)$ hence $f\iota_i = f_i$. If $g\iota_i = f_i$ for all $i \in I$ then $g(a) = g(\sum \iota_i a_i) = \sum g\iota_i a_i = \sum f_i(a_i)$ hence $f = g$. $\square$

Proposition 3.8. *Let $(M_i|i \in I)$ be a family of submodules of M. The following statements are equivalent:*

(1) $(M, (\iota_i : M_i \longrightarrow M))$ *is a coproduct of R-modules.*
(2) $M = \sum_{i \in I} M_i$ *and* $(\sum m_i = 0 \Longrightarrow \forall i \in I : m_i = 0)$.
(3) $M = \sum_{i \in I} M_i$ *and* $(\sum m_i = \sum m_i' \Longrightarrow \forall i \in I : m_i = m_i')$.
(4) $M = \sum_{i \in I} M_i$ *and* $\forall i \in I : M_i \cap \sum_{j \neq i, j \in I} M_j = 0$.

Definition 3.9. Is one of the equivalent conditions of Proposition 3.8 is satisfied then M is called an *internal direct sum* of the M_i and we write $M = \oplus_{i \in I} M_i$.

Proof of Proposition 3.8: (1) $\Longrightarrow$ (2): Use the commutative diagram

$$\begin{tikzcd} M_j \arrow[r, "\iota_j"] \arrow[dr, "0"'] & M \arrow[d, shift right, "0"'] \arrow[d, shift left, "\nu"] \\ & M/\sum M_i \end{tikzcd}$$

to conclude $\nu = 0$ and $M = \sum M_i$. If $\sum m_i = 0$ then use the diagram

$$\begin{tikzcd} M_j \arrow[r, "\iota_j"] \arrow[dr, "\delta_{jk}"'] & M \arrow[d, "p_k"] \\ & M_k \end{tikzcd}$$

to show $0 = p_k(0) = p_k(\sum m_j) = \sum_j p_k \iota_j(m_j) = \sum_j \delta_{jk}(m_j) = m_k$.
(2) $\Longrightarrow$ (3): trivial.
(3) $\Longrightarrow$ (4): Let $m_i = \sum_{j \neq i} m_j$. Then $m_i = 0$ and $m_j = 0$ for all $j \neq i$.
(4) $\Longrightarrow$ (2): If $\sum m_j = 0$ then $m_i = \sum_{j \neq i} -m_j = 0 \in M_i \cap \sum_{j \neq i} M_j$.
(3) $\Longrightarrow$ (1): Define f for the diagram

$$\begin{tikzcd} M_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

by $f(\sum m_i) := \sum f_i(m_i)$. Then f is a well defined homomorphism and we have $f \iota_j(m_j) = f(m_j) = f_j(m_j)$. Furthermore f is uniquely determined since $g \iota_j = f_j \Longrightarrow g(\sum m_i) = \sum g(m_i) = \sum g \iota_i(m_i) = \sum f_i(m_i) = f(\sum m_i) \Longrightarrow f = g$. □

Proposition 3.10. *Let* $(\coprod M_i, (\iota_j : M_j \longrightarrow \coprod_{i \neq j} M_i))$ *be a coproduct of R-modules. Then* $\coprod M_i$ *is an internal direct sum of the* $\iota_j(M_j)$.

Proof. ι_j is injective $\Longrightarrow M_j \cong \iota_j(M_j) \Longrightarrow$

$$\begin{tikzcd} M_j \arrow[r, phantom, "\cong"] \arrow[drr] & \iota_j(M_j) \arrow[r] \arrow[dr] & \coprod M_i \arrow[d] \\ & & N \end{tikzcd}$$

defines a coproduct. By 3.8 we have an internal direct sum. □

Definition 3.11. A submodule $M \subseteq N$ is called a *direct summand* of N if there is a submodule $M' \subseteq N$ such that $N = M \oplus M'$ is an internal direct sum.

Proposition 3.12. *For a submodule $M \subseteq N$ the following are equivalent:*

(1) *M is a direct summand of N.*
(2) *There is $p \in \mathrm{Hom}_R(.N, .M)$ with*

$$(M \xrightarrow{\iota} N \xrightarrow{p} M) = \mathrm{id}_M .$$

(3) *There is $f \in \mathrm{Hom}_R(.N, .N)$ with $f^2 = f$ and $f(N) = M$.*

Proof. (1) $\Longrightarrow$ (2): Let $M_1 := M$ and $M_2 \subseteq N$ with $N = M_1 \oplus M_2$. We define $p = p_1 : N \longrightarrow M_1$ by

$$\begin{tikzcd} M_i \arrow[r, "\iota_i"] \arrow[dr, "\delta_{ij}"'] & N \arrow[d, "p_j"] \\ & M_j \end{tikzcd}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = \mathrm{id}_{M_i}$ for $i = j$. Then $p_1\iota_1 = \delta_{11} = \mathrm{id}_M$.
(2) $\Longrightarrow$ (3): For $f := \iota p : N \longrightarrow N$ we have $f^2 = \iota p \iota p = \iota p = f$ since $p\iota = \mathrm{id}$. Furthermore $f(N) = \iota p(N) = M$ since p is surjective.
(3) $\Longrightarrow$ (1): Let $M' = \mathrm{Ke}(f)$. We first show $N = M + M'$. Take $n \in N$. Then we have $n = f(n) + (n - f(n))$ with $f(n) \in M$. Since $f(n - f(n)) = f(n) - f^2(n) = 0$ we get $n - f(n) \in \mathrm{Ke}(f) = M'$ so that $N = M + M'$. Now let $n \in M \cap M'$. Then $f(n) = 0$ and $n = f(n')$ for $n' \in N$ hence $n = f(n') = f^2(n') = f(n) = 0$. $\square$

Problem 3.1. Discuss the definition and the properties of products of groups.

Problem 3.2. Show that the tensor product of two commutative $\mathbb{K}$-algebras is a coproduct.

Problem 3.3. Show that the disjoint union of two sets is a coproduct.

3.2. **Projective modules.**

Definition 3.13. An R-module P is called *projective* if for each epimorphism $f : M \longrightarrow N$ and for each homomorphism $g : P \longrightarrow N$ there exists a homomorphism $h : P \longrightarrow M$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "g"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commutes.

Example 3.14. All vector spaces are projective. $\mathbb{Z}/n\mathbb{Z}$ $(n > 1)$ is not a projective $\mathbb{Z}$-module.

Lemma 3.15. *Let $P = \oplus_{i \in I} P_i$. P is projective iff all P_i, $i \in I$ are projective.*

Proof. Let P be projective. We show that P_i is projective. Let $f : M \longrightarrow N$ be an epimorphism and $g : P_i \longrightarrow N$ be a homomorphism. Consider the diagram

$$\begin{tikzcd} P_i \arrow[r, "\iota_i"] \arrow[dr, "h\iota_i"'] & P \arrow[r, "p_i"] \arrow[d, "h"] & P_i \arrow[d, "g"] \\ & M \arrow[r, "f"] & N \end{tikzcd}$$

where p_i and ι_i are projections and injections of the direct sum, in particular $p_i\iota_i = \mathrm{id}_{P_i}$. Since f is an epimorphism there is an $h : P \longrightarrow M$ with $fh = gp_i$ hence $g = gp_i\iota_i = fh\iota_i$. Thus P_i is projective.
Assume that all P_i are projective. Let $f : M \longrightarrow N$ be an epimorphism and $g : P \longrightarrow N$ be a homomorphism. Consider the diagram

$$\begin{tikzcd} P_i \arrow[r, "\iota_i"] \arrow[d, "h_i"'] & P \arrow[dl, "h"'] \arrow[d, "g"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Since f is surjective there are $h_i : P_i \longrightarrow M$, $i \in I$ with $fh_i = g\iota_i$. Since P is the coproduct of the P_i there is a (unique) $h : P \longrightarrow M$ with $h\iota_i = h_i$ for all $i \in I$. Thus $fh\iota_i = fh_i = g\iota_i$ for all $i \in I$ hence $fh = g$. So P is projective. □

Proposition 3.16. *Let P be an R-module. Then the following are equivalent*

(1) *P is projective.*
(2) *Each epimorphism $f : M \longrightarrow P$* splits*, i.e. for each R-module M and each epimorphism $f : M \longrightarrow P$ there is a homomorphism $g : P \longrightarrow M$ such that $fg = \mathrm{id}_P$.*
(3) *P is isomorphic to a direct summand of a free R-module RX.*

Proof. (1) $\Longrightarrow$ (2): The diagram

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "\mathrm{id}_P"] \\ M \arrow[r, "f"'] & P \end{tikzcd}$$

implies the existence of g with $fg = \mathrm{id}_P$.
(2) $\Longrightarrow$ (3): Let $\iota : P \longrightarrow RP$ be the free module over (the set) P with ι a *map*. Then there is a homomorphism $f : RP \longrightarrow P$ such that

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[dr, "\mathrm{id}_P"'] & RP \arrow[d, "f"] \\ & P \end{tikzcd}$$

commutes. Obviously f is surjective. By (2) there is a *homomorphism* $g : P \longrightarrow RP$ with $fg = \mathrm{id}_P$. By 3.12 P is a direct summand of RP (up to an isomorphism).
(3) $\Longrightarrow$ (1): Let $f : M \longrightarrow N$ be surjective. Let $\iota : X \longrightarrow RX$ be a free module and let $g : RX \longrightarrow N$ be a homomorphism. In the following diagram let $k = g\iota : X \longrightarrow N$. Since f is surjective there is a map $h : X \longrightarrow M$ with $fh = k$. Hence there is a homomorphism $l : RX \longrightarrow M$ with $l\iota = h$. This implies $fl\iota = fh = k = g\iota$ and thus $fl = g$ since RX is free. So

RX is projective. The transition to a direct summand follows from 3.15.

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[d, "h"'] \arrow[dr, "k"] & RX \arrow[d, "g"] \arrow[dl, "l"'] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

□

3.3. **Dual basis.**

Remark 3.17. Let P_R be a right R-module. Then $E := \mathrm{End}_R(P.) = \mathrm{Hom}_R(P., P.)$ is a ring and P is an E-R-bimodule because of $f(pr) = (fp)r$. Let $P^* := \mathrm{Hom}_R(P., R.)$ be the *dual* of P. Then $P^* = {}_R\mathrm{Hom}_R({}_EP., {}_RR.)_E$ is an R-E-bimodule. The following maps are bimodule homomorphisms

$$\mathrm{ev} : {}_RP^* \otimes_E P_R \ni f \otimes p \mapsto f(p) \in {}_RR_R,$$

the *evaluation homomorphism*, and

$$\mathrm{db} : {}_EP \otimes_R P^*_E \longrightarrow {}_EE_E = {}_E\mathrm{End}_R(P.)_E$$

with $\mathrm{db}(p \otimes f)(q) = pf(q)$, the *dual basis homomorphism*. We check the bilinearity: $\mathrm{ev}(fe, p) = (fe)(p) = f(e(p)) = \mathrm{ev}(f, ep)$ and $\mathrm{db}(pr, f)(q) = (pr)f(q) = p(rf(q)) = \mathrm{db}(p, rf)(q)$. We also check that db is a bimodule homomorphism: $\mathrm{db}(ep \otimes f)(q) = e(p)f(q) = e(pf(q)) = e\,\mathrm{db}(p \otimes f)(q)$ and $\mathrm{db}(p \otimes fe)(q) = pfe(q) = \mathrm{db}(p \otimes f)e(q)$.

Lemma 3.18. *The following diagrams commute*

$$\begin{tikzcd}
P^* \otimes_E P \otimes_R P^* \arrow[r, "1 \otimes \mathrm{db}"] \arrow[d, "\mathrm{ev} \otimes 1"'] & P^* \otimes_E E \arrow[d, "\mu"] \\
R \otimes_R P^* \arrow[r, "\mu"'] & P^*
\end{tikzcd}
\qquad
\begin{tikzcd}
P \otimes_R P^* \otimes_E P \arrow[r, "1 \otimes \mathrm{ev}"] \arrow[d, "\mathrm{db} \otimes 1"'] & P \otimes_R R \arrow[d, "\mu"] \\
E \otimes_E P \arrow[r, "\mu"'] & P
\end{tikzcd}$$

Proof. The proof follows from the associative law: $\mu(1 \otimes \mathrm{db})(f \otimes p \otimes g)(q) = \mu(f \otimes pg)(q) = f(pg)(q) = f(pg(q)) = f(p)g(q) = \mu(f(p) \otimes g)(q) = \mu(\mathrm{ev} \otimes 1)(f \otimes p \otimes g)(q)$ and $\mu(\mathrm{db} \otimes 1)(p \otimes f \otimes q) = \mu(pf \otimes q) = pf(q) = \mu(p \otimes f(q)) = \mu(1 \otimes \mathrm{ev})(p \otimes f \otimes q)$. □

Proposition 3.19. (dual basis Lemma) *Let P_R be a right R-module. Then the following are equivalent:*

(1) *P is finitely generated and projective,*
(2) (dual basis) *There are $f_1, \ldots, f_n \in \mathrm{Hom}_R(P., R.) = P^*$ and $p_1, \ldots, p_n \in P$ so that*

$$p = \sum p_i f_i(p)$$

for all $p \in P$

(3) *The dual basis homomorphism*

$$\mathrm{db} : P \otimes_R P^* \longrightarrow \mathrm{Hom}_R(P., P.)$$

is an isomorphism.

Proof. (1) $\Longrightarrow$ (2): Let P be generated by $\{p_1, \ldots, p_n\}$. Let RX be a free right R-module over the set $X = \{x_1, \ldots, x_n\}$. Let $\pi_i : RX \longrightarrow R$ be the projections induced by

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\sigma_i"'] & RX \arrow[d, "\pi_i"] \\ & R \end{tikzcd}$$

where $\sigma_i(x_j) = \delta_{ij}$. By Proposition 1.10 we have $z = \sum x_i\pi_i(z)$ for all $z \in RX$. Let $g : RX \longrightarrow P$ be the R-module homomorphism with $g(x_i) = p_i$. Since the p_i generate P as a module, the homomorphism g is surjective. P is projective hence there is a homomorphism $h : P \longrightarrow RX$ with $gh = \mathrm{id}_P$ by 3.16. Define $f_i := \pi_i h$. Then $\sum p_i\pi_i h(p) = \sum g(x_i)\pi_i h(p) = g(\sum x_i\pi_i(h(p))) = gh(p) = p$.
(2) $\Longrightarrow$ (3): The homomorphism $\psi : \mathrm{Hom}_R(P., P.) \longrightarrow P \otimes_R P^*$ defined by $\psi(e) = \sum e(p_i) \otimes f_i$ is the inverse map of db. In fact we have $\mathrm{db} \circ \psi(e)(p) = \sum e(p_i)f_i(p) = e(\sum p_i f_i(p)) = e(p)$, hence $\mathrm{db} \circ \psi = \mathrm{id}$. Furthermore we have $\psi \circ \mathrm{db}(p \otimes f) = \psi(pf) = \sum pf(p_i) \otimes f_i = p \otimes \sum f(p_i)f_i = p \otimes f$ since $\sum f(p_i)f_i(q) = f(\sum p_i f_i(q)) = f(q)$, hence we have also $\psi \circ \mathrm{db} = \mathrm{id}$.
(3) $\Longrightarrow$ (2): $\sum p_i \otimes f_i = \mathrm{db}^{-1}(\mathrm{id}_P)$ is a dual basis, because $\sum p_i f_i(p) = \mathrm{db}(\sum p_i \otimes f_i)(p) = \mathrm{id}_P(p) = p$.
(2) $\Longrightarrow$ (1): The p_i generate P since $\sum p_i f_i(p) = p$ for all $p \in P$. Thus P is finitely generated. Furthermore the homomorphism $g : RX \longrightarrow P$ with $g(x_i) = p_i$ is surjective. Let $h : P \longrightarrow RX$ be defined by $h(p) = \sum x_i f_i(p)$. Then $gh(p) = p$, hence P is a direct summand of RX, and consequently P is projective. $\square$

Remark 3.20. Observe that analogous statements hold for left R- modules. The problem that in that situation two rings R and $\mathrm{End}_R(.P)$ operate from the left on P is best handled by considering P as a right $\mathrm{End}_R(.P)^{op}$- module where $\mathrm{End}_R(.P)^{op}$ has the opposite multiplication $*$ given by $f * g := g \circ f$. We leave it to the reader to verify the details. The evaluation and dual basis homomorphisms are in this case $\mathrm{ev} : {}_RP \otimes_{E^{op}} P_R^* \ni p \otimes f \mapsto f(p) \in {}_RR_R$, and $\mathrm{db} : P^* \otimes_R P \longrightarrow \mathrm{Hom}_R(.P, .P)$.

Proposition 3.21. *Let R be a commutative ring and P be an R-module. Then the following are equivalent*

(1) *${}_RP$ is finitely generated and projective,*
(2) *there exists an R-module P', and homomorphisms $\mathrm{db}' : R \longrightarrow P \otimes_R P'$ and $\mathrm{ev} : P' \otimes_R P \longrightarrow R$ such that*

$$(P \xrightarrow{\mathrm{db}' \otimes \mathrm{id}} P \otimes_R P' \otimes_R P \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} P) = \mathrm{id}_P,$$
$$(P' \xrightarrow{\mathrm{id} \otimes \mathrm{db}'} P' \otimes_R P \otimes_R P' \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} P') = \mathrm{id}_{P'}.$$

Proof. "$\Longleftarrow$": $\mathrm{ev} \in \mathrm{Hom}_R(P' \otimes_R P, R) \cong \mathrm{Hom}_R(P', \mathrm{Hom}_R(P, R))$ induces a homomorphism $\epsilon : P' \longrightarrow P^*$ by $\epsilon(f)(p) = \mathrm{ev}(f \otimes p) = fp$ for $f \in P'$. Let $\mathrm{db}'(1) = \sum p_i \otimes f_i$. Then $p = \mathrm{id}_P(p) = (\mathrm{id} \otimes_R \mathrm{ev})(\mathrm{db}' \otimes_R \mathrm{id})(p) = (\mathrm{id} \otimes_R \mathrm{ev})(\sum p_i \otimes f_i \otimes p) = \sum p_i f_i p$. By 3.19 P is finitely generated and projective.
"$\Longrightarrow$": Define $P' := P^*$ and $(\mathrm{ev} : P' \otimes_R P \longrightarrow R) = (\mathrm{ev} : P^* \otimes_R P \longrightarrow R)$. Let $\mathrm{db}'(1) = \sum p_i \otimes f_i$ be the dual basis for P. Then we have $(\mathrm{id} \otimes_R \mathrm{ev})(\mathrm{db}' \otimes_R \mathrm{id})(p) = (\mathrm{id} \otimes_R \mathrm{ev})(\sum p_i \otimes f_i \otimes p) = \sum p_i f_i(p) = p$. Furthermore we have $\sum f(p_i)f_i(p) = f(\sum p_i f_i(p)) = f(p)$, hence $\sum f(p_i)f_i = f$. This implies $(\mathrm{ev} \otimes_R \mathrm{id})(\mathrm{id} \otimes_R \mathrm{db}')(f) = (\mathrm{ev} \otimes_R \mathrm{id})(\sum f \otimes p_i \otimes f_i) = \sum f(p_i)f_i = f$. $\square$

Example 3.22. of a projective module, that is not free:

Let $S^2 =$ 2-sphere $= \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$. Let R be the ring of all continuous real-valued functions on S^2. Let $F = \{f : S^2 \longrightarrow \mathbb{R}^3 | f \text{ continuous}\} = \{(f_1, f_2, f_3) | f_i \in R\} = R^3$ be the free R-module on three generators. F is a set of vector valued functions, the vectors starting in the point of S^2 where their counterimage is. These are vector fields over S^2. Let $P = \{\text{tangential vector fields}\}$ and $Q = \{\text{normal vector fields}\}$. Then $F = P \oplus Q$ as R-modules. So P and Q are projective. Furthermore $Q \cong R$. Hence $F \cong P \oplus R$. Suppose P were free. Evaluating all elements of P in a given point $p \in S^2$ we get the tangent plane at p which is $\mathbb{R}^2$. If P is free then it has a basis e_1, e_2 (see later remarks on the rank of free modules over a commutative ring). For $p \in S^2$ we have $e_1(p), e_2(p)$ generates the tangent plane, hence is a basis for the tangent plane. So $e_1(p) \neq 0$ for all $p \in S^2$. By the "egg theorem" this is impossible.

3.4. **Generators.**

Definition 3.23. A right R-module G_R is called a *generator* if for each homomorphism $f : M \longrightarrow N$ with $f \neq 0$ there exists a homomorphism $g : G \longrightarrow M$ such that $fg \neq 0$.

Proposition 3.24. *Let G_R be an R-module. The following are equivalent*

(1) *G is a generator,*
(2) *for each R-module M_R there is a set I and an epimorphism $h : \coprod_I G \longrightarrow M$,*
(3) *R is isomorphic to a direct summand of $\coprod_I G$ (for an appropriate set I),*
(4) *there are $f_1, \ldots, f_n \in G^* = \mathrm{Hom}_R(G., R.)$ and $q_1, \ldots, q_n \in G$ with $\sum f_i(q_i) = 1$.*

Proof. (1) $\Longrightarrow$ (2): Define $I := \mathrm{Hom}_R(G., M.)$. Then the diagram

$$\begin{tikzcd} G = G_f \arrow[r, "\iota_f"] \arrow[dr, "f"'] & \coprod_I G_f \arrow[d, "h"] \\ & M \end{tikzcd}$$

defines a unique homomorphism h with $h\iota_f = f$ for all $f \in I$. Let $N = \mathrm{Im}(h)$. Consider $\nu : M \longrightarrow M/N$. If $N \neq M$ then $\nu \neq 0$. Since G is a generator there exists an f such that $\nu f \neq 0$. This implies $\nu h \neq 0$ a contradiction to $N = \mathrm{Im}(h)$. Hence $N = M$ so that h is an epimorphism.
(2) $\Longrightarrow$ (3): Let $\coprod G \longrightarrow R$ be an epimorphism. Since R is a free module hence projective, 3.16 implies that R is a direct summand of $\coprod G$ up to isomorphism.
(3) $\Longrightarrow$ (4): Since R is (isomorphic to) a direct summand of $\coprod_I G$ there is $p : \coprod_I G \longrightarrow R$ with $p\iota = \mathrm{id}_R$. Let $p((g_i)) = 1$ and $f_i = p\iota_i : G \longrightarrow R$. Then $1 = p((g_i)) = p(\sum \iota_i(g_i)) = \sum p\iota_i(g_i) = \sum f_i(q_i)$.
(4) $\Longrightarrow$ (1): Assume $(g : M \longrightarrow N) \neq 0$. Then there is an $m \in M$ with $g(m) \neq 0$. Define $f : R \longrightarrow M$ by $f(1) = m$, $f(r) = rm$. Let f_i, q_i be given with $\sum f_i(q_i) = 1$. Then we have $0 \neq g(m) = gf(1) = \sum gff_i(q_i)$, so we have the existence of a homomorphism $ff_i : G \longrightarrow M$ with $gff_i \neq 0$. $\square$

4. Categories and Functors

4.1. **Categories.** In the preceding sections we saw that certain constructions like products can be performed for different kinds of mathematical structures, e.g. modules, rings, Abelian groups, groups, etc. In order to indicate the kind of structure that one uses the notion of a category has been invented.

Definition 4.1. Let $\mathcal{C}$ consist of

(1) a class $\mathrm{Ob}\,\mathcal{C}$ whose elements $A, B, C, \ldots \in \mathrm{Ob}\,\mathcal{C}$ are called *objects*,
(2) a family $(\mathrm{Mor}_{\mathcal{C}}(A,B) | A, B \in \mathrm{Ob}\,\mathcal{C})$ of mutually disjoint sets whose elements $f, g, \ldots \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ are called *morphisms*, and
(3) a family $(\mathrm{Mor}_{\mathcal{C}}(A,B) \times \mathrm{Mor}_{\mathcal{C}}(B,C) \ni (f,g) \mapsto gf \in \mathrm{Mor}_{\mathcal{C}}(A,C) | A, B, C \in \mathrm{Ob}\,\mathcal{C})$ of maps called *compositions*.

$\mathcal{C}$ is called a *category* if the following axioms hold for $\mathcal{C}$

(1) Associative Law:
$\forall A, B, C, D \in \mathrm{Ob}\,\mathcal{C}, f \in \mathrm{Mor}_{\mathcal{C}}(A,B), g \in \mathrm{Mor}_{\mathcal{C}}(B,C), h \in \mathrm{Mor}_{\mathcal{C}}(C,D):$
$$h(gf) = (hg)f;$$
(2) Identity Law:
$\forall A \in \mathrm{Ob}\,\mathcal{C}\ \exists 1_A \in \mathrm{Mor}_{\mathcal{C}}(A,A)\ \forall B, C \in \mathrm{Ob}\,\mathcal{C},\ \forall f \in \mathrm{Mor}_{\mathcal{C}}(A,B),\ \forall g \in \mathrm{Mor}_{\mathcal{C}}(C,A):$
$$1_A g = g \qquad \text{and} \qquad f1_A = f.$$

Examples 4.2. (1) The category of sets Set.
(2) The categories of R-modules R-Mod, $\mathbb{K}$-vector spaces $\mathbb{K}$-Vec or $\mathbb{K}$-Mod, groups Gr, Abelian groups Ab, monoids Mon, commutative monoids cMon, rings Ri, fields Field, topological spaces Top.
(3) The left A-modules in the sense of Definition 2.17 and their homomorphisms form the category A-Mod of A-modules.
(4) The $\mathbb{K}$-algebras in the sense of Definition 2.1 and their homomorphisms form the category $\mathbb{K}$-Alg of $\mathbb{K}$-algebras.
(5) The category of commutative $\mathbb{K}$-algebras will be denoted by $\mathbb{K}$-cAlg.
(6) The $\mathbb{K}$-coalgebras in the sense of Definition 2.18 and their homomorphisms form a category $\mathbb{K}$-Coalg of $\mathbb{K}$-coalgebras.
(7) The category of cocommutative $\mathbb{K}$-coalgebras will be denoted by $\mathbb{K}$-cCoalg.

For arbitrary categories we adopt many of the customary notations.

Notation 4.3. $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ will be written as $f : A \longrightarrow B$ or $A \xrightarrow{f} B$. A is called the *domain*, B the *range* of f.
The *composition* of two morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is written as $gf : A \longrightarrow C$ or as $g \circ f : A \longrightarrow C$.

Definition and Remark 4.4. A morphism $f : A \longrightarrow B$ is called an *isomorphism* if there exists a morphism $g : B \longrightarrow A$ in $\mathcal{C}$ such that $fg = 1_B$ and $gf = 1_A$. The morphism g is uniquely determined by f since $g' = g'fg = g$. We write $f^{-1} := g$.
An object A is said to be *isomorphic* to an object B if there exists an isomorphism $f : A \longrightarrow B$. If f is an isomorphism then so is f^{-1}. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms in $\mathcal{C}$ then so is $gf : A \longrightarrow C$. We have: $(f^{-1})^{-1} = f$ and $(gf)^{-1} = f^{-1}g^{-1}$. The relation of being isomorphic between objects is an equivalence relation.

Example 4.5. In the categories Set, R-Mod, k-Vec, Gr, Ab, Mon, cMon, Ri, Field the isomorphisms are exactly those morphisms which are bijective as set maps.

In Top the set $M = \{a, b\}$ with $\mathfrak{T}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and with $\mathfrak{T}_2 = \{\emptyset, M\}$ defines two different topological spaces. The map $f = \mathrm{id} : (M, \mathfrak{T}_1) \longrightarrow (M, \mathfrak{T}_2)$ is bijective and continuous. The inverse map, however, is not continuous, hence f is no isomorphism (homeomorphism).

Many well known concepts can be defined for arbitrary categories. We are going to introduce some of them. Here are two examples.

Definition 4.6. (1) A morphism $f : A \longrightarrow B$ is called a *monomorphism* if $\forall C \in \mathrm{Ob}\,\mathcal{C},\ \forall g, h \in \mathrm{Mor}_{\mathcal{C}}(C, A)$:

$$fg = fh \Longrightarrow g = h \quad (f \text{ is left cancellable}).$$

(2) A morphism $f : A \longrightarrow B$ is called an *epimorphism* if $\forall C \in \mathrm{Ob}\,\mathcal{C},\ \forall g, h \in \mathrm{Mor}_{\mathcal{C}}(B, C)$:

$$gf = hf \Longrightarrow g = h \quad (f \text{ is right cancellable}).$$

Definition 4.7. Given $A, B \in \mathcal{C}$. An object $A \times B$ in $\mathcal{C}$ together with morphisms $p_A : A \times B \longrightarrow A$ and $p_B : A \times B \longrightarrow B$ is called a (categorical) *product* of A and B if for every (test) object $T \in \mathcal{C}$ and every pair of morphisms $f : T \longrightarrow A$ and $g : T \longrightarrow B$ there exists a unique morphism $(f, g) : T \longrightarrow A \times B$ such that the diagram

$$\begin{tikzcd}
& T \arrow[dl, "f"'] \arrow[d, "{(f,g)}"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B
\end{tikzcd}$$

commutes.
An object $E \in \mathcal{C}$ is called a *final object* if for every (test) object $T \in \mathcal{C}$ there exists a unique morphism $e : T \longrightarrow E$ (i.e. $\mathrm{Mor}_{\mathcal{C}}(T, E)$ consists of exactly one element).
A category $\mathcal{C}$ which has a product for any two objects A and B and which has a final object is called a *category with finite products*.

Remark 4.8. If the product $(A \times B, p_A, p_B)$ of two objects A and B in $\mathcal{C}$ exists then it is unique up to isomorphism.
If the final object E in $\mathcal{C}$ exists then it is unique up to isomorphism.

Problem 4.1. Let $\mathcal{C}$ be a category with finite products. Give a definition of a product of a family $A_1, \ldots, A_n$ $(n \geq 0)$. Show that products of such families exist in $\mathcal{C}$.

Definition and Remark 4.9. Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{op}$ with the following data $\mathrm{Ob}\,\mathcal{C}^{op} := \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}^{op}}(A, B) := \mathrm{Mor}_{\mathcal{C}}(B, A)$, and $f \circ_{op} g := g \circ f$ defines a new category, the *dual category* of $\mathcal{C}$.

Remark 4.10. Any notion expressed in categorical terms (with objects, morphisms, and their composition) has a *dual notion*, i.e. the given notion in the dual category.
Monomorphisms f in the dual category $\mathcal{C}^{op}$ are epimorphisms in the original category $\mathcal{C}$ and conversely. A final object I in the dual category $\mathcal{C}^{op}$ is an *initial object* in the original category $\mathcal{C}$.

Definition 4.11. The *coproduct* of two objects in the category $\mathcal{C}$ is defined to be a product of the objects in the dual category $\mathcal{C}^{op}$.

Remark 4.12. Equivalent to the preceding definition is the following definition.
Given $A, B \in \mathcal{C}$. An object $A \amalg B$ in $\mathcal{C}$ together with morphisms $j_A : A \longrightarrow A \amalg B$ and $j_B : B \longrightarrow A \amalg B$ is a (categorical) coproduct of A and B if for every (test) object $T \in \mathcal{C}$ and every

pair of morphisms $f : A \longrightarrow T$ and $g : B \longrightarrow T$ there exists a unique morphism $[f,g] : A \amalg B \longrightarrow T$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & A \amalg B \arrow[d, "{[f,g]}"] & B \arrow[l, "j_B"'] \arrow[dl, "g"] \\ & T & \end{tikzcd}$$

commutes.

The category $\mathcal{C}$ is said to have *finite coproducts* if $\mathcal{C}^{op}$ is a category with finite products. In particular coproducts are unique up to isomorphism.

4.2. **Functors.**

Definition 4.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $\mathcal{F}$ consist of

(1) a map $\mathrm{Ob}\,\mathcal{C} \ni A \mapsto \mathcal{F}(A) \in \mathrm{Ob}\,\mathcal{D}$,
(2) a family of maps

$$(\mathcal{F}_{A,B} : \mathrm{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \mathcal{F}_{A,B}(f) \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(A),\mathcal{F}(B)) | A, B \in \mathcal{C})$$

$$[\text{ or } (\mathcal{F}_{A,B} : \mathrm{Mor}_{\mathcal{C}}(A,B) \ni f \mapsto \mathcal{F}_{A,B}(f) \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(B),\mathcal{F}(A)) | A, B \in \mathcal{C})]$$

$\mathcal{F}$ is called a *covariant* [*contravariant*] *functor* if

(1) $\mathcal{F}_{A,A}(1_A) = 1_{\mathcal{F}(A)}$ for all $A \in \mathrm{Ob}\,\mathcal{C}$,
(2) $\mathcal{F}_{A,C}(gf) = \mathcal{F}_{B,C}(g)\mathcal{F}_{A,B}(f)$ for all $A, B, C \in \mathrm{Ob}\,\mathcal{C}$.
[$\mathcal{F}_{A,C}(gf) = \mathcal{F}_{A,B}(f)\mathcal{F}_{B,C}(g)$ for all $A, B, C \in \mathrm{Ob}\,\mathcal{C}$].

Notation: We write

$$\begin{array}{lcc} A \in \mathcal{C} & \text{instead of} & A \in \mathrm{Ob}\,\mathcal{C} \\ f \in \mathcal{C} & \text{instead of} & f \in \mathrm{Mor}_{\mathcal{C}}(A,B) \\ \mathcal{F}(f) & \text{instead of} & \mathcal{F}_{A,B}(f). \end{array}$$

Examples 4.14. The following define functors

(1) $\mathrm{Id} : \mathrm{Set} \longrightarrow \mathrm{Set}$;
(2) Forget : R-Mod $\longrightarrow$ Set;
(3) Forget : Ri $\longrightarrow$ Ab;
(4) Forget : Ab $\longrightarrow$ Gr;
(5) $\mathcal{P} : \mathrm{Set} \longrightarrow \mathrm{Set}, \mathcal{P}(M) :=$ power set of M. $\mathcal{P}(f)(X) := f^{-1}(X)$ for $f : M \longrightarrow N, X \subseteq N$ is a contravariant functor;
(6) $\mathcal{Q} : \mathrm{Set} \longrightarrow \mathrm{Set}, \mathcal{Q}(M) :=$ power set of M. $\mathcal{Q}(f)(X) := f(X)$ for $f : M \longrightarrow N, X \subseteq M$ is a covariant functor;
(7) $- \otimes_R N : \text{Mod-}R \longrightarrow \mathrm{Ab}$;
(8) $M \otimes_R - : R\text{-Mod} \longrightarrow \mathrm{Ab}$;
(9) $- \otimes_R - : \text{Mod-}R \times R\text{-Mod} \longrightarrow \mathrm{Ab}$;
(10) the embedding functor $\iota : \mathbb{K}\text{-Mod} \longrightarrow \mathbb{K}\text{-Mod-}\mathbb{K}$.
(11) the tensor product over $\mathbb{K}$ in $\mathbb{K}$-Mod-$\mathbb{K}$ can be restricted to $\mathbb{K}$-Mod so that the following diagram of functors commutes:

$$\begin{tikzcd} \mathbb{K}\text{-Mod} \times \mathbb{K}\text{-Mod} \arrow[r, "\iota \times \iota"] \arrow[d, "\otimes_{\mathbb{K}}"'] & \mathbb{K}\text{-Mod-}\mathbb{K} \times \mathbb{K}\text{-Mod-}\mathbb{K} \arrow[d, "\otimes_{\mathbb{K}}"] \\ \mathbb{K}\text{-Mod} \arrow[r, "\iota"'] & \mathbb{K}\text{-Mod-}\mathbb{K}. \end{tikzcd}$$

Proof of (9). $(f \times g) \circ (f' \times g') = ff' \times gg'$ implies $(f \otimes_R g) \circ (f' \otimes_R g') = ff' \otimes_R gg'$. Furthermore $1_M \times 1_N = 1_{M \times N}$ implies $1_M \otimes_R 1_N = 1_{M \otimes_R N}$. □

Lemma 4.15. (1) *Let $X \in \mathcal{C}$. Then*

$$\operatorname{Ob}\mathcal{C} \ni A \mapsto \operatorname{Mor}_{\mathcal{C}}(X, A) \in \operatorname{Ob}\operatorname{Set}$$

$$\operatorname{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \operatorname{Mor}_{\mathcal{C}}(X, f) \in \operatorname{Mor}_{\operatorname{Set}}(\operatorname{Mor}_{\mathcal{C}}(X, A), \operatorname{Mor}_{\mathcal{C}}(X, B)),$$

with $\operatorname{Mor}_{\mathcal{C}}(X, f) : \operatorname{Mor}_{\mathcal{C}}(X, A) \ni g \mapsto fg \in \operatorname{Mor}_{\mathcal{C}}(X, B)$ *or* $\operatorname{Mor}_{\mathcal{C}}(X, f)(g) = fg$ *is a covariant functor* $\operatorname{Mor}_{\mathcal{C}}(X, \text{-})$.

(2) *Let $X \in \mathcal{C}$. Then*

$$\operatorname{Ob}\mathcal{C} \ni A \mapsto \operatorname{Mor}_{\mathcal{C}}(A, X) \in \operatorname{Ob}\operatorname{Set}$$

$$\operatorname{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \operatorname{Mor}_{\mathcal{C}}(f, X) \in \operatorname{Mor}_{\operatorname{Set}}(\operatorname{Mor}_{\mathcal{C}}(B, X), \operatorname{Mor}_{\mathcal{C}}(A, X))$$

with $\operatorname{Mor}_{\mathcal{C}}(f, X) : \operatorname{Mor}_{\mathcal{C}}(B, X) \ni g \mapsto gf \in \operatorname{Mor}_{\mathcal{C}}(A, X)$ *or* $\operatorname{Mor}_{\mathcal{C}}(f, X)(g) = gf$ *is a contravariant functor* $\operatorname{Mor}_{\mathcal{C}}(\text{-}, X)$.

Proof. (1) $\operatorname{Mor}_{\mathcal{C}}(X, 1_A)(g) = 1_A g = g = \operatorname{id}(g), \operatorname{Mor}_{\mathcal{C}}(X, f)\operatorname{Mor}_{\mathcal{C}}(X, g)(h) = fgh = \operatorname{Mor}_{\mathcal{C}}(X, fg)(h)$.
(2) analogously. □

Remark 4.16. The preceding lemma shows that $\operatorname{Mor}_{\mathcal{C}}(\text{-}, \text{-})$ is a functor in both arguments. A functor in two arguments is called a *bifunctor*. We can regard the bifunctor $\operatorname{Mor}_{\mathcal{C}}(\text{-}, \text{-})$ as a covariant functor

$$\operatorname{Mor}_{\mathcal{C}}(\text{-}, \text{-}) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \operatorname{Set}.$$

The use of the dual category removes the fact that the bifunctor $\operatorname{Mor}_{\mathcal{C}}(\text{-}, \text{-})$ is contravariant in the first variable.
Obviously the composition of two functors is again a functor and this composition is associative. Furthermore for each category $\mathcal{C}$ there is an identity functor $\operatorname{Id}_{\mathcal{C}}$.
Functors of the form $\operatorname{Mor}_{\mathcal{C}}(X, \text{-})$ resp. $\operatorname{Mor}_{\mathcal{C}}(\text{-}, X)$ are called *representable functors* (covariant resp. contravariant) and X is called the *representing object* (see also section 5).

4.3. Natural Transformations.

Definition 4.17. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. A *natural transformation* or a *functorial morphism* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a family of morphisms $(\varphi(A) : \mathcal{F}(A) \longrightarrow \mathcal{G}(A) | A \in \mathcal{C})$ such that the diagram

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\varphi(A)"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(A) \arrow[d, "\mathcal{G}(f)"] \\ \mathcal{F}(B) \arrow[r, "\varphi(B)"'] & \mathcal{G}(B) \end{tikzcd}$$

commutes for all $f : A \longrightarrow B$ in $\mathcal{C}$, i.e. $\mathcal{G}(f)\varphi(A) = \varphi(B)\mathcal{F}(f)$.

Lemma 4.18. *Given covariant functors* $\mathcal{F} = \operatorname{Id}_{\operatorname{Set}} : \operatorname{Set} \longrightarrow \operatorname{Set}$ *and*

$$\mathcal{G} = \operatorname{Mor}_{\operatorname{Set}}(\operatorname{Mor}_{\operatorname{Set}}(-, A), A) : \operatorname{Set} \longrightarrow \operatorname{Set}$$

for a set A. Then $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ with

$$\varphi(B) : B \ni b \mapsto (\operatorname{Mor}_{\operatorname{Set}}(B, A) \ni f \mapsto f(b) \in A) \in \mathcal{G}(B)$$

is a natural transformation.

Proof. Given $g : B \longrightarrow C$. Then the following diagram commutes

$$\begin{tikzcd}
B \arrow[r, "\varphi(B)"] \arrow[d, "g"'] & \mathrm{Mor}_{\mathrm{Set}}(\mathrm{Mor}_{\mathrm{Set}}(B,A),A) \arrow[d, "\mathrm{Mor}_{\mathrm{Set}}(\mathrm{Mor}_{\mathrm{Set}}(g,A),A)"] \\
C \arrow[r, "\varphi(C)"] & \mathrm{Mor}_{\mathrm{Set}}(\mathrm{Mor}_{\mathrm{Set}}(C,A),A)
\end{tikzcd}$$

since

$$\begin{aligned}\varphi(C)\mathcal{F}(g)(b)(f) &= \varphi(C)g(b)(f) = fg(b) = \varphi(B)(b)(fg) \\ &= [\varphi(B)(b)\,\mathrm{Mor}_{\mathrm{Set}}(g,A)](f) = [\mathrm{Mor}_{\mathrm{Set}}(\mathrm{Mor}_{\mathrm{Set}}(g,A),A)\varphi(B)(b)](f).\end{aligned}$$

□

Lemma 4.19. *Let* $f : A \longrightarrow B$ *be a morphism in* $\mathcal{C}$. *Then* $\mathrm{Mor}_{\mathcal{C}}(f, \text{-}) : \mathrm{Mor}_{\mathcal{C}}(B, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(A, \text{-})$ *given by* $\mathrm{Mor}_{\mathcal{C}}(f, C) : \mathrm{Mor}_{\mathcal{C}}(B, C) \ni g \mapsto gf \in \mathrm{Mor}_{\mathcal{C}}(A, C)$ *is a natural transformation of covariant functors.*
Let $f : A \longrightarrow B$ *be a morphism in* $\mathcal{C}$. *Then* $\mathrm{Mor}_{\mathcal{C}}(\text{-}, f) : \mathrm{Mor}_{\mathcal{C}}(\text{-}, A) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, B)$ *given by* $\mathrm{Mor}_{\mathcal{C}}(C, f) : \mathrm{Mor}_{\mathcal{C}}(C, A) \ni g \mapsto fg \in \mathrm{Mor}_{\mathcal{C}}(C, B)$ *is a natural transformation of contravariant functors.*

Proof. Let $h : C \longrightarrow C'$ be a morphism in $\mathcal{C}$. Then the diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(B,C) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(f,C)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(B,h)"'] & \mathrm{Mor}_{\mathcal{C}}(A,C) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(A,h)"] \\
\mathrm{Mor}_{\mathcal{C}}(B,C') \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(f,C')"'] & \mathrm{Mor}_{\mathcal{C}}(A,C')
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(C',A) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(C',f)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(h,A)"'] & \mathrm{Mor}_{\mathcal{C}}(C',B) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(h,B)"] \\
\mathrm{Mor}_{\mathcal{C}}(C,A) \arrow[r, "\mathrm{Mor}_{\mathcal{C}}(C,f)"'] & \mathrm{Mor}_{\mathcal{C}}(C,B)
\end{tikzcd}$$

commute. □

Remark 4.20. The composition of two natural transformations is again a natural transformation. The identity $\mathrm{id}_{\mathcal{F}}(A) := 1_{\mathcal{F}(A)}$ is also a natural transformation.

Definition 4.21. A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is called a *natural isomorphism* if there exists a natural transformation $\psi : \mathcal{G} \longrightarrow \mathcal{F}$ such that $\varphi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$. The natural transformation ψ is uniquely determined by φ. We write $\varphi^{-1} := \psi$.
A functor $\mathcal{F}$ is said to be *isomorphic* to a functor $\mathcal{G}$ if there exists a natural isomorphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$.

Remark 4.22. The isomorphisms given in Theorem 1.22 for ${}_RM_S$, ${}_SN_T$, and ${}_TP_U$ are natural isomorphisms:

(1) *Associativity Law:* $\alpha : (M \otimes_S N) \otimes_T P \cong M \otimes_S (N \otimes_T P)$ with $\alpha((m \otimes n) \otimes p) := m \otimes (n \otimes p)$;
(2) *Law of the Left Unit:* $\lambda : R \otimes_R M \cong M$ with $\lambda(r \otimes m) := rm$;
(3) *Law of the Right Unit:* $\rho : M \otimes_S S \cong M$ with $\rho(m \otimes r) := mr$;
(4) *Symmetry Law:* $\tau : M \otimes N \cong N \otimes M$ for $\mathbb{K}$-modules M and N with $\tau(m \otimes n) := n \otimes m$;

(5) *Inner Hom-Functors:*

$$\phi : \mathrm{Hom}_{S\text{-}T}(.P \otimes_R M.,.N.) \cong \mathrm{Hom}_{S\text{-}R}(.P.,.\,\mathrm{Hom}_T(M.,N.).)$$

with $\phi(f)(p)(m) := f(p \otimes m)$ and

$$\psi : \mathrm{Hom}_{S\text{-}T}(.P \otimes_R M.,.N.) \cong \mathrm{Hom}_{R\text{-}T}(.M.,.\,\mathrm{Hom}_S(.P,.N).)$$

with $\psi(f)(m)(p) := f(p \otimes m)$ for bimodules ${}_RM_T$, ${}_SN_T$, and ${}_SP_R$.

Problem 4.2. (1) Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. Show that a natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a natural isomorphism if and only if $\varphi(A)$ is an isomorphism for all objects $A \in \mathcal{C}$.
(2) Let $(A \times B, p_A, p_B)$ be the product of A and B in $\mathcal{C}$. Then there is a natural isomorphism

$$\mathrm{Mor}(\text{-}, A \times B) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, A) \times \mathrm{Mor}_{\mathcal{C}}(\text{-}, B).$$

(3) Let $\mathcal{C}$ be a category with finite products. For each object A in $\mathcal{C}$ show that there exists a morphism $\Delta_A : A \longrightarrow A \times A$ satisfying $p_1\Delta_A = 1_A = p_2\Delta_A$. Show that this defines a natural transformation. What are the functors?
(4) Let $\mathcal{C}$ be a category with finite products. Show that there is a bifunctor $\text{-} \times \text{-} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ such that $(\text{-} \times \text{-})(A, B)$ is the object of a product of A and B. We denote elements in the image of this functor by $A \times B := (\text{-} \times \text{-})(A, B)$ and similarly $f \times g$.
(5) With the notation of the preceding problem show that there is a natural transformation $\alpha(A, B, C) : (A \times B) \times C \cong A \times (B \times C)$. Show that the diagram (*coherence or constraints*)

$$\begin{tikzcd}
((A \times B) \times C) \times D \arrow[r, "\alpha(A,B,C)\times 1"] \arrow[d, "\alpha(A\times B,C,D)"] & (A \times (B \times C)) \times D \arrow[r, "\alpha(A,B\times C,D)"] & A \times ((B \times C) \times D) \arrow[d, "1\times\alpha(B,C,D)"] \\
(A \times B) \times (C \times D) \arrow[rr, "\alpha(A,B,C\times D)"] & & A \times (B \times (C \times D))
\end{tikzcd}$$

commutes.
(6) With the notation of the preceding problem show that there are a natural transformations $\lambda(A) : E \times A \longrightarrow A$ and $\rho(A) : A \times E \longrightarrow A$ such that the diagram (*coherence or constraints*)

$$\begin{tikzcd}
(A \times E) \times B \arrow[rr, "\alpha(A,E,B)"] \arrow[dr, "\rho(A)\times 1"'] & & A \times (E \times B) \arrow[dl, "1\times\lambda(B)"] \\
& A \times B &
\end{tikzcd}$$

Definition 4.23. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called an *equivalence of categories* if there exists a covariant functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\varphi : \mathcal{G}\mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$ and $\psi : \mathcal{F}\mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$.
A contravariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called a *duality of categories* if there exists a contravariant functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\varphi : \mathcal{G}\mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$ and $\psi : \mathcal{F}\mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$.
A category $\mathcal{C}$ is said to be *equivalent* to a category $\mathcal{D}$ if there exists an equivalence $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. A category $\mathcal{C}$ is said to be *dual* to a category $\mathcal{D}$ if there exists a duality $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$.

Problem 4.3. (1) Show that the dual category $\mathcal{C}^{op}$ is dual to the category $\mathcal{C}$.
(2) Let $\mathcal{D}$ be a category dual to the category $\mathcal{C}$. Show that $\mathcal{D}$ is equivalent to the dual category $\mathcal{C}^{op}$.
(3) Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be an equivalence with respect to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$, $\varphi : \mathcal{G}\mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$, and $\psi : \mathcal{F}\mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$. Show that $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ is an equivalence. Show that $\mathcal{G}$ is uniquely determined by $\mathcal{F}$ up to a natural isomorphism.

5. Representable and Adjoint Functors, the Yoneda Lemma

5.1. Representable functors.

Definition 5.1. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Set}$ be a covariant functor. A pair (A, x) with $A \in \mathcal{C}, x \in \mathcal{F}(A)$ is called a *representing (generic, universal) object* for $\mathcal{F}$ and $\mathcal{F}$ is called a *representable functor*, if for each $B \in \mathcal{C}$ and $y \in \mathcal{F}(B)$ there exists a unique $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ such that $\mathcal{F}(f)(x) = y$:

$$\begin{tikzcd}
A \arrow[d, "f"'] & \mathcal{F}(A) \ni x \arrow[d, "\mathcal{F}(f)"'] & x \arrow[d, mapsto] \\
B & \mathcal{F}(B) \ni y & y
\end{tikzcd}$$

Proposition 5.2. *Let (A, x) and (B, y) be representing objects for $\mathcal{F}$. Then there exists a unique isomorphism $h : A \longrightarrow B$ such that $\mathcal{F}(h)(x) = y$.*

$$\begin{tikzcd}
A \arrow[d, "h"] \arrow[dd, bend right=60, "1_A"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"'] \arrow[dd, bend left=60, "1_{\mathcal{F}(A)}"] & x \arrow[d, mapsto] \\
B \arrow[d, "k"] \arrow[dd, bend right=60, "1_B"'] & \mathcal{F}(B) \arrow[d, "\mathcal{F}(k)"'] \arrow[dd, bend left=60, "1_{\mathcal{F}(B)}"] & y \arrow[d, mapsto] \\
A \arrow[d, "h"] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"'] & x \arrow[d, mapsto] \\
B & \mathcal{F}(B) & y
\end{tikzcd}$$

Examples 5.3. (1) Let R be a ring. Let $X \in \mathrm{Set}$ be a set. $\mathcal{F} : R\text{-Mod} \longrightarrow \mathrm{Set}$, $\mathcal{F}(M) := \mathrm{Map}(X, M)$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the free R-module $(RX, x : X \longrightarrow RX)$ with the property, that for all $(M, y : X \longrightarrow M)$ there exists a unique $f \in \mathrm{Hom}_R(RX, M)$ such that $\mathcal{F}(f)(x) = \mathrm{Map}(X, f)(x) = fx = y$

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[dr, "y"'] & RX \arrow[d, "f"] \\
& M.
\end{tikzcd}$$

(2) Given modules M_R and ${}_RN$. Define $\mathcal{F} : \mathrm{Ab} \longrightarrow \mathrm{Set}$ by $\mathcal{F}(A) := \mathrm{Bil}_R(M, N; A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the tensor product $(M \otimes_R N, \otimes : M \times N \longrightarrow M \otimes_R N)$ with the property that for all $(A, f : M \times N \longrightarrow A)$ there exists a unique $g \in \mathrm{Hom}(M \otimes_R N, A)$ such that $\mathcal{F}(g)(\otimes) = \mathrm{Bil}_R(M, N; g)(\otimes) = g\otimes = f$

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "g"] \\
& A.
\end{tikzcd}$$

(3) Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbb{K}\text{-Alg} \longrightarrow \text{Set}$ by $\mathcal{F}(A) := \text{Hom}(V, A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the tensor algebra $(T(V), \iota : V \longrightarrow T(V))$ with the property that for all $(A, f : V \longrightarrow A)$ there exists a unique $g \in \text{Mor}_{\text{Alg}}(T(V), A)$ such that $\mathcal{F}(g)(\iota) = \text{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

(4) Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbb{K}\text{-cAlg} \longrightarrow \text{Set}$ by $\mathcal{F}(A) := \text{Hom}(V, A)$. Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the symmetric algebra $(S(V), \iota : V \longrightarrow S(V))$ with the property that for all $(A, f : V \longrightarrow A)$ there exists a unique $g \in \text{Mor}_{\text{cAlg}}(S(V), A)$ such that $\mathcal{F}(g)(\iota) = \text{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

(5) Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbb{K}\text{-Alg} \longrightarrow \text{Set}$ by

$$\mathcal{F}(A) := \{f \in \text{Hom}(V, A) | \forall v, v' \in V : f(v)f(v') = f(v')f(v)\}.$$

Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the symmetric algebra $(S(V), \iota : V \longrightarrow S(V))$ with the property that for all $(A, f : V \longrightarrow A)$ such that $f(v)f(v') = f(v')f(v)$ for all $v, v' \in V$ there exists a unique $g \in \text{Mor}_{\text{Alg}}(S(V), A)$ such that $\mathcal{F}(g)(\iota) = \text{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

(6) Given a $\mathbb{K}$-module V. Define $\mathcal{F} : \mathbb{K}\text{-Alg} \longrightarrow \text{Set}$ by

$$\mathcal{F}(A) := \{f \in \text{Hom}(V, A) | \forall v \in V : f(v)^2 = 0\}.$$

Then $\mathcal{F}$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the exterior algebra $(E(V), \iota : V \longrightarrow E(V))$ with the property that for all $(A, f : V \longrightarrow A)$ such that $f(v)^2 = 0$ for all $v \in V$ there exists a unique $g \in \text{Mor}_{\text{Alg}}(E(V), A)$ such that $\mathcal{F}(g)(\iota) = \text{Hom}(V, g)(\iota) = g\iota = f$

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & E(V) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

(7) Let $\mathbb{K}$ be a commutative ring. Let $X \in \text{Set}$ be a set. $\mathcal{F} : \mathbb{K}\text{-cAlg} \longrightarrow \text{Set}$, $\mathcal{F}(A) := \text{Map}(X, A)$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the polynomial ring $(\mathbb{K}[X], \iota : X \longrightarrow \mathbb{K}[X])$ with the property, that for all $(A, f : X \longrightarrow A)$ there exists a

unique $g \in \mathrm{Mor}_{\mathrm{cAlg}}(\mathbb{K}[X], A)$ such that $\mathcal{F}(g)(\iota) = \mathrm{Map}(X, g)(x) = g\iota = f$

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}[X] \arrow[d, "g"] \\ & A. \end{tikzcd}$$

(8) Let $\mathbb{K}$ be a commutative ring. Let $X \in \mathrm{Set}$ be a set. $\mathcal{F} : \mathbb{K}\text{-Alg} \longrightarrow \mathrm{Set}$, $\mathcal{F}(A) := \mathrm{Map}(X, A)$ is a covariant functor. A representing object for $\mathcal{F}$ is given by the noncommutative polynomial ring $(\mathbb{K}\langle X\rangle, \iota : X \longrightarrow \mathbb{K}\langle X\rangle)$ with the property, that for all $(A, f : X \longrightarrow A)$ there exists a unique $g \in \mathrm{Mor}_{\mathrm{Alg}}(\mathbb{K}\langle X\rangle, A)$ such that $\mathcal{F}(g)(\iota) = \mathrm{Map}(X, g)(x) = g\iota = f$

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}\langle X\rangle \arrow[d, "g"] \\ & A. \end{tikzcd}$$

Problem 5.1. (1) Given $V \in \mathbb{K}\text{-Mod}$. For $A \in \mathbb{K}\text{-Alg}$ define

$$F(A) := \{f : V \longrightarrow A | f\ \mathbb{K}\text{-linear}, \forall v, w \in V : f(v) \cdot f(w) = 0\}.$$

Show that this defines a functor $F : \mathbb{K}\text{-Alg} \longrightarrow \mathrm{Set}$.

(2) Show that F has the algebra $D(V)$ as constructed in Exercise 2.1 (3) as a representing object.

Proposition 5.4. *$\mathcal{F}$ has a representing object (A, a) if and only if there is a natural isomorphism $\varphi : \mathcal{F} \cong \mathrm{Mor}_{\mathcal{C}}(A, -)$ (with $a = \varphi(A)^{-1}(1_A)$).*

Proof. $\Longrightarrow$: The map

$$\varphi(B) : \mathcal{F}(B) \ni y \mapsto f \in \mathrm{Mor}_{\mathcal{C}}(A, B) \text{ with } \mathcal{F}(f)(a) = y$$

is bijective with the inverse map

$$\psi(B) : \mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathcal{F}(f)(a) \in \mathcal{F}(B).$$

In fact we have $y \mapsto f \mapsto \mathcal{F}(f)(a) = y$ and $f \mapsto y := \mathcal{F}(f)(a) \mapsto g$ such that $\mathcal{F}(g)(a) = y$ but then $\mathcal{F}(g)(a) = y = \mathcal{F}(f)(a)$. By uniqueness we get $f = g$. Hence all $\varphi(B)$ are bijective with inverse map $\psi(B)$. It is sufficient to show that ψ is a natural transformation.
Given $g : B \longrightarrow C$. Then the following diagram commutes

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\psi(B)"] \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(A,g)"'] & \mathcal{F}(B) \arrow[d, "\mathcal{F}(g)"] \\ \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[r, "\psi(C)"'] & \mathcal{F}(C) \end{tikzcd}$$

since $\psi(C)\,\mathrm{Mor}_{\mathcal{C}}(A, g)(f) = \psi(C)(gf) = \mathcal{F}(gf)(a) = \mathcal{F}(g)\mathcal{F}(f)(a) = \mathcal{F}(g)\psi(B)(f)$.
$\Leftarrow$: Let A be given. Let $a := \varphi(A)^{-1}(1_A)$. For $y \in \mathcal{F}(B)$ we get $y = \varphi(B)^{-1}(f) = \varphi(B)^{-1}(f1_A) = \varphi(B)^{-1}\,\mathrm{Mor}_{\mathcal{C}}(A, f)(1_A) = \mathcal{F}(f)\varphi(A)^{-1}(1_A) = \mathcal{F}(f)(a)$ for a uniquely determined $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$. □

Proposition 5.5. *Let $\mathcal{D}$ be a category. Given a representable functor $\mathcal{F}_X : \mathcal{C} \longrightarrow \mathrm{Set}$ for each $X \in \mathcal{D}$. Given a natural transformation $\mathcal{F}_g : \mathcal{F}_Y \longrightarrow \mathcal{F}_X$ for each $g : X \longrightarrow Y$ (contravariant!) such that $\mathcal{F}$ depends functorially on X, i.e. $\mathcal{F}_{1_X} = 1_{\mathcal{F}_X}, \mathcal{F}_{hg} = \mathcal{F}_g\mathcal{F}_h$. Then the representing objects (A_X, a_X) for $\mathcal{F}_X$ depend functorially on X, i.e. for each $g : X \longrightarrow Y$ there is a unique* *morphism* $A_g : A_X \longrightarrow A_Y$ *(with* $\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_g(A_Y)(a_Y)$*) and the following identities hold* $A_{1_X} = 1_{A_X}, A_{hg} = A_h A_g$*. So we get a covariant functor* $\mathcal{D} \ni X \longrightarrow A_X \in \mathcal{C}$*.*

Proof. Choose a representing object (A_X, a_X) for $\mathcal{F}_X$ for each $X \in \mathcal{D}$ (by the axiom of choice). Then there is a unique morphism $A_g : A_X \longrightarrow A_Y$ with

$$\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_g(A_Y)(a_Y) \in \mathcal{F}_X(A_Y),$$

for each $g : X \longrightarrow Y$ because $\mathcal{F}_g(A_Y) : \mathcal{F}_Y(A_Y) \longrightarrow \mathcal{F}_X(A_Y)$ is given. We have $\mathcal{F}_X(A_1)(a_X) = \mathcal{F}_1(A_X)(a_X) = a_X = \mathcal{F}_X(1)(a_X)$ hence $A_1 = 1$, and $\mathcal{F}_X(A_{hg})(a_X) = \mathcal{F}_{hg}(A_Z)(a_Z) = \mathcal{F}_g(A_Z)\mathcal{F}_h(A_Z)(a_Z) = \mathcal{F}_g(A_Z)\mathcal{F}_Y(A_h)(a_Y) = \mathcal{F}_X(A_h)\mathcal{F}_g(A_Y)(a_Y) = \mathcal{F}_X(A_h)\mathcal{F}_X(A_g)(a_X) = \mathcal{F}_X(A_h A_g)(a_X)$ hence $A_h A_g = A_{hg}$ for $g : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ in $\mathcal{D}$. $\square$

Corollary 5.6. (1) $\mathrm{Map}(X, M) \cong \mathrm{Hom}_R(RX, M)$ *is a natural transformation in* M *(and in* X*!). In particular* $\mathrm{Set} \ni X \mapsto RX \in R\text{-Mod}$ *is a functor.*
(2) $\mathrm{Bil}_R(M, N; A) \cong \mathrm{Hom}(M \otimes_R N, A)$ *is a natural transformation in* A *(and in* $(M, N) \in \text{Mod-}R \times R\text{-Mod}$*). In particular* $\text{Mod-}R \times R\text{-Mod} \ni M, N \mapsto M \otimes_r N \in \mathrm{Ab}$ *is a functor.*
(3) $R\text{-Mod-}S \times S\text{-Mod-}T \ni (M, N) \mapsto M \otimes_S N \in R\text{-Mod-}T$ *is a functor.*

5.2. **The Yoneda Lemma.**

Theorem 5.7. *(Yoneda Lemma) Let* $\mathcal{C}$ *be a category. Given a covariant functor* $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Set}$ *and an object* $A \in \mathcal{C}$*. Then the map*

$$\pi : \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}) \ni \phi \mapsto \phi(A)(1_A) \in \mathcal{F}(A)$$

is bijective with the inverse map

$$\pi^{-1} : \mathcal{F}(A) \ni a \mapsto h^a \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}),$$

where $h^a(B)(f) = \mathcal{F}(f)(a)$*.*

Proof. For $\phi \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F})$ we have a map $\phi(A) : \mathrm{Mor}_{\mathcal{C}}(A, A) \longrightarrow \mathcal{F}(A)$, hence π with $\pi(\phi) := \phi(A)(1_A)$ is a well defined map. For π^{-1} we have to check that h^a is a natural transformation. Given $f : B \longrightarrow C$ in $\mathcal{C}$. Then the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(A, B) \arrow[r, "\mathrm{Mor}(A,f)"] \arrow[d, "h^a(B)"'] & \mathrm{Mor}_{\mathcal{C}}(A, C) \arrow[d, "h^a(C)"] \\
\mathcal{F}(B) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(C)
\end{tikzcd}$$

is commutative. In fact if $g \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ then $h^a(C)\,\mathrm{Mor}_{\mathcal{C}}(A, f)(g) = h^a(C)(fg) = \mathcal{F}(fg)(a) = \mathcal{F}(f)\mathcal{F}(g)(a) = \mathcal{F}(f)h^a(B)(g)$. Thus π^{-1} is well defined.
Let $\pi^{-1}(a) = h^a$. Then $\pi\pi^{-1}(a) = h^a(A)(1_A) = \mathcal{F}(1_A)(a) = a$. Let $\pi(\phi) = \phi(A)(1_A) = a$. Then $\pi^{-1}\pi(\phi) = h^a$ and $h^a(B)(f) = \mathcal{F}(f)(a) = \mathcal{F}(f)(\phi(A)(1_A)) = \phi(B)\,\mathrm{Mor}_{\mathcal{C}}(A, f)(1_A) = \phi(B)(f)$, hence $h^a = \phi$. $\square$

Corollary 5.8. *Given* $A, B \in \mathcal{C}$*. Then the following hold*
(1) $\mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathrm{Mor}_{\mathcal{C}}(f, \text{-}) \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B, \text{-}), \mathrm{Mor}_{\mathcal{C}}(A, \text{-}))$ *is a bijective map.*
(2) *Under the bijective map from (1) the isomorphisms in* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *correspond to natural isomorphisms in* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B, \text{-}), \mathrm{Mor}_{\mathcal{C}}(A, \text{-}))$.
(3) *For contravariant functors* $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Set}$ *we have* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathcal{F}) \cong \mathcal{F}(A)$*.*
(4) $\mathrm{Mor}_{\mathcal{C}}(A, B) \ni f \mapsto \mathrm{Mor}_{\mathcal{C}}(\text{-}, f) \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathrm{Mor}_{\mathcal{C}}(\text{-}, B))$ *is a bijective map that defines a one-to-one correspondence between the isomorphisms in* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *and the natural isomorphisms in* $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, A), \mathrm{Mor}_{\mathcal{C}}(\text{-}, B))$.

Proof. (1) follows from the Yoneda Lemma with $\mathcal{F} = \mathrm{Mor}_{\mathcal{C}}(A, \text{-})$.
(2) Observe that $h^f(C)(g) = \mathrm{Mor}_{\mathcal{C}}(A, g)(f) = gf = \mathrm{Mor}_{\mathcal{C}}(f, C)(g)$ hence $h^f = \mathrm{Mor}_{\mathcal{C}}(f, \text{-})$. Since we have $\mathrm{Mor}_{\mathcal{C}}(f, \text{-}) \mathrm{Mor}_{\mathcal{C}}(g, \text{-}) = \mathrm{Mor}_{\mathcal{C}}(gf, \text{-})$ and $\mathrm{Mor}_{\mathcal{C}}(f, \text{-}) = \mathrm{id}_{\mathrm{Mor}_{\mathcal{C}}(A,\text{-})}$ if and only if $f = 1_A$ we get the one-to-one correspondence between the isomorphisms from (1).
(3) and (4) follow by dualizing. □

Remark 5.9. The map π is a natural transformation in the arguments A and $\mathcal{F}$. More precisely: if $f : A \longrightarrow B$ and $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ are given then the following diagrams commute

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Nat}(\mathrm{Mor}(A,\text{-}),\phi)"'] & \mathcal{F}(A) \arrow[d, "\phi(A)"] \\
\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{G}) \arrow[r, "\pi"'] & \mathcal{G}(A)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \mathcal{F}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Nat}(\mathrm{Mor}(f,\text{-}),\mathcal{F})"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"] \\
\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(B, \text{-}), \mathcal{F}) \arrow[r, "\pi"'] & \mathcal{F}(B).
\end{tikzcd}$$

This can be easily checked. Indeed we have for $\psi : \mathrm{Mor}_{\mathcal{C}}(A, \text{-}) \longrightarrow \mathcal{F}$

$$\pi \, \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(A, \text{-}), \phi)(\psi) = \pi(\phi\psi) = (\phi\psi)(A)(1_A) = \phi(A)\psi(A)(1_A) = \phi(A)\pi(\psi)$$

and

$$\begin{aligned}
&\pi \, \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(f, \text{-}), \mathcal{F})(\psi) = \pi(\psi \, \mathrm{Mor}_{\mathcal{C}}(f, \text{-})) = (\psi \, \mathrm{Mor}_{\mathcal{C}}(f, \text{-}))(B)(1_B) = \psi(B)(f) \\
&= \psi(B) \, \mathrm{Mor}_{\mathcal{C}}(A, f)(1_A) = \mathcal{F}(f)\psi(A)(1_A) = \mathcal{F}(f)\pi(\psi).
\end{aligned}$$

Remark 5.10. By the previous corollary the representing object A is uniquely determined up to isomorphism by the isomorphism class of the functor $\mathrm{Mor}_{\mathcal{C}}(A, \text{-})$.

Proposition 5.11. *Let $\mathcal{G} : \mathcal{C} \times \mathcal{D} \longrightarrow \mathrm{Set}$ be a covariant bifunctor such that the functor $\mathcal{G}(C, \text{-}) : \mathcal{D} \longrightarrow \mathrm{Set}$ is representable for all $C \in \mathcal{C}$. Then there exists a contravariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathcal{G} \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})$ holds. Furthermore $\mathcal{F}$ is uniquely determined by $\mathcal{G}$ up to isomorphism.*

Proof. For each $C \in \mathcal{C}$ choose an object $\mathcal{F}(C) \in \mathcal{D}$ and an isomorphism $\xi_C : \mathcal{G}(C, \text{-}) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), \text{-})$. Given $f : C \longrightarrow C'$ in $\mathcal{C}$ then let $\mathcal{F}(f) : \mathcal{F}(C') \longrightarrow \mathcal{F}(C)$ be the uniquely determined morphism (by the Yoneda Lemma) in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd}
\mathcal{G}(C, \text{-}) \arrow[r, "\xi_C"] \arrow[d, "\mathcal{G}(f,\text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), \text{-}) \arrow[d, "\mathrm{Mor}(\mathcal{F}(f),\text{-})"] \\
\mathcal{G}(C', \text{-}) \arrow[r, "\xi_{C'}"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C'), \text{-})
\end{tikzcd}$$

commutes. Because of the uniqueness of $\mathcal{F}(f)$ and because of the functoriality of $\mathcal{G}$ it is easy to see that $\mathcal{F}(fg) = \mathcal{F}(g)\mathcal{F}(f)$ and $\mathcal{F}(1_C) = 1_{\mathcal{F}(C)}$ hold and that $\mathcal{F}$ is a contravariant functor.
If $\mathcal{F}' : \mathcal{C} \longrightarrow \mathcal{D}$ is given with $\mathcal{G} \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-})$ then $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-})$. Hence by the Yoneda Lemma $\psi(C) : \mathcal{F}(C) \cong \mathcal{F}'(C)$ is an isomorphism for all $C \in \mathcal{C}$. With these isomorphisms induced by ϕ the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'(C), \text{-}) \arrow[r, "\mathrm{Mor}(\psi(C),\text{-})"] \arrow[d, "\mathrm{Mor}(\mathcal{F}'(f),\text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), \text{-}) \arrow[d, "\mathrm{Mor}(\mathcal{F}(f),\text{-})"] \\
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'(C'), \text{-}) \arrow[r, "\mathrm{Mor}(\psi(C'),\text{-})"'] & \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C'), \text{-})
\end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd}
\mathcal{F}(C') \arrow[r, "\psi(C')"] \arrow[d, "\mathcal{F}'(f)"'] & \mathcal{F}'(C') \arrow[d, "\mathcal{F}(f)"] \\
\mathcal{F}(C) \arrow[r, "\psi(C)"'] & \mathcal{F}'(C)
\end{tikzcd}$$

commutes. Thus $\psi : \mathcal{F} \longrightarrow \mathcal{F}'$ is a natural isomorphism. $\square$

5.3. **Adjoint functors.**

Definition 5.12. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be covariant functors. $\mathcal{F}$ is called *left adjoint* to $\mathcal{G}$ and $\mathcal{G}$ *right adjoint* to $\mathcal{F}$ if there is a natural isomorphism of bifunctors $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})$ from $\mathcal{C}^{op} \times \mathcal{D}$ to Set.

Lemma 5.13. *If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is left adjoint to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ then $\mathcal{F}$ is uniquely determined by $\mathcal{G}$ up to isomorphism. Similarly $\mathcal{G}$ is uniquely determined by $\mathcal{F}$ up to isomorphism.*

Proof. We only prove the first claim. Assume that also $\mathcal{F}'$ is left adjoint to $\mathcal{G}$ with $\phi' : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})$. Then we have a natural isomorphism $\phi'^{-1}\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}'\text{-}, \text{-})$. By Proposition 5.11 we get $\mathcal{F} \cong \mathcal{F}'$. $\square$

Lemma 5.14. *A functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint functor iff all functors $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}\text{-})$ are representable.*

Proof. follows from 5.11. $\square$

Lemma 5.15. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be covariant functors. Then*

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathcal{GF}) \ni \Phi \mapsto \mathcal{G}\text{-}\Phi\text{-} \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}), \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}))$$

is a bijective map with inverse map

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}), \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})) \ni \phi \mapsto \phi(\text{-}, \mathcal{F}\text{-})(1_{\mathcal{F}\text{-}}) \in \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathcal{GF}).$$

Furthermore

$$\mathrm{Nat}(\mathcal{FG}, \mathrm{Id}_{\mathcal{C}}) \ni \Psi \mapsto \Psi\text{-}\mathcal{F}\text{-} \in \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}), \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}))$$

is a bijective map with inverse map

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}), \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})) \ni \psi \mapsto \psi(\mathcal{G}\text{-}, \text{-})(1_{\mathcal{G}\text{-}}) \in \mathrm{Nat}(\mathcal{FG}, \mathrm{Id}_{\mathcal{C}}).$$

Proof. The natural transformation $\mathcal{G}\text{-}\Phi\text{-}$ is defined as follows. Given $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D)$ then let $(\mathcal{G}\text{-}\Phi\text{-})(C, D)(f) := \mathcal{G}(f)\Phi(C) : C \longrightarrow \mathcal{GF}(C) \longrightarrow \mathcal{G}(D)$. It is easy to check the properties of a natural transformation.
Given Φ then one obtains by applying the two maps $\mathcal{G}(1_{\mathcal{F}(C)})\Phi(C) = \mathcal{GF}(1_C)\Phi(C) = \Phi(C)$.
Given ϕ one obtains

$$\begin{aligned}
&\mathcal{G}(f)(\phi(C, \mathcal{F}(C))(1_{\mathcal{F}(C)}) = \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(f))\phi(C, \mathcal{F}(C))(1_{\mathcal{F}(C)}) \\
&= \phi(C, D) \,\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), f)(1_{\mathcal{F}(C)}) = \phi(C, D)(f).
\end{aligned}$$

So the two maps are inverses of each other.
The second part of the lemma is proved similarly. $\square$

Proposition 5.16. *Let*

$$\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \quad \textit{and} \quad \psi : \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})$$

be natural transformations with associated natural transformations (by Lemma 5.15) $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{GF}$ *resp.* $\Psi : \mathcal{FG} \longrightarrow \mathrm{Id}_{\mathcal{D}}$.

(1) *Then we have* $\phi\psi = \mathrm{id}_{\mathrm{Mor}(\text{-},\mathcal{G}\text{-})}$ *if and only if* $(\mathcal{G} \xrightarrow{\Phi\mathcal{G}} \mathcal{GFG} \xrightarrow{\mathcal{G}\Psi} \mathcal{G}) = \mathrm{id}_{\mathcal{G}}$.

(2) *Furthermore we have* $\psi\phi = \mathrm{id}_{\mathrm{Mor}(\mathcal{F}\text{-},\text{-})}$ *if and only if* $(\mathcal{F} \xrightarrow{\mathcal{F}\Phi} \mathcal{FGF} \xrightarrow{\Psi\mathcal{F}} \mathcal{F}) = \mathrm{id}_{\mathcal{F}}$.

Proof. We get

$$\begin{aligned}
&\mathcal{G}\Psi(D)\Phi\mathcal{G}(D) = \mathcal{G}\Psi(D)\phi(\mathcal{G}(D), \mathcal{FG}(D))(1_{\mathcal{FG}(D)})\\
&= \mathrm{Mor}_{\mathcal{C}}(\mathcal{G}(D), \mathcal{G}\Psi(D))\phi(\mathcal{G}(D), \mathcal{FG}(D))(1_{\mathcal{FG}(D)})\\
&= \phi(\mathcal{G}(D), D)\,\mathrm{Mor}_{\mathcal{D}}(\mathcal{FG}(D), \Psi(D))(1_{\mathcal{FG}(D)})\\
&= \phi(\mathcal{G}(D), D)(\Psi(D))\\
&= \phi(\mathcal{G}(D), D)\psi(\mathcal{G}(D), D)(1_{\mathcal{G}(D)})\\
&= \phi\psi(\mathcal{G}(D), D)(1_{\mathcal{G}(D)}).
\end{aligned}$$

Similarly we get

$$\begin{aligned}
&\phi\psi(C, D)(f) = \phi(C, D)\psi(C, D)(f) = \mathcal{G}(\Psi(D)\mathcal{F}(f))\Phi(C)\\
&= \mathcal{G}\Psi(D)\mathcal{GF}(f)\Phi(C) = \mathcal{G}\Psi(D)\Phi\mathcal{G}(D)f.
\end{aligned}$$ $\square$

Corollary 5.17. *Let* $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *and* $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ *be functors.* $\mathcal{F}$ *is left adjoint to* $\mathcal{G}$ *if and only if there are natural transformations* $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{GF}$ *and* $\Psi : \mathcal{FG} \longrightarrow \mathrm{Id}_{\mathcal{D}}$ *such that* $(\mathcal{G}\Psi)(\Phi\mathcal{G}) = \mathrm{id}_{\mathcal{G}}$ *and* $(\Psi\mathcal{F})(\mathcal{F}\Phi) = \mathrm{id}_{\mathcal{F}}$.

Definition 5.18. The natural transformations $\Phi : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{GF}$ and $\Psi : \mathcal{FG} \longrightarrow \mathrm{Id}_{\mathcal{D}}$ given in 5.17 are called *unit* and *counit* resp. for the adjoint functors $\mathcal{F}$ and $\mathcal{G}$.

Problem 5.2. (1) Let ${}_RM_S$ be a bimodule. Show that the functor $M \otimes_S \text{-} : {}_S\mathcal{M} \longrightarrow {}_R\mathcal{M}$ is left adjoint to $\mathrm{Hom}_R(M, \text{-}) : {}_R\mathcal{M} \longrightarrow {}_S\mathcal{M}$. Determine the associated unit and counit.
(2) Show that there is a natural isomorphism $\mathrm{Map}(A \times B, C) \cong \mathrm{Map}(B, \mathrm{Map}(A, C))$. Determine the associated unit and counit.
(3) Show that there is a natural isomorphism $\mathbb{K}\text{-Alg}(\mathbb{K}G, A) \cong \mathrm{Gr}(G, U(A))$ where $U(A)$ is the group of units of the algebra A and $\mathbb{K}G$ is the group ring (see Section 12). Determine the associated unit and counit.
(4) Use Section 12 to show that there is a natural isomorphism

$$\mathbb{K}\text{-Alg}(U(\mathfrak{g}), A) \cong \text{Lie-Alg}(\mathfrak{g}, A^L).$$

Determine the corresponding left adjoint functor and the associated unit and counit.

5.4. **Universal problems.**

Definition 5.19. Let $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ be a covariant functor. $\mathcal{G}$ generates a *(co-)universal problem* a follows:
Given $C \in \mathcal{C}$. Find an object $\mathcal{F}(C) \in \mathcal{D}$ and a morphism $\iota : C \longrightarrow \mathcal{G}(\mathcal{F}(C))$ in $\mathcal{C}$ such that for each object $D \in \mathcal{D}$ and for each morphism $f : C \longrightarrow \mathcal{G}(D)$ in $\mathcal{C}$ there is a unique morphism $g : \mathcal{F}(C) \longrightarrow D$ in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathcal{G}(\mathcal{F}(C)) \arrow[d, "\mathcal{G}(g)"] \\
& \mathcal{G}(D)
\end{tikzcd}$$

commutes.
A pair $(\mathcal{F}(C), \iota)$ that satisfies the above conditions is called a *universal solution* of the (co-)universal problem defined by $\mathcal{G}$ and C.
Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor. $\mathcal{F}$ generates a *universal problem* a follows:
Given $D \in \mathcal{D}$. Find an object $\mathcal{G}(D) \in \mathcal{C}$ and a morphism $\nu : \mathcal{F}(\mathcal{G}(D)) \longrightarrow D$ in $\mathcal{D}$ such that for each object $C \in \mathcal{C}$ and for each morphism $f : \mathcal{F}(C) \longrightarrow D$ in $\mathcal{D}$ there is a unique morphism $g : C \longrightarrow \mathcal{G}(D)$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\mathcal{F}(C) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "f"] & \\
\mathcal{F}\mathcal{G}(D) \arrow[r, "\nu"'] & D
\end{tikzcd}$$

commutes.
A pair $(\mathcal{G}(D), \nu)$ that satisfies the above conditions is called a *universal solution* of the universal problem defined by $\mathcal{F}$ and D.

Proposition 5.20. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be left adjoint to $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$. Then $\mathcal{F}(C)$ and the unit $\iota = \Phi(C) : C \longrightarrow \mathcal{G}\mathcal{F}(C)$ form a universal solution for the (co-)universal problem defined by $\mathcal{G}$ and C.*
Furthermore $\mathcal{G}(D)$ and the counit $\nu = \Psi(D) : \mathcal{F}\mathcal{G}(D) \longrightarrow D$ form a universal solution for the universal problem defined by $\mathcal{F}$ and D.

Proof. By Theorem 5.16 the morphisms $\phi : \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-})$ and $\psi : \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\text{-}) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}\text{-}, \text{-})$ are inverses of each other. Using unit and counit they are defined as $\phi(C, D)(g) = \mathcal{G}(g)\Phi(C)$ resp. $\psi(C, D)(f) = \Psi(D)\mathcal{F}(f)$. Hence for each $f : C \longrightarrow \mathcal{G}(D)$ there is a unique $g : \mathcal{F}(C) \longrightarrow D$ such that $f = \phi(C, D)(g) = \mathcal{G}(g)\Phi(C) = \mathcal{G}(g)\iota$.
The second statement follows analogously. $\square$

Remark 5.21. If $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ and $C \in \mathcal{C}$ are given then the universal solution $(\mathcal{F}(C), \iota : C \longrightarrow \mathcal{G}(D))$ can be considered as the best (co-)approximation of the object C in $\mathcal{C}$ by an object D in $\mathcal{D}$ with the help of a functor $\mathcal{G}$. The object $D \in \mathcal{D}$ turns out to be $\mathcal{F}(C)$.
If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $D \in \mathcal{D}$ are given then the universal solution $(\mathcal{G}(D), \nu : \mathcal{F}\mathcal{G}(D) \longrightarrow D)$ can be considered as the best approximation of the object D in $\mathcal{D}$ by an object C in $\mathcal{C}$ with the help of a functor $\mathcal{F}$. The object $C \in \mathcal{C}$ turns out to be $\mathcal{G}(D)$.

Proposition 5.22. *Given $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$. Assume that for each $C \in \mathcal{C}$ the universal problem defined by $\mathcal{G}$ and C has a universal solution. Then there is a left adjoint functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ to $\mathcal{G}$.*
Given $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. Assume that for each $D \in \mathcal{D}$ the universal problem defined by $\mathcal{F}$ and D has a universal solution. Then there is a right adjoint functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ to $\mathcal{F}$.

Proof. Assume that the (co-)universal problem defined by $\mathcal{G}$ and C is solved by $\iota : C \longrightarrow \mathcal{G}\mathcal{F}(C)$. Then the map $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D)) \ni f \mapsto g \in \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D)$ with $\mathcal{G}(g)\iota = f$ is bijective. The inverse map is given by $g \mapsto \mathcal{G}(g)\iota$. This is a natural transformation since the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D) \arrow[r, "\mathcal{G}(\text{-})\iota"] \arrow[d, "\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C),h)"'] & \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D)) \arrow[d, "\mathrm{Mor}_{\mathcal{C}}(C,\mathcal{G}(h))"] \\
\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(C), D') \arrow[r, "\mathcal{G}(\text{-})\iota"'] & \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(D'))
\end{tikzcd}$$

commutes for each $h \in \mathrm{Mor}_D(D, D')$. In fact we have

$$\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(h))(\mathcal{G}(g)\iota) = \mathcal{G}(h)\mathcal{G}(g)\iota = \mathcal{G}(hg)\iota = \mathcal{G}(\mathrm{Mor}_{\mathcal{C}}(\mathcal{F}(C), h)(g))\iota.$$

Hence for all $C \in \mathcal{C}$ the functor $\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{G}(\text{-})) : \mathcal{D} \longrightarrow \mathrm{Set}$ induced by the bifunctor $\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\text{-})) : \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathrm{Set}$ is representable. By Theorem 5.11 there is a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\text{-})) \cong \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(\text{-}), \text{-})$.
The second statement follows analogously. □

Remark 5.23. One can characterize the properties that $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ (resp. $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$) must have in order to possess a left (right) adjoint functor. One of the essential properties for this is that $\mathcal{G}$ preserves limits (and thus preserves direct products and difference kernels).

Proposition 5.24. *The construction of tensor algebras $T(V)$ defines a functor $T : \mathbb{K}\text{-Mod} \longrightarrow \mathbb{K}\text{-Alg}$ that is left adjoint to the underlying functor $U : \mathbb{K}\text{-Alg} \longrightarrow \mathbb{K}\text{-Mod}$.*

Proof. Follows from the universal property and 5.22. □

Proposition 5.25. *The construction of symmetric algebras $S(V)$ defines a functor $S : \mathbb{K}\text{-Mod} \longrightarrow \mathbb{K}\text{-cAlg}$ that is left adjoint to the underlying functor $U : \mathbb{K}\text{-cAlg} \longrightarrow \mathbb{K}\text{-Mod}$.*

Proof. Follows from the universal property and 5.22. □

6. Limits and Colimits, Products and Equalizers

6.1. **Limits of diagrams.** Limit constructions are a very important tool in category theory. We will introduce the basic facts on limits and colimits in this section.

Definition 6.1. A *diagram scheme* $\mathcal{D}$ is a small category (i. e. the class of objects is a set). Let $\mathcal{C}$ be an arbitrary category. A *diagram* in $\mathcal{C}$ over the diagram scheme $\mathcal{D}$ is a covariant functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$.

Example 6.2. (for diagram schemes)

(1) The empty category $\mathcal{D}$.
(2) The category with precisely one object D and precisely one morphism 1_D.
(3) The category with two objects D_1, D_2 and one morphism $f : D_1 \longrightarrow D_2$ (apart from the two identities).
(4) The category with two objects D_1, D_2 and two morphisms $f, g : D_1 \longrightarrow D_2$ between them.
(5) The category with a family of objects $(D_i | i \in I)$ and the associated identities.
(6) The category with four objects $D_1, \ldots, D_4$ and morphisms f, g, h, k such that the diagram

$$\begin{tikzcd} D_1 \arrow[r, "f"] \arrow[d, "g"'] & D_2 \arrow[d, "k"] \\ D_3 \arrow[r, "h"'] & D_4 \end{tikzcd}$$

commutes, i. e. $kf = hg$.

Definition 6.3. Let $\mathcal{D}$ be a diagram scheme and $\mathcal{C}$ a category. Each object $C \in \mathcal{C}$ defines a *constant diagram* $\mathcal{K}_C : \mathcal{D} \longrightarrow \mathcal{C}$ with $\mathcal{K}_C(D) := C$ for all $D \in \mathcal{D}$ and $\mathcal{K}(f) := 1_C$ for all morphisms in $\mathcal{D}$. Each morphism $f : C \longrightarrow C'$ in $\mathcal{C}$ defines a *constant natural transformation* $\mathcal{K}_f : \mathcal{K}_C \longrightarrow \mathcal{K}_{C'}$ with $\mathcal{K}_f(D) = f$. This defines a *constant functor* $\mathcal{K} : \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{D}, \mathcal{C})$ from the category $\mathcal{C}$ into the category of diagrams $\mathrm{Funct}(\mathcal{D}, \mathcal{C})$.
Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram. An object C together with a natural transformation $\pi : \mathcal{K}_C \longrightarrow \mathcal{F}$ is called a *limit* or a *projective limit* of the diagram $\mathcal{F}$ with the *projection* π if for each object $C' \in \mathcal{C}$ and for each natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ there is a unique morphism $f : C' \longrightarrow C$ such that

$$\begin{tikzcd} \mathcal{K}_{C'} \arrow[d, "\mathcal{K}_f"'] \arrow[dr, "\varphi"] & \\ \mathcal{K}_C \arrow[r, "\pi"'] & \mathcal{F} \end{tikzcd}$$

commutes, this means in particular that the diagrams

$$\begin{tikzcd} C \arrow[r, "\pi_i"] \arrow[dr, "\pi_j"'] & \mathcal{F}(D_i) \arrow[d, "\mathcal{F}(g)"] \\ & \mathcal{F}(D_j) \end{tikzcd}$$

commute for all morphisms $g : D_i \longrightarrow D_j$ in $\mathcal{D}$ (π is a natural transformation) and the diagrams

$$\begin{tikzcd} C' \arrow[d, "f"'] \arrow[dr, "\varphi_i"] & \\ C \arrow[r, "\pi_i"'] & \mathcal{F}(D_i). \end{tikzcd}$$

commute for all objects D_i in $\mathcal{D}$.
A category $\mathcal{C}$ *has limits for diagrams over a diagram scheme* $\mathcal{D}$ if for each diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ over $\mathcal{D}$ there is a limit in $\mathcal{C}$. A category $\mathcal{C}$ is called *complete* if each diagram in $\mathcal{C}$ has a limit.

Example 6.4. (1) Let $\mathcal{D}$ be a diagram scheme consisting of two objects D_1, D_2 and the identities. A diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is defined by giving two objects C_1 and C_2 in $\mathcal{C}$. An object $C_1 \times C_2$ together with two morphisms $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ and $\pi_2 : C_1 \times C_2 \longrightarrow C_2$ is called a *product* of the two objects if $C_1 \times C_2, \pi : \mathcal{K}_{C_1 \times C_2} \longrightarrow \mathcal{F}$ is a limit, i. e. if for each object C' in $\mathcal{C}$ and for any two morphisms $\varphi_1 : C' \longrightarrow C_1$ and $\varphi_2 : C' \longrightarrow C_2$ there is a unique morphism $f : C' \longrightarrow C_1 \times C_2$ such that

$$\begin{tikzcd} & C' \arrow[dl, "\varphi_1"'] \arrow[d, "f"] \arrow[dr, "\varphi_2"] & \\ C_1 & C_1 \times C_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C_2 \end{tikzcd}$$

commutes. The two morphisms $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ and $\pi_2 : C_1 \times C_2 \longrightarrow C_2$ are called the *projections* from the product to the two factors.
(2) Let $\mathcal{D}$ a diagram scheme consisting of a finite (non empty) set of objects $D_1, \ldots, D_n$ and the associated identities. A limit of a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is called a *finite product* of the objects $C_1 := \mathcal{F}(D_1), \ldots, C_n := \mathcal{F}(D_n)$ and is denoted by $C_1 \times \ldots \times C_n = \prod_{i=1}^n C_i$.
(3) A limit over a discrete diagram (i. e. $\mathcal{D}$ has only the identities as morphisms) is called *product* of the $C_i := \mathcal{F}(D_i)$, $i \in I$ and is denoted by $\prod_I C_i$.
(4) Let $\mathcal{D}$ be the empty diagram scheme and $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ the (only possible) empty diagram. The limit $C, \pi : \mathcal{K}_C \longrightarrow \mathcal{F}$ of $\mathcal{F}$ is called the *final object*. It has the property that for each object C' in $\mathcal{C}$ (the uniquely determined natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ does not have to be mentioned) there is a unique morphism $f : C' \longrightarrow C$. In Set the one-point set is a final object. In Ab, Gr, Vec the zero group 0 is a final object.
(5) Let $\mathcal{D}$ be the diagram scheme from 6.2 (4) with two objects D_1, D_2 and two morphisms (different from the two identities) $a, b : D_1 \longrightarrow D_2$. A diagram over $\mathcal{D}$ consists of two objects C_1 and C_2 and two morphisms $g, h : C_1 \longrightarrow C_2$. The limit of such a diagram is called *equalizer* of the two morphisms and is given by an object $\mathrm{Ke}(g, h)$ and a morphism $\pi_1 : \mathrm{Ke}(g, h) \longrightarrow C_1$. The second morphism to C_2 arises from the composition $\pi_2 = g\pi_1 = h\pi_1$. The equalizer has the following universal property. For each object C' and each morphism $\varphi_1 : C' \longrightarrow C_1$ with $g\varphi_1 = h\varphi_1 (= \varphi_2)$ there is a unique morphism $f : C' \longrightarrow \mathrm{Ke}(g, h)$ with $\pi_1 f = \varphi_1$ (and thus $\pi_2 f = \varphi_2$), i. e. the diagram

$$\begin{tikzcd} & C' \arrow[dl, "f"'] \arrow[d, "\varphi_1"] & \\ \mathrm{Ke}(g,h) \arrow[r, "\pi_1"] & C_1 \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C_2 \end{tikzcd}$$

commutes.

Problem 6.1. (1) Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$ be a discrete diagram. Show that the Cartesian product over $\mathcal{F}$ coincides with the categorical product.
(2) Let $\mathcal{D}$ be a pair of morphisms as in 6.4 (5) and let $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$ be a diagram. Show that the set $\{x \in \mathcal{F}(D_1) | \mathcal{F}(f)(x) = \mathcal{F}(g)(x)\}$ with the inclusion map into $\mathcal{F}(D_1)$ is an equalizer of $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$.
(3) Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$ be a diagram. Show that the set

$$\{(x_D | D \in \mathrm{Ob}\, D, x_D \in \mathcal{F}(D)) | \forall (f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)(x_D) = x_{D'}\}$$

with the projections into the single components of the families is the limit of $\mathcal{F}$.
(4) Given a homomorphism $f : M \longrightarrow N$ in R-Mod. Show that $(K, \iota : K \longrightarrow M)$ is a kernel of f iff it is the equalizer of the pair of homomorphisms $f, 0 : M \longrightarrow N$ iff the sequence

$$0 \longrightarrow K \stackrel{\iota}{\longrightarrow} M \stackrel{f}{\longrightarrow} N$$

is exact.

6.2. **Colimits of diagrams.**

Definition 6.5. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram. An object C and a natural transformation $\iota : \mathcal{F} \longrightarrow \mathcal{K}_C$ is called *colimit* or *inductive limit* of the diagram $\mathcal{F}$ with the *injection* ι if for each object $C' \in \mathcal{C}$ and for each natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{K}_{C'}$ there is a unique morphism $f : C \longrightarrow C'$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathcal{K}_C \arrow[d, "\mathcal{K}_f"] \\ & \mathcal{K}_{C'} \end{tikzcd}$$

commutes, i. e. the diagram

$$\begin{tikzcd} \mathcal{F}(D_i) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "\iota_i"] & \\ \mathcal{F}(D_j) \arrow[r, "\iota_j"'] & C \end{tikzcd}$$

commutes for all morphisms $g : D_i \longrightarrow D_j$ in $\mathcal{D}$ (ι is a natural transformation) and the diagram

$$\begin{tikzcd} \mathcal{F}(D_i) \arrow[r, "\iota_i"] \arrow[dr, "\varphi_i"'] & C \arrow[d, "f"] \\ & C'. \end{tikzcd}$$

commutes for all objects D_i in $\mathcal{D}$.
The special colimits that can be formed over the diagrams as in Example 6.4 are called *coproduct, initial object,* resp. *coequalizer.*

Example 6.6. In $\mathbb{K}$-Vec the object 0 is an initial object. In $\mathbb{K}$-Alg the object $\mathbb{K}$ is an initial object. In $\mathbb{K}$-Alg the object $\{a \in A | f(a) = g(a)\}$ is the equalizer of the two algebra homomorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B$. In $\mathbb{K}$-Alg the Cartesian (set of pairs) and the categorical products coincide.

Remark 6.7. A colimit of a diagram $\mathcal{C}$ is a limit of the corresponding (dual) diagram in the dual category $\mathcal{C}^{op}$. Thus theorems about limits in arbitrary categories automatically also produce (dual) theorems about colimits. However, observe that theorems about limits in a particular category (for example the category of vector spaces) translate only into theorems about colimits in the *dual* category, which most often is not too useful.

Proposition 6.8. *Limits and colimits of diagrams are unique up to isomorphism.*

Proof. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram and let C, π and $\tilde{C}, \tilde{\pi}$ be limits of $\mathcal{F}$. Then there are unique morphisms $f : \tilde{C} \longrightarrow C$ and $g : C \longrightarrow \tilde{C}$ with $\pi\mathcal{K}_f = \tilde{\pi}$ and $\tilde{\pi}\mathcal{K}_g = \pi$. This implies $\pi\mathcal{K}_{1_C} = \pi\, \mathrm{id}_{\mathcal{K}_C} = \pi = \tilde{\pi}\mathcal{K}_g = \pi\mathcal{K}_f\mathcal{K}_g = \pi\mathcal{K}_{fg}$ and analogously $\tilde{\pi}\mathcal{K}_{1_{\tilde{C}}} = \tilde{\pi}\mathcal{K}_{gf}$. Because of the uniqueness this implies $1_C = fg$ and $1_{\tilde{C}} = gf$. □

Remark 6.9. Now that we have the uniqueness of the limit resp. colimit (up to isomorphism) we can introduce a unified notation. The limit of a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ will be denoted by $\varprojlim(\mathcal{F})$, the colimit by $\varinjlim(\mathcal{F})$.

Problem 6.2. Given a homomorphism $f : M \longrightarrow N$ in R-Mod. Show that $(Q, \nu : N \longrightarrow Q)$ is a cokernel of f iff it is the coequalizer of the pair of homomorphisms $f, 0 : M \longrightarrow N$ iff the sequence

$$M \xrightarrow{f} N \xrightarrow{\nu} Q \longrightarrow 0$$

is exact.

6.3. **Completeness.**

Theorem 6.10. *If $\mathcal{C}$ has arbitrary products and equalizers then $\mathcal{C}$ has arbitrary limits. In this case we say that $\mathcal{C}$ is* complete.

Proof. Let $\mathcal{D}$ be a diagram scheme and $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ a diagram. First we form the products $\prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ and $\prod_{f \in \mathrm{Mor}\,\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$ where $\mathrm{Codom}(f)$ is the codomain (range) of the morphism $f : D' \longrightarrow D''$ in $\mathcal{D}$ so in this case $\mathrm{Codom}(f) = D''$. We define for each morphism $f : D' \longrightarrow D''$ two morphisms as follows

$$p_f := \pi_{\mathcal{F}(D'')} : \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \mathcal{F}(D'') = \mathcal{F}(\mathrm{Codom}(f))$$

and

$$q_f := \mathcal{F}(f)\pi_{\mathcal{F}(D')} : \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \mathcal{F}(D') \longrightarrow \mathcal{F}(D'') = \mathcal{F}(\mathrm{Codom}(f)).$$

These two families of morphisms induce two morphisms into the corresponding product

$$p, q : \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \prod_{f \in \mathrm{Mor}\,\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$$

with $\pi_f q = q_f$ and $\pi_f p = p_f$. Now we show that the equalizer of these two morphisms

$$\begin{tikzcd} \mathrm{Ke}(p,q) \arrow[r, "\psi"] & \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{f \in \mathrm{Mor}\,\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f)) \end{tikzcd}$$

is the limit of the diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$. We have $p\psi = q\psi$. The morphism $\rho(D) := \pi_{\mathcal{F}(D)}\psi :$ $\mathrm{Ke}(p,q) \longrightarrow \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$ defines a family of morphisms for $D \in \mathrm{Ob}\,\mathcal{D}$. If $f : D'$ $\longrightarrow D''$ is in $\mathcal{D}$ then the diagram

$$\begin{tikzcd} \mathrm{Ke}(p,q) \arrow[r, "\rho(D')"] \arrow[dr, "\rho(D'')"'] & \mathcal{F}(D') \arrow[d, "\mathcal{F}(f)"] \\ & \mathcal{F}(D'') \end{tikzcd}$$

is commutative because of $\mathcal{F}(f)\rho(D') = \mathcal{F}(f)\pi_{\mathcal{F}(D')}\psi = q_f\psi = \pi_f q\psi = \pi_f p\psi = p_f\psi = \pi_{\mathcal{F}(D'')}\psi = \rho(D'')$. Thus we have obtained a natural transformation $\rho : \mathcal{K}_{\mathrm{Ke}(p,q)} \longrightarrow \mathcal{F}$.
Now let an object C' and a natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ be given. Then this defines a unique morphism $g : C' \longrightarrow \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ with $\pi_{\mathcal{F}(D)}g = \varphi(D)$ for all $D \in \mathcal{D}$. Since φ is a natural transformation we have $\varphi(D'') = \mathcal{F}(f)\varphi(D')$ for each morphism $f : D' \longrightarrow D''$. Thus we obtain $\pi_f pg = p_f g = \pi_{\mathcal{F}(D'')}g = \varphi(D'') = \mathcal{F}(f)\varphi(D') = \mathcal{F}(f)\pi_{\mathcal{F}(D')}g = q_f g = \pi_f qg$ for all morphisms $f : D' \longrightarrow D''$ hence $pg = qg$. Thus g can be uniquely factorized through the equalizer $\psi : \mathrm{Ke}(p, q) \longrightarrow \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ in the form $g = \psi h$ with $h : C' \longrightarrow \mathrm{Ke}(p, q)$. Then we have $\rho(D)h = \pi_{\mathcal{F}(D)}\psi h = \pi_{\mathcal{F}(D)}g = \varphi(D)$ for all $D \in \mathcal{D}$ hence $\rho\mathcal{K}_h = \varphi$.
Finally let another morphism $h' : C' \longrightarrow \mathrm{Ke}(p, q)$ with $\rho\mathcal{K}_{h'} = \varphi$ be given. Then we have $\pi_{\mathcal{F}(D)}\psi h' = \rho(D)h' = \varphi(D) = \rho(D)h = \pi_{\mathcal{F}(D)}\psi h$ hence $\psi h' = \psi h = g$. Because of the uniqueness of the factorization of g through ψ we get $h = h'$. Thus $(\mathrm{Ke}(p, q), \rho)$ is the limit of $\mathcal{F}$. □

Remark 6.11. The proof of the preceding Theorem gives an explicit construction of the limit of $\mathcal{F}$ as an equalizer

$$\mathrm{Ke}(p,q) \xrightarrow{\psi} \prod_{D\in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{f\in \mathrm{Mor}\,\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$$

Hence the limit can be represented as a subobject of a suitable product. Dually the colimit can be represented as a quotient object of a suitable coproduct.

6.4. **Adjoint functors and limits.** Another fact is very important for us, the fact that certain functors preserve limits resp. colimits. We say that a functor $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{C}'$ *preserves* limits over the diagram scheme $\mathcal{D}$ if $\varprojlim(\mathcal{G}\mathcal{F}) \cong \mathcal{G}(\varprojlim(\mathcal{F}))$ for each diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$.

Proposition 6.12. *Covariant representable functors preserve limits. Contravariant representable functors map colimits into limits.*

Proof. We only prove the first assertion. The second assertion is dual to the first one. For a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$ the set

$$\{(x_D|D \in \mathrm{Ob}\,\mathcal{D}, x_D \in \mathcal{F}(D))|\forall(f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)(x_D) = x_{D'}\}$$

is a limit of $\mathcal{F}$ by Problem 6.1 (3). Now let a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be given and let $\varprojlim(\mathcal{F})$ be the limit. Furthermore let $\mathrm{Mor}_{\mathcal{C}}(C', \text{-}) : \mathcal{C} \longrightarrow \mathrm{Set}$ be a representable functor. By the definition of the limit of $\mathcal{F}$ there is a unique morphism $f : C' \longrightarrow \varprojlim(\mathcal{F})$ with $\pi\mathcal{K}_f = \varphi$ for each natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$. This defines an isomorphism $\mathrm{Nat}(\mathcal{K}_{C'}, \mathcal{F}) \cong \mathrm{Mor}_{\mathcal{C}}(C', \varprojlim(\mathcal{F}))$. Hence we have

$$\begin{aligned}&\varprojlim(\mathrm{Mor}_{\mathcal{C}}(C', \mathcal{F})) \cong\\ &\{(\varphi(D) : C' \longrightarrow \mathcal{F}(D)|D \in \mathcal{D})|\forall(f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)\varphi(D) = \varphi(D')\}\\ &= \mathrm{Nat}(\mathcal{K}_{C'}, \mathcal{F}) \cong \mathrm{Mor}_{\mathcal{C}}(C', \varprojlim(\mathcal{F})). \quad \square\end{aligned}$$

Corollary 6.13. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}'$ be left adjoint to $\mathcal{G} : \mathcal{C}' \longrightarrow \mathcal{C}$. Then $\mathcal{F}$ preserves colimits and $\mathcal{G}$ preserves limits.*

Proof. For a diagram $\mathcal{H} : \mathcal{D} \longrightarrow \mathcal{C}$ we have

$$\begin{gathered}\mathrm{Mor}_{\mathcal{C}}(\text{-}, \varprojlim(\mathcal{G}\mathcal{H})) \cong \varprojlim \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\mathcal{H}) \cong \varprojlim \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}\text{-}, \mathcal{H}) \cong\\ \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}\text{-}, \varprojlim(\mathcal{H})) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\varprojlim(\mathcal{H}))),\end{gathered}$$

hence $\varprojlim(\mathcal{G}\mathcal{H}) \cong \mathcal{G}(\varprojlim(\mathcal{H}))$ as representing objects. The proof for the left adjoint functor is analogous. □

7. The Morita Theorems

Throughout this section let $\mathbb{K}$ be a commutative ring.

Definition 7.1. A category $\mathcal{C}$ is called a $\mathbb{K}$-*category*, if $\mathrm{Mor}_{\mathcal{C}}(M,N)$ is a $\mathbb{K}$-module and $\mathrm{Mor}_{\mathcal{C}}(f,g)$ is a homomorphism of $\mathbb{K}$-modules for all $M,\ N,\ f,\ g \in \mathcal{C}$.
A functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ between $\mathbb{K}$-categories $\mathcal{C}$ and $\mathcal{D}$ is called a $\mathbb{K}$-*functor*, if $\mathcal{F} : \mathrm{Mor}_{\mathcal{C}}(M,N) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(M),\mathcal{F}(N))$ for all $M,N \in \mathcal{C}$ is a homomorphism of $\mathbb{K}$-modules.
If $\mathbb{K} = \mathbb{Z}$, then $\mathbb{K}$-categories are called *(pre-)additive categories* and $\mathbb{K}$-functors are called *additive functors*.

Remark 7.2. In this section 7 we always write homomorphisms at the opposite side from where ring elements act on the modules: $f : {}_RM \longrightarrow {}_RN$ with $(rm)f = r(mf)$.

Let A and B be $\mathbb{K}$-algebras. Let ${}_AM$ be a left A-module. Then it is also a $\mathbb{K}$-module by $\kappa m := (\kappa \cdot 1_A)\cdot m$. Analogously a right B-module is also a $\mathbb{K}$-module. We redefine the notion of a bimodule as follows:

Definition 7.3. A $\mathbb{K}$-*bimodule* ${}_AM_B$ is an A-B-bimodule satisfying $(\kappa \cdot 1_A)\cdot m = \kappa m = m\kappa = m\cdot(1_B \cdot \kappa)$ i.e. the induced right and left structures of a $\mathbb{K}$-module coincide.

Definition 7.4. A *Morita context* consists of a 6-tuple $(A, B, {}_AP_B, {}_BQ_A, f, g)$ with $\mathbb{K}$-algebras A, B, $\mathbb{K}$-bimodules ${}_AP_B$, ${}_BQ_A$ and homomorphisms of $\mathbb{K}$-bimodules

$$f : {}_AP \otimes_B Q_A \longrightarrow {}_AA_A, \quad g : {}_BQ \otimes_A P_B \longrightarrow {}_BB_B,$$

such that:

(1) $qf(p\otimes q') = g(q\otimes p)q'$ oder $q(pq') = (qp)q'$,
(2) $f(p\otimes q)p' = pg(q\otimes p')$ oder $(pq)p' = p(qp')$,

where we will use the following notation $pq := f(p\otimes q)$ and $qp := g(q\otimes p)$.

Remark 7.5. With this convention all products are associative e.g. $(pb)q = p(bq)$, $(qa)p = q(ap)$.

Lemma 7.6. *Let A be a $\mathbb{K}$-algebra and ${}_AP$ be an A-module. Then* $(A,B,P,Q,f = \mathrm{ev}, g = \mathrm{db})$ *is a Morita context with*

$$\begin{array}{ll} B := \mathrm{Hom}_A(.P,.P) & {}_BQ_A := {}_B\mathrm{Hom}_A(.P_B,.A_A)_A \\ f(p\otimes q) := (p)q & (p')[g(q\otimes p)] := (p')qp. \end{array}$$

Proof. as in 3.18. □

Definition 7.7. A $\mathbb{K}$-*equivalence of* $\mathbb{K}$-*categories* $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of $\mathbb{K}$-functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$, $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ such that $\mathrm{Id}_{\mathcal{D}} \cong \mathcal{F}\mathcal{G}$ and $\mathrm{Id}_{\mathcal{C}} \cong \mathcal{G}\mathcal{F}$.

Theorem 7.8. (Morita I)
Let (A,B,P,Q,f,g) be a Morita context. Let f and g be epimorphisms. Then the following statements hold

(1) *P is a finitely generated projective generator in A-Mod and in Mod-B.*
Q is a finitely generated projective generator in Mod-A and in B-Mod.
(2) *f and g are isomorphisms.*
(3) $Q \cong \mathrm{Hom}_A(.P,.A) \cong \mathrm{Hom}_B(P.,B.)$
$P \cong \mathrm{Hom}_B(.Q,.B) \cong \mathrm{Hom}_A(Q.,A.)$
as bimodules.
(4) $A \cong \mathrm{Hom}_B(.Q,.Q) \cong \mathrm{Hom}_B(P.,P.)$
$B \cong \mathrm{Hom}_A(.P,.P) \cong \mathrm{Hom}_A(Q.,Q.)$
as $\mathbb{K}$-algebras and as bimodules.

(5) $P \otimes_B$ - : B-Mod $\longrightarrow$ A-Mod *and* $Q \otimes_A$ - : A-Mod $\longrightarrow$ B-Mod *are mutually inverse* $\mathbb{K}$*-equivalences. Symmetrically* - $\otimes_A P$: Mod-A $\longrightarrow$ Mod-B *and* - $\otimes_B Q$: Mod-B $\longrightarrow$ Mod-A *are mutually inverse* $\mathbb{K}$*-equivalences. Furthermore the following functors are naturally isomorphic:*

$$\begin{aligned} P \otimes_B \text{-} &\cong \mathrm{Hom}_B(.Q, .\text{-}), \\ Q \otimes_A \text{-} &\cong \mathrm{Hom}_A(.P, .\text{-}), \\ \text{-} \otimes_A P &\cong \mathrm{Hom}_A(Q., \text{-}.), \\ \text{-} \otimes_B Q &\cong \mathrm{Hom}_B(P., \text{-}.). \end{aligned}$$

(6) *We have the following isomorphisms of lattices (ordered sets):*

$$\begin{array}{rclcrcl} \mathcal{V}({}_AP) & \cong & \mathcal{V}({}_BB), & & \mathcal{V}(P_B) & \cong & \mathcal{V}(A_A), \\ \mathcal{V}({}_BQ) & \cong & \mathcal{V}({}_AA), & & \mathcal{V}(Q_A) & \cong & \mathcal{V}(B_B), \\ \mathcal{V}({}_BQ_A) & \cong & \mathcal{V}({}_AA_A) & \cong & \mathcal{V}({}_BB_B) & \cong & \mathcal{V}({}_AP_B). \end{array}$$

(7) *The following centers are isomorphic* $\mathrm{Cent}(A) \cong \mathrm{Cent}(B)$.

Proof. (1) The isomorphisms from Theorem 1.22 (5) map $g \in \mathrm{Hom}_{B\text{-}B}(.Q \otimes_A P., .B.)$ to homomorphisms of bimodules $g_1 : P \longrightarrow \mathrm{Hom}_B(.Q, .B)$ and $g_2 : Q \longrightarrow \mathrm{Hom}_B(P., B.)$. Furthermore f induces homomorphisms of bimodules $f_1 : P \longrightarrow \mathrm{Hom}_A(Q., A.)$ and $f_2 : Q \longrightarrow \mathrm{Hom}_A(.P, .A)$.
If g is an epimorphism then there is an element $\sum q_i \otimes p_i \in Q \otimes_A P$ with $g(\sum q_i \otimes p_i) = 1_B = \mathrm{id}_P$. Hence $p = \sum pq_ip_i = \sum (p)[f_2(q_i)]p_i$ for each $p \in P$. By the dual basis Lemma 3.19 ${}_AP$ is finitely generated and projective.
If f is an epimorphism then there is an element $\sum x_i \otimes y_i \in P \otimes_B Q$ with $f(\sum x_i \otimes y_i) = 1_A = \sum (x_i)[f_2(y_i)]$. By 3.24 ${}_AP$ is a generator. The claims for P_B, ${}_BQ$, and Q_A follow by symmetry.
(2) If $f(\sum a_i \otimes b_i) = 0$ then $\sum_i a_i \otimes b_i = \sum_{i,j} a_i \otimes b_i f(x_j \otimes y_j) = \sum a_i \otimes g(b_i \otimes x_j)y_j = \sum a_i g(b_i \otimes x_j) \otimes y_j = \sum f(a_i \otimes b_i)x_j \otimes y_j = 0$. Hence f is injective. By symmetry we get that g is an isomorphism.
(3) The homomorphism $f_2 : Q \longrightarrow \mathrm{Hom}_A(.P, .A)$ defined as in (1) satisfies $(p)[f_2(q)] = f(p \otimes q) = pq$. Let $\varphi \in \mathrm{Hom}_A(.P, .A)$. Then $(p)\varphi = (p \sum q_ip_i)\varphi = \sum (pq_i)(p_i)\varphi$ hence $\varphi = \sum q_i(p_i)\varphi = \sum f_2(q_i(p_i)\varphi)$. Thus f_2 is an epimorphism. Let $(p)[f_2(q)] = pq = 0$ for all $p \in P$. Then we get $q = 1_Bq = \sum q_ip_iq = 0$. Hence f_2 is an isomorphism.
(4) The structure of a B-module on P induces $B \longrightarrow \mathrm{Hom}_A(.P, .P)$. Let $pb = 0$ for all $p \in P$. Then $b = 1_B \cdot b = \sum q_ip_ib = 0$. If $\varphi \in \mathrm{Hom}_A(.P, .P)$ then we have $(p)\varphi = (p1_B)\varphi = (\sum p(q_ip_i))\varphi = \sum (pq_i)(p_i)\varphi = \sum p(q_i(p_i)\varphi)$ and thus $\varphi = \sum q_i(p_i)\varphi$. This shows that we have an isomorphism $B \longrightarrow \mathrm{Hom}_A(.P, .P)$ of $\mathbb{K}$-algebras and bimodules.
(5) ${}_AP \otimes_B Q \otimes_A X \cong {}_AA \otimes_A X \cong {}_AX$ is natural in X and ${}_BQ \otimes_A P \otimes_B Y \cong {}_BB \otimes_B Y \cong {}_BY$ is natural in Y thus we get the claim. Furthermore ${}_BQ \otimes_A U \cong {}_B\mathrm{Hom}_A(.P, .A) \otimes_A U \cong {}_B\mathrm{Hom}_A(.P, .A \otimes_A U) \cong {}_B\mathrm{Hom}_A(.P, .U)$ is natural in U since the homomorphism $\varphi : \mathrm{Hom}_A(.P, .A) \otimes_A U \longrightarrow \mathrm{Hom}_A(.P, .A \otimes_A U)$ with $(p)[\varphi(f \otimes u)] := ((p)f) \otimes u$ is an isomorphism. More generally we show:

Lemma 7.9. *If* ${}_AP$ *is finitely generated projective and* ${}_AV_B$ *and* ${}_BU$ *are (bi-)modules then the natural transformation (in* U *and* V*)*

$$\varphi : \mathrm{Hom}_A(.P, .V) \otimes_B U \longrightarrow \mathrm{Hom}_A(.P, .V \otimes_B U)$$

is an isomorphism.

Proof. Let $\sum f_i \otimes p_i \in \mathrm{Hom}_A(.P, .A) \otimes_A P$ be a dual basis for P. Then

$$\varphi^{-1} : \mathrm{Hom}_A(.P, .V \otimes_B U) \longrightarrow \mathrm{Hom}_A(.P, .V) \otimes_B U$$

defined by $\varphi^{-1}(g) = \sum_{i,j}()f_i v_{ij} \otimes u_{ij}$ with $(p_i)g =: \sum_j v_{ij} \otimes u_{ij}$ is inverse to φ defined by $(p)[\varphi(f \otimes u)] = (p)f \otimes u$. Since φ is a homomorphism $(p)[\varphi(fb \otimes u)] = (p)fb \otimes u = (p)f \otimes bu = (p)[\varphi(f \otimes bu)]$ it suffices to show that φ^{-1} is a map. Now we have $(p_i)\varphi(f \otimes u) = (p_i)f \otimes u$ hence $\varphi^{-1}\varphi(f \otimes u) = \sum()f_i(p_i)f \otimes u = \sum(()f_i p_i)f \otimes u = f \otimes u$. Furthermore we have $\varphi\varphi^{-1}(g) = \varphi(\sum()f_i(p_i)g) = \sum()f_i(p_i)g = (\sum()f_i p_i)g = g$. $\square$

Proof of 7.8: (continued)
(6) Under the equivalence of categories ${}_AP$ is mapped to $\mathrm{Hom}_A(.P, .P) \cong {}_BB$. This implies $\mathcal{V}({}_AP) \cong \mathcal{V}({}_BB)$. In fact, a submodule of ${}_AP$ is an isomorphism class of monomorphisms ${}_AU \longrightarrow {}_AP$, two such isomorphisms being called isomorphic, if there is a (necessarily unique) isomorphism $U \cong U'$, such that

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ U' \arrow[r] & P \end{tikzcd}$$

commutes. Obviously such subobjects are being preserved under an equivalence of categories. For subobjects of ${}_AP_B$ we have furthermore that

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\cdot b"'] & P \arrow[d, "\cdot b"] \\ U \arrow[r] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Hom}_A(.P, .U) \arrow[r] \arrow[d, "\cdot b"'] & B \arrow[d, "\cdot b"] \\ \mathrm{Hom}_A(.P, .U) \arrow[r] & B \end{tikzcd}$$

commute. Hence ${}_AU_B \in \mathcal{V}({}_AP_B)$ iff $\mathrm{Hom}_A(.P, .U) \in \mathcal{V}({}_BB_B)$.
(7) The proof of this part will consist of two steps. We use the algebra $\mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}})$ of natural endomorphisms of $\mathrm{Id}_{A\text{-}\mathrm{Mod}}$ with the addition of morphisms and the composition of morphisms as the operations of the algebra. Obviously this defines an algebra.
In a first step we show that the center of A is isomorphic to $\mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}})$. In a second step we show that $\mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}}) \cong \mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{B\text{-}\mathrm{Mod}})$. This last step is almost trivial since all terms defined by categorical means are preserved by an equivalence. Then we have $\mathrm{Cent}(A) \cong \mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}}) \cong \mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{B\text{-}\mathrm{Mod}}) \cong \mathrm{Cent}(B)$.
Let $z \in Z(A)$. For $M = \mathrm{Id}_{A\text{-}\mathrm{Mod}}(M)$ we have $zam = azm$ hence $z = z\cdot \in \mathrm{End}_A(M)$. Thus $z\cdot$ defines an endomorphism of $\mathrm{Id}_{A\text{-}\mathrm{Mod}}(M)$, for

$$\begin{tikzcd} M \arrow[r, "z\cdot"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "z\cdot"'] & N \end{tikzcd}$$

commutes. So we have defined a homomorphism $\mathrm{Cent}(A) \longrightarrow \mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}})$. Let $\varphi \in \mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}})$. Then the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi(A)"] \arrow[d, "f_m"'] & A \arrow[d, "f_m"] \\ M \arrow[r, "\varphi(M)"'] & M \end{tikzcd}$$

commutes, where $(a)f_m = am$. Each $f \in \mathrm{Hom}_A(.A, .M)$ is of this form. For $M = A$ we have $a(1)[\varphi(A)] = (a)[\varphi(A)] = (1)[f_a\varphi(A)] = (1)[\varphi(A)f_a] = (1)[\varphi(A)]a$ hence $(1)[\varphi(A)] \in Z(A)$. For an arbitrary $M \in A\text{-Mod}$ we then have $(m)[\varphi(M)] = (1)[f_m\varphi(M)] = (1)[\varphi(A)f_m] = (1)[\varphi(A)]m$ i.e. $\varphi(M)$ is of the form $z\cdot$ with $z = (1)[\varphi(A)]$. The maps defined in this way obviously are inverses of each other: $z \mapsto z\cdot \mapsto z \cdot 1 = z$ and $\varphi \mapsto (1)[\varphi(A)] \mapsto (1)[\varphi(A)]\cdot$.
In order to show that $\mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}})$ and $\mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{B\text{-}\mathrm{Mod}})$ are isomorphic, let $\varphi \in \mathrm{End}_{\mathrm{funkt}}(\mathrm{Id}_{A\text{-}\mathrm{Mod}}) =: E(A)$. We define $\varphi' \in E(B)$ by

$$\begin{tikzcd}
{}_BM \arrow[r, "\varphi'(M)"] \arrow[d, "\beta(M)"'] & {}_BM \arrow[d, "\beta(M)"] \\
ST({}_BM) \arrow[r, "S\varphi T(M)"'] & ST({}_BM)
\end{tikzcd}$$

where $S : A\text{-Mod} \longrightarrow B\text{-Mod}$, $T : B\text{-Mod} \longrightarrow A\text{-Mod}$ are the mutually inverse equivalences from (5), and $\alpha : \mathrm{Id}_{A\text{-}\mathrm{Mod}} \longrightarrow TS$ and $\beta : \mathrm{Id}_{B\text{-}\mathrm{Mod}} \longrightarrow ST$ resp. are the associated isomorphisms. Analogously we associate with each $\psi \in E(B)$ an element $\psi' \in E(A)$ by

$$\begin{tikzcd}
{}_AN \arrow[r, "\psi'(N)"] \arrow[d, "\alpha(N)"'] & {}_AN \arrow[d, "\alpha(N)"] \\
TS({}_AN) \arrow[r, "T\psi S(N)"'] & TS({}_AN)
\end{tikzcd} .$$

The compositions of $\psi \mapsto \psi'$ and $\varphi \mapsto \varphi'$ in each direction define isomorphisms, hence each single map is an isomorphism. One of the two compositions is contained in the following diagram.

$$\begin{tikzcd}
N \arrow[r, "\varphi''(N)"] \arrow[d, "\alpha(N)"'] & N \arrow[d, "\alpha(N)"] \\
TS(N) \arrow[r, "T\varphi' S(N)"] \arrow[d, "T\beta S(N)"'] & TS(N) \arrow[d, "T\beta S(N)"] \\
T(ST)S(N) \arrow[r, "TS\varphi TS(N)"] & T(ST)S(N) \\
TS(N) \arrow[r, "TS\varphi(N)"] \arrow[u, "TS\alpha(N)"] & TS(N) \arrow[u, "TS\alpha(N)"'] \\
N \arrow[r, "\varphi(N)"] \arrow[u, "\alpha(N)"] & N \arrow[u, "\alpha(N)"']
\end{tikzcd}$$

Thus the map $\varphi \mapsto \varphi''$ is an inner automorphism of $E(A)$, hence it is bijective. $\square$

Theorem 7.10. (Morita II)
Let $S : A\text{-Mod} \longrightarrow B\text{-Mod}$ and $T : B\text{-Mod} \longrightarrow A\text{-Mod}$ be mutually inverse $\mathbb{K}$-equivalences. Let ${}_AP_B := T(B)$ and ${}_BQ_A := S(A)$. Then there are isomorphisms $f : {}_AP \otimes_B Q_A \longrightarrow {}_AA_A$ and $g : {}_BQ \otimes_A P_B \longrightarrow {}_BB_B$, such that (A, B, P, Q, f, g) is a Morita context.
Furthermore the following hold $S \cong Q \otimes_A$ - and $T \cong P \otimes_B$ -.

Theorem 7.11. (Morita III)
Let $P \in A$-Mod *be a finitely generated projective generator* (= progenerator). *Then the Morita context* $(A, \mathrm{Hom}_A(.P, .P), P, Q, f = \mathrm{ev}, g = \mathrm{db})$ *is strict, i.e.* f *and* g *are epimorphisms.*

Proof. Since ${}_AP$ is finitely generated projective, $g = \mathrm{db}$ is an isomorphism (3.19). Since ${}_AP$ is a generator, $f = \mathrm{ev}$ is an epimorphism (3.24). $\square$

Proof of 7.10: 1. Given S, T. Then $S : \mathrm{Hom}_A(.M, .N) \ni f \mapsto S(f) \in \mathrm{Hom}_B(.SM, .SN)$ is an isomorphism. Let $\alpha : TS \cong \mathrm{Id}_{A\text{-}\mathrm{Mod}}$. Then

$$\mathrm{Hom}_A(.M, .N) \xrightarrow{S} \mathrm{Hom}_B(.SM, .SN) \xrightarrow{T} \mathrm{Hom}_A(.TSM, .TSN) \xrightarrow{\mathrm{Hom}(\alpha^{-1},\alpha)} \mathrm{Hom}_A(.M, .N)$$

is the identity, since $\mathrm{Hom}(\alpha^{-1}, \alpha)TS(f) = \alpha \circ TSf \circ \alpha^{-1} = f$. This holds since

$$\begin{tikzcd} TSM \arrow[r, "TDf"] \arrow[d, "\alpha"'] & TSN \arrow[d, "\alpha"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commutes. So S is a monomorphism and $\mathrm{Hom}(\alpha^{-1}, \alpha) \circ T$ is an epimorphism. Since $\mathrm{Hom}(\alpha^{-1}, \alpha)$ is an isomorphism, T is an epimorphism where $T : \mathrm{Hom}_B(.SM, .SN) \longrightarrow \mathrm{Hom}_A(.TSM, .TSN)$. By symmetry T is a monomorphism. Hence T is an isomorphism in the above map. Thus S is an isomorphism.
2. $\mathrm{Hom}_B(.SM, .N) \xrightarrow{T} \mathrm{Hom}_A(.TSM, .TN) \xrightarrow{\mathrm{Hom}(\alpha^{-1},\mathrm{id})} \mathrm{Hom}_A(.M, .TN)$ is a natural isomorphism. It is clear that this is an isomorphism. Since T is a functor, the first map is a natural transformation. The second map is a natural transformation, since α is a natural transformation. In particular, S is left adjoint to T.
3. $S(\oplus_{i\in I} M_i) \cong \oplus_{i\in I} S(M_i)$, since S is a left adjoint functor and thus preserves direct coproducts.
4. If $f \in B$-Mod is an epimorphism, then $Tf \in A$-Mod is an epimorphism, too. In fact, let $f : M \longrightarrow N$ be an epimorphism. Let $g, h \in A$-Mod be given with $g \circ Tf = h \circ Tf$. Then we have a commutative diagram

$$\begin{tikzcd} STM \arrow[r, "STf"] \arrow[d, "\beta"'] & STN \arrow[r, shift left, "Sg"] \arrow[r, shift right, "Sh"'] \arrow[d, "\beta"] & SM \\ M \arrow[r, "f"] & N & \end{tikzcd}$$

with $Sg \circ STf = Sh \circ STf$. Since f is an epimorphism this implies $Sg = Sh$, hence $g = h$.
5. If $P \in A$-Mod is projective, then $SP \in B$-Mod is projective. In fact given an epimorphism $f : M \longrightarrow N$ in B-Mod and a homomorphism $g : SP \longrightarrow N$. Then $Tf : TM \longrightarrow TN$ is an epimorphism and $Tg : TSP \longrightarrow TN$ is in A-Mod. Since $\alpha : TSP \cong P$, there is an $h : P \longrightarrow TM$ with $Tf \circ h = Tg \circ \alpha^{-1}$ or $Tf \circ h \circ \alpha = Tg$. We apply S and get $STf \circ S(h \circ \alpha) = STg$, where $S(h \circ \alpha) \in \mathrm{Hom}_B(.STSP, .STM)$. Since $\beta : STM \cong M$, we have an isomorphism $\mathrm{Hom}(\beta^{-1}, \beta) : \mathrm{Hom}_B(.STSP, .STM) \longrightarrow \mathrm{Hom}_B(.SP, .M)$ with inverse $\mathrm{Hom}(\beta, \beta^{-1})$. For $k : SP \longrightarrow M$ with $k = \beta \circ S(h \circ \alpha) \circ \beta^{-1}$ we then have $\beta \circ ST(k) = k \circ \beta = \beta \circ S(h \circ \alpha) \circ \beta^{-1} \circ \beta = \beta \circ S(h \circ \alpha)$, hence $ST(k) = S(h \circ \alpha)$ and $T(k) = h \circ \alpha$. So we get $STf \circ STk = STg = ST(f \circ k)$ and thus $g = f \circ k$. So SP is projective.

6. SA is finitely generated as a B-module: Since SA is projective, we have $SA \oplus X \cong \bigoplus_{i \in I} B$. By (3) applied to T we get $A \oplus TX \cong TSA \oplus TX \cong \bigoplus_{i \in I} TB$. Since A is finitely generated, the image of A in $\bigoplus_{i \in I} TB$ is already a direct summand in a finite direct subsum $\bigoplus_{i \in E} TB$, so $A \oplus Y \cong \bigoplus_{i \in E} TB$. Hence $SA \oplus SY \cong \bigoplus_{i \in E} STB \cong \bigoplus_{i \in E} B$ and thus SA is finitely generated.
7. If $G \in A$-Mod is a generator then $SG \in B$-Mod is also a generator. In fact let $(f : M \longrightarrow N) \neq 0$ in B-Mod. Then $Tf \neq 0$, hence there is a $g : G \longrightarrow TM$ with $Tf \circ g \neq 0$. Consequently $STf \circ Sg \neq 0$ and $f \circ (\alpha \circ Sg) = \alpha \circ STf \circ Sg \neq 0$.
8. This shows that $S(A)$ is a finitely generated projective generator.
(Remark: An equivalence S always maps finitely generated modules to finitely generated modules. We will give the proof further down in Proposition 7.12.)
9. $A \cong \mathrm{Hom}_B(.SA, .SA)$ as algebras, since $A \cong \mathrm{Hom}_A(.A, .A) \xrightarrow{S} \mathrm{Hom}_B(.SA, .SA)$.
10. $TB \cong \mathrm{Hom}_B(.SA, .B)$, since $\mathrm{Hom}_B(.SA, .B) \xrightarrow{T} \mathrm{Hom}_A(.TSA, .TB) \cong \mathrm{Hom}_A(.A, .TB) \cong TB$.
11. (B, A, SA, TB, f, g) defines a strict Morita context by Morita III.
12. The functor S is isomorphic to $SA \otimes_A -$. Infact we have $\mathrm{Hom}_B(.SA \otimes_A M, .N) \cong \mathrm{Hom}_A(.M, .\mathrm{Hom}_B(.SA, .N))$

$$\begin{aligned} &\cong \mathrm{Hom}_A(.M, .\mathrm{Hom}_A(.A, .TN)) \\ &\cong \mathrm{Hom}_A(.M, .TN) \\ &\cong \mathrm{Hom}_B(.SM, .N). \end{aligned}$$

The representing object ${}_BSM \cong {}_BSA \otimes_A M$ depends functorially on M by 5.5. □

Proposition 7.12. *${}_AM$ is finitely generated iff in each set of submodules $\{A_i | i \in I\}$ with $A_i \subseteq M$ and $\sum_{i \in I} A_i = M$ there is a finite subset $\{A_i | i \in I_0\}$ ($I_0 \subseteq I$ finite) such that $\sum_{i \in I_0} A_i = M$.*

Proof. Let $M = Am_1 + \ldots + Am_n$. Each m_j is contained in a finite sum of the A_i, hence all of the m_j and hence the module M itself. Conversely consider $\{Am | m \in M\}$. Then $M = \sum Am$, hence M is a sum of finitely many of the Am and thus is finitely generated. □

Corollary 7.13. *Under an equivalence of categories $T : A$-Mod $\longrightarrow B$-Mod finitely generated modules are mapped into finitely generated modules.*

Proof. The lattice of submodules $\mathcal{V}(M)$ is isomorphic to the lattice of submodules $\mathcal{V}(TM)$. □

Problem 7.1. Let A-Mod be equivalent to B-Mod. Show that Mod-A and Mod-B are also equivalent.

Problem 7.2. Show that an equivalence of arbitrary categories preserves monomorphisms.

Problem 7.3. Show that an equivalence of module categories preserves projective modules, but not free modules.

8. Simple and Semisimple rings and Modules

8.1. **Simple and Semisimple rings.**

Definition 8.1. An ideal ${}_RI \subseteq {}_RR$ is called *nilpotent*, if there is $n \geq 1$ such that $I^n = 0$.
A module ${}_RM$ is called *Artinian* (Emil Artin, 1898-1962), if each non empty set of submodules of M contains a minimal element.
A module ${}_RM$ is called *Noetherian* (Emmy Noether, 1882-1935), if each non empty set of submodules of M contains a maximal element.
A ring R is called *simple*, if ${}_RR$ as a module is Artinian and if R does not have non trivial ($\neq 0, R$) two sided ideals.
A ring R is called *semisimple*, if ${}_RR$ is Artinian and if R does not have non trivial ($\neq 0$) nilpotent left ideals.

Lemma 8.2. *Each simple ring is semisimple.*

Proof. $C := \sum(I|{}_RI \subseteq {}_RR$ nilpotent) is a two sided ideal. In fact take $a \in I$ and $r \in R$. Then

$$(r_1ar)(r_2ar)\ldots(r_nar) = (r_1a)(rr_2a)\ldots(rr_na)r \in I^nR = 0.$$

Hence we have $(Rar)^n = 0 \Longrightarrow Rar \subseteq C$, so $ar \in C$ and C is a two sided ideal. Thus $C = 0$ or $C = R$. If $C = 0$ then there are no non trivial nilpotent ideals. If $C = R$ then there are ideals and elements $a_i \in I_i$ such that $1 = a_1 + \ldots + a_n$. The ideal $I_1 + I_2$ is nilpotent since $(a_1 + b_1)(a_2 + b_2)\ldots(a_{2n} + b_{2n})$ consists of monomials either in $I_1^n \cdot R$ or in $I_2^n \cdot R$. But $I_1^n = 0 = I_2^n \Longrightarrow (I_1 + I_2)^{2n} = 0$. Hence 1 is nilpotent. Contradiction. □

Definition 8.3. A module ${}_RM$ is called *simple* iff $M \neq 0$ and M has only the modules 0 and M as submodules. An ideal ${}_RI$ is called *simple* or *minimal*, if it is simple as a module.

Lemma 8.4. *Let R be semisimple. Then each left ideal of R is a direct summand of R.*

Proof. Let I be an ideal in R, that is not a direct summand, and let I be minimal with respect to this property. Such an ideal exists, since R Artinian.
Case 1: Let $I \subseteq R$ be an ideal that is not minimal (simple), i.e. there is an ideal $J \subseteq I$ with $0 \neq J \neq I$. Then J is a direct summand of R, i.e. there is a homomorphism $f : R \longrightarrow J$ with $(J \longrightarrow I \longrightarrow R \xrightarrow{f} J) = \mathrm{id}_J$. This implies $I = J \oplus K$ for $K := \mathrm{Ke}(I \longrightarrow R \xrightarrow{f} J)$. Since $K \neq I$, there is also a $g : R \longrightarrow K$ with $(K \longrightarrow I \longrightarrow R \xrightarrow{g} K) = \mathrm{id}_K$. The map $f + g - gf : I \longrightarrow R \longrightarrow I$ satisfies $(f + g - gf)(j) = f(j) + g(j) - gf(j) = j + g(j) - g(j) = j$ for all $j \in J$ and $(f+g-gf)(k) = f(k)+g(k)-gf(k) = 0+k-0 = k$ for all $k \in K$, hence $(f+g-gf : I \longrightarrow R \longrightarrow I) = \mathrm{id}_I$. Thus I is a direct summand of R. Contradiction.
Case 2: Let I be a minimal or simple ideal. Since I is not nilpotent and $0 \neq I^2 \subseteq I$ holds, we get $I^2 = I$. In particular there exists an $a \in I$ with $Ia = I$, since Ia is also an ideal. Thus $\cdot a : I \longrightarrow I$ is an epimorphism and even an isomorphism, for $\mathrm{Ke}(\cdot a)$ must be zero as an ideal (see Lemma of Schur 8.5.) So there is an $e \in I$, $e \neq 0$ with $ea = a$. $\Longrightarrow (e^2 - e)a = eea - ea = a - a = 0 \Longrightarrow e^2 - e = 0 \in I \Longrightarrow e^2 = e \in I$. From $I = Re$ we get $R = Re \oplus R(1 - e)$, since $R = Re + R(1 - e)$ and $re = s(1 - e) \in Re \cap R(1 - e) \Longrightarrow re = re^2 = s(1 - e)e = 0$. Thus I is a direct summand of R. Contradiction. □

Lemma 8.5. (Schur) *Let ${}_RM$, ${}_RN$ be simple modules. Then the following hold:*
1. *If $M \ncong N$, then* $\mathrm{Hom}_R(.M, .N) = 0$.
2. $\mathrm{Hom}_R(.M, .M)$ *is a skew-field (= division algebra = non commutative field).*

Proof. Let $f : M \longrightarrow N$ be a homomorphism with $f \neq 0$. Then $\mathrm{Im}(f) = N$, since N is simple and $\mathrm{Ke}(f) = 0$, since M is simple, hence f is an isomorphism. This implies (1).

Furthermore we have (2), since each endomorphism $f : M \longrightarrow M$ with $f \neq 0$ is invertible under the multiplication of $\mathrm{Hom}_R(.M, .M)$. Observe that a *skew-field* is a ring, whose non zero elements form a group under the multiplication. $\square$

Remark 8.6. Let ${}_RM$ be simple. Then $\mathrm{End}_R(.M) = D$ is a skew-field. Hence the R-module structure of M can be characterized by $R \longrightarrow \mathrm{End}_D(M.) = M_n(D)$.

Theorem 8.7. (Artin-Wedderburn) *The following are equivalent:*

(1) *R is simple.*
(2) *R possesses a simple ideal that is an R-progenerator.*
(3) *$R \cong M_n(D)$ is a full matrix ring over a skew-field D. (n is unique, D is unique up to isomorphism.)*
(4) *$R = I_1 \oplus \ldots \oplus I_n$ with isomorphic simple left ideals $I_1, \ldots, I_n$.*

Proof. (1) $\Longrightarrow$ (2): Since R is Artinian there is a simple ideal $0 \neq I \subseteq R$. Let $J := \sum\{I' | I'$ ideal in R and $I' \cong I\}$. Then J is a two sided ideal, since $I' \cdot r \neq 0 \Longrightarrow \cdot r : I' \longrightarrow R$ with $\mathrm{Ke}(\cdot r) = 0$, hence $\cdot r$ is injective and the image $I' \cdot r$ is isomorphic to I' resp. I, hence is in J. Since R is simple we have $R = J = \sum I_i$. Since $1 \in I_1 + \ldots + I_n$, there is an epimorphism $I_1 \oplus \ldots \oplus I_n \longrightarrow R$ (exterior direct sum), that splits since R is projective. Hence R is a direct summand of $I_1 \oplus \ldots \oplus I_n$ up to isomorphism, and thus I is a generator. Furthermore I is a direct summand of R by 8.4, hence it is finitely generated projective, thus I is an R-progenerator.
(2) $\Longrightarrow$ (3): By the Lemma of Schur $\mathrm{End}_R(.I) =: D$ is a skew-field. ${}_RI_D$ generates an equivalence of categories. Hence $R \cong \mathrm{End}_D(I.) \cong M_n(D)$.
(3) $\Longrightarrow$ (4): $R \cong M_n(D) \Longrightarrow R \cong \mathrm{End}_D(V.)$ with an n-dimensional D-vector space V. V_D is a progenerator. Hence we have $\mathcal{V}({}_RR) \cong \mathcal{V}({}_DV^*)$. Since $V^* \cong D \oplus \ldots \oplus D$, we have ${}_RR \cong I_1 \oplus \ldots \oplus I_n$ with $I_1 \cong \ldots \cong I_n \cong {}_RV \otimes_D D \cong {}_RV$.
(4) $\Longrightarrow$ (2): I_1 is obviously an R-progenerator.
(2) $\Longrightarrow$ (1): R-Mod $\cong$ D-Mod with $D \cong \mathrm{End}_R(I)$. Hence $\mathcal{V}({}_RR) \cong \mathcal{V}({}_D\mathrm{Hom}_D(I., {}_DD.))$ is Artinian, and we have $\mathcal{V}({}_RR_R) \cong \mathcal{V}({}_DD_D) = \{0, D\}$. Thus R is simple. $\square$

Corollary 8.8. *Let R be a simple ring and let ${}_RM \neq 0$ be finitely generated. Then the following hold*

(1) *${}_RM$ is an R-progenerator.*
(2) *$S := \mathrm{End}_R(.M)$ is a simple ring.*
(3) $\mathrm{Cent}(R) \cong \mathrm{Cent}(\mathrm{End}_R(.M))$.
(4) $R \cong \mathrm{End}_S(M.)$.

Proof. (1) The claim follows from the fact that R-Mod $\cong$ D-Mod and since each finitely generated D-module is a progenerator.
(2) S-Mod $\cong$ R-Mod $\cong$ D-Mod implies that $\mathcal{V}({}_SS) \cong \mathcal{V}({}_DP)$ is Artinian. Furthermore $\mathcal{V}({}_SS_S) \cong \mathcal{V}({}_DD_D)$, hence S is a simple ring.
(3)+(4) follow from the Morita theorems. $\square$

8.2. **Injective Modules.**

Definition and Remark 8.9. An R-module ${}_RJ$ is called *injective*, if for each monomorphism $f : M \longrightarrow N$ and for each homomorphism $g : M \longrightarrow J$ there exists a homomorphism

$h : N \longrightarrow J$ with $hf = g$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, "h"] \\ J & \end{tikzcd}.$$

Vector spaces are injective. ${}_{\mathbb{Z}}\mathbb{Z}$ is not injective. The injective $\mathbb{Z}$-modules are exactly the divisible Abelian groups. ${}_{\mathbb{Z}}\mathbb{Q}$ is injective.

Theorem 8.10. *(The Baer criterion): The following are equivalent for $Q \in R$-Mod:*

(1) *Q is injective.*
(2) *$\forall_R I \subseteq {}_R R,\ \forall g : I \longrightarrow Q\ \exists h : R \longrightarrow Q$ with $h\iota = g$*

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[d, "g"'] & R \arrow[dl, "h"] \\ Q & \end{tikzcd}.$$

(3) *Each monomorphism $f : Q \xrightarrow{f} M$ splits, i.e. there is an epimorphism $g : M \longrightarrow Q$ with $gf = 1_Q$.*

Proof. (1) $\Longrightarrow$ (2): follows immediately from the definition.
(1) $\Longrightarrow$ (3): The diagram

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "1_Q"'] & M \arrow[dl, "g"] \\ Q & \end{tikzcd}$$

defines the required g.
(3) $\Longrightarrow$ (1): In the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[d, "\psi"] \\ Q \arrow[r, shift left, "\varphi"] & P \arrow[l, shift left, "\rho"] \end{tikzcd}$$

assume that f is a monomorphism and $P := N \oplus Q/\{(f(m), -g(m)) | m \in M\}$ with φ resp. ψ are canonical maps to the left resp. the right components: $\varphi(q) := \overline{(0, q)}, \psi(n) := \overline{(n, 0)}$. Since $\psi f(m) = \overline{(f(m), 0)} = \overline{(0, g(m))} = \varphi g(m)$ we have $\psi f = \varphi g$. Let $\varphi(q) = \overline{(0, q)} = 0$. Then there exists an $m \in M$ with $f(m) = 0$ and $g(m) = q$. Since f is an injective map, we have $m = 0$ and thus φ injective. By (3) there is a ρ with $\rho\varphi = 1_Q$. Then $\rho\psi f = \rho\varphi g = g$, and thus Q is injective.
(2) $\Longrightarrow$ (1): Given a monomorphism $f : M \longrightarrow N$ and a homomorphism $g : N \longrightarrow Q$. Consider the set $\mathcal{S} := \{(N_i, \varphi_i)\}$, where $N_i \subseteq N$ is a submodule with $\mathrm{Im}(f) \subseteq N_i$ and $\varphi_i : N_i$

$\longrightarrow Q$ is a homomorphism such that

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "g"'] & N_i \arrow[r] \arrow[dl, "\varphi_i"] & N \\
Q & &
\end{tikzcd}$$

commutes. We have $\mathcal{S} \neq \emptyset$, since $(\mathrm{Im}(f), gf^{-1}) \in S$. Furthermore $\mathcal{S}$ is ordered by $(N_i, \varphi_i) \leq (N_j, \varphi_j)$ if $N_i \subseteq N_j$ and $\varphi_j|_{N_i} = \varphi_i$. Let $\{(N_i, \varphi_i) | i \in J\}$ be a chain in $\mathcal{S}$. Then $\cup N_i \subseteq N$ is a submodule. $\psi : \cup N_i \longrightarrow Q$ with $\psi(n_i) = \varphi_i(n_i)$ is a well defined homomorphism and $(\cup N_i, \psi) \in \mathcal{S}$. Furthermore we have $(N_j, \varphi_j) \leq (\cup N_i, \psi)$ for all $j \in J$. By Zorn's Lemma there exists a maximal element (N', φ') in $\mathcal{S}$. We show that $N' = N$, for then the continuation of g to N exists. Let $x \in N \setminus N'$. Then $N' \subsetneqq N' + Rx$. Let $I := \{r \in R | rx \in N'\}$. Then I is an ideal and we have a commutative diagram

$$\begin{tikzcd}
& I \arrow[r, "\iota"] \arrow[d, "\cdot x"'] & R \arrow[d, "\rho"] \arrow[ddl, "\sigma"'] \\
M \arrow[r, "f"] \arrow[dr, "g"'] & N' \arrow[r] \arrow[d, "\varphi'"'] & N' + Rx \arrow[dl, "\tau"] \\
& Q &
\end{tikzcd}$$

with $\rho(r) := r \cdot x$. Then we have $\rho(I) \subseteq N'$. Thus by (2) there is a homomorphism $\sigma : R \longrightarrow Q$ with $\sigma\iota = \varphi' \circ (\cdot x)$. We define $\tau : N' + Rx \longrightarrow Q$ by $\tau(n' + rx) := \varphi'(n') + \sigma(r)$. This is a well defined map, for if $n' + rx = n'_1 + r_1 x$ then $(r - r_1)x = n'_1 - n' \in N'$ hence $r - r_1 \in I$. Thus $\sigma(r - r_1) = \varphi'((r - r_1)x) = \varphi'(n'_1 - n')$ and $\varphi'(n') + \sigma(r) = \varphi'(n'_1) + \sigma(r_1)$. It is easy to see that τ is also a homomorphism. Since $\tau|_{N'} = \varphi'$ holds we have $(N' + Rx, \tau) \in \mathcal{S}$ and $(N', \varphi') \lneqq (N' + Rx, \tau)$ a contradiction to the maximality of (N', φ'). Thus $N' = N$. $\square$

Corollary 8.11. *If R is a semisimple ring then each R-module is projective and injective.*

Proof. Let Q be an R-module. By 8.4 each ideal is a direct summand of R. The following diagram together with the Baer criterion shows that Q is injective:

$$\begin{tikzcd}
I \arrow[r, shift left] \arrow[d] & R \arrow[l, shift left] \arrow[dl] \\
Q. &
\end{tikzcd}$$

Let $f : N \longrightarrow P$ be surjective. Since $\mathrm{Ke}(f) \subseteq N$ is a submodule and injective there is a $g : N \longrightarrow \mathrm{Ke}(f)$ with $g(n) = n$ for all $n \in \mathrm{Ke}(f)$. We define $k : P \longrightarrow N$ by $k(p) = n - g(n)$ for $n \in N$ with $f(n) = p$. If also $f(n') = p$ then $f(n - n') = 0$ hence $n - n' \in \mathrm{Ke}(f)$ and $g(n-n') = n-n'$. This implies $n - g(n) = n' - g(n')$. So k is a well defined map. Furthermore $fk(p) = f(n - g(n)) = f(n) - fg(n) = p - 0$, hence $fk = 1_P$. In order to show that k is a homomorphism let $f(n) = p$, $f(n') = p'$. Then we get $f(rn + r'n') = rp + r'p'$. This implies $k(rp + r'p') = rn + r'n' - g(rn + r'n') = r(n - g(n)) + r'(n' - g(n')) = rk(p) + r'k(p')$. Thus P is projective. $\square$

Lemma 8.12. *Let* $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ *be a short exact sequence.* M *and* P *are Artinian if and only if* N *is Artinian. In particular if* M *and* N *are Artinian then* $M \oplus N$ *is Artinian.*

Proof. Let N be Artinian. This implies immediately that M is Artinian. If $\{L_i\}$ is a set of submodules of P then $\{g^{-1}(L_i)\}$ is a set of submodules of N. Let $g^{-1}(L_0)$ be minimal in this set. Since $gg^{-1}(L_i) = L_i$ we have that L_0 is minimal in $\{L_i\}$.
Let M and P be Artinian. Let $\{L_i\}$ be a set of submodules of N. Let L_0 be chosen such that $g(L_0)$ is minimal in the set $\{g(L_i)\}$. Let L be chosen such that $f^{-1}(L)$ is minimal in the set $\{f^{-1}(L_j)|L_j \in \{L_i\}$ and $g(L_j) = g(L_0)\}$. We show that L is minimal in $\{L_i\}$. Let $L' \in \{L_i\}$ with $L \supseteq L'$. Then $g(L_0) = g(L) \supseteq g(L')$, hence $g(L') = g(L_0)$. Furthermore we have $f^{-1}(L) \supseteq f^{-1}(L')$, hence $L = L'$. $\square$

8.3. **Simple and Semisimple Modules.**

Lemma 8.13. *Let* $R_1, \ldots, R_n$ *be semisimple rings. Then* $R_1 \times \ldots \times R_n$ *is a semisimple ring.*

Proof. (Only for the case $R_1 \times R_2$) By Lemma 8.12 $R_1 \times R_2$ is Artinian. Let $I \subseteq R$ be nilpotent. From $I^n = 0$ we get for each $a \in I$ the equation $(Ra)^n = 0$. From $a = (a_1, a_2)$ follows $0 = (Ra)^n = (R_1a_1, R_2a_2)^n$. Hence $R_1a_1 = 0$ and $R_2a_2 = 0$, i.e. $Ra = 0$ and thus $I = 0$. $\square$

Lemma 8.14. *Each proper submodule* N *of a finitely generated module* M *is contained in a maximal submodule of* M*. In particular* M *possesses a simple quotient module.*

Proof. Let $N \subsetneqq M$ be a proper submodule of M. Let $\mathcal{M}$ be the set of submodules U with $N \subseteq U \subsetneqq M$. $\mathcal{M}$ is ordered by inclusion. Let (U_i) be a chain in $\mathcal{M}$ and $U' := \cup U_i$. Then U' is again a submodule and $N \subseteqq U'$. If $U' = M$ then all generating elements $m_1, \ldots, m_t$ are in U', hence there is a module U_i with $m_1, \ldots, m_t \in U_i$. Thus $U_i = M$. This is impossible. So $U' \neq M$ and thus in $\mathcal{M}$. Furthermore U' is an upper bound of (U_i). By Zorn's Lemma there is a maximal submodule of M (in $\mathcal{M}$), that contains N. $\square$

Lemma 8.15. (1) *If* $X \subseteq {}_{\mathbb{Z}}\mathbb{Q}$ *is a set of generating elements of* $\mathbb{Q}$ *over* $\mathbb{Z}$ *and* $x \in X$ *then* $X \setminus \{x\}$ *is also a set of generating elements of* $\mathbb{Q}$*.*
(2) ${}_{\mathbb{Z}}\mathbb{Q}$ *possesses no maximal submodules.*

Proof. (1) Let $B = \langle X \setminus \{x\} \rangle$. Then $\mathbb{Q} = \mathbb{Z}x + B$. There is a $y \in \mathbb{Q}$ with $2y = x$. We represent y as $y = nx + b$ with $n \in \mathbb{Z}$, $b \in B$. This implies $x = 2y = 2nx + 2b$ and thus $(1-2n)x = 2b \in B$. Furthermore there is a $z \in \mathbb{Q}$ with $(1-2n)z = x$, since obviously $1-2n \neq 0$. We represent z as $z = mx + b'$. This implies $x = (1-2n)z = (1-2n)mx + (1-2n)b' = 2mb + (1-2n)b' \in B$. Thus $B = \mathbb{Q}$ and we can omit x from the set of generating elements.
(2) Let $N \subseteq \mathbb{Q}$ be a maximal submodule and $x \in \mathbb{Q} \setminus N$. Then $N \cup \{x\}$ is a set of generating elements of $\mathbb{Q}$, hence also N. Contradiction. $\square$

Lemma 8.16. *Let* ${}_RM$ *be a module in which each submodule is a direct summand. Then each submodule* $0 \neq N \subseteq M$ *contains a simple submodule. Furthermore* M *is a sum of simple submodules.*

Proof. Let $x \in N$, $x \neq 0$. It suffices to show that Rx has a simple submodule. Since Rx is finitely generated Rx possesses a maximal submodule L. Since L is a direct summand of M, there is $f : M \longrightarrow L$ with $(L \longrightarrow Rx \longrightarrow M \xrightarrow{f} L) = 1_L$, hence $L \oplus I = Rx$, where $I = \mathrm{Ke}(Rx \longrightarrow M \longrightarrow L)$. If $0 \neq J \subsetneqq I$ then $L \subsetneqq L + J \subsetneqq Rx$ in contradiction to L maximal in Rx. Hence I is simple with $I \subseteq Rx \subseteq N$.

Let $N := \sum I_j$ be the sum of all simple submodules of M. Then $M = N \oplus K$. If $K \neq 0$ then K contains a simple submodule I and we have $I \subseteq N \cap K$. Contradiction. Thus $K = 0$ and $M = \sum I_j$. □

Lemma 8.17. *Let ${}_RM$ be a sum of simple submodules: $M = \sum_{j \in X} I_j$. Let $N \subseteq M$ be a submodule. Then there is a set $Y \subseteq X$ with $M = N \oplus \bigoplus_{j \in Y} I_j$ and a set $Z \subseteq X$ with $N \cong \bigoplus_{j \in Z} I_j$. In particular each submodule N of M is a direct sum of simple submodules.*

Proof. Let $\mathcal{S} = \{Z \subseteq X | N + (\sum_{j \in Z} I_j) = N \oplus (\bigoplus_{j \in Z} I_j)\}$. The set $\mathcal{S}$ is ordered by inclusion and not empty since $\emptyset \in \mathcal{S}$. Let (Z_i) be a chain in $\mathcal{S}$. Then $Z' := \cup Z_i \in \mathcal{S}$. In order to show this let $n + \sum_{j \in Z'} a_j = 0$. Then at most finitely many $a_j \in I_j$ are different from 0. Hence there is a Z_i in the chain with $j \in Z_i$ for all $a_j \neq 0$ in the sum. From $N + (\sum_{j \in Z_i} I_j) = N \oplus (\bigoplus_{j \in Z_i} I_j)$ we get $n = 0 = a_j$ for all $j \in Z'$. By Zorn's Lemma there is a maximal element $Z'' \in \mathcal{S}$, and we have $P := N + (\sum_{j \in Z''} I_j) = N \oplus (\bigoplus_{j \in Z''} I_j)$. Let I_k be simple with $k \in X \setminus Z''$. If $P + I_k = P \oplus I_k$, then $N + (\sum_{j \in Z''} I_j) + I_k = N \oplus (\bigoplus_{j \in Z''} I_j) \oplus I_R$ in contradiction to the maximality of Z''. Hence $0 \neq P \cap I_k \subseteq I_k$, or $I_k \subseteq P$. This implies $P = N + \sum_{j \in X} I_j = M$.
Now we apply the first claim to $\bigoplus_{j \in Y} I_j$ and obtain $N \oplus (\bigoplus_{j \in Y} I_j) = M = (\bigoplus_{j \in Y} I_j) \oplus (\bigoplus_{j \in Z} I_j)$. This implies $N \cong M / (\bigoplus_{j \in Y} I_j) \cong \bigoplus_{j \in Z} I_j$. □

Theorem 8.18. *(Structure Theorem for Semisimple Modules): For ${}_RM$ the following are equivalent*

1. *Each submodule of M is a sum of simple submodules.*
2. *M is a sum of simple submodules.*
3. *M is a direct sum of simple submodules.*
4. *Each submodule of M is a direct summand.*

Proof. (1) $\Longrightarrow$ (2): trivial.
(2) $\Longrightarrow$ (3): Lemma 8.17.
(3) $\Longrightarrow$ (1): Lemma 8.17.
(2) $\Longrightarrow$ (4): Lemma 8.17.
(4) $\Longrightarrow$ (2): Lemma 8.16. □

Definition 8.19. A module ${}_RM$ is called *semisimple*, if it satisfies one of the equivalent conditions of Theorem 8.18.

Corollary 8.20. (1) *Each submodule of a semisimple module is semisimple.*
(2) *Each quotient (residue class) module of a semisimple module is semisimple.*
(3) *Each sum of semisimple modules is semisimple.*

Proof. (1) trivial.
(2) Let $N \subseteq M$. Then $M \cong N \oplus M/N$, in particular M/N is isomorphic to a submodule of M.
(3) trivial. □

Remark 8.21. With the notion of a semisimple module we have obtained a particularly suitable generalization of the notion of a vector space. Important *theorems of linear algebra* have been generalized in Theorem 8.18. The simple modules over a field are exactly the one dimensional vector spaces. Condition (2) of Theorem 8.18 is trivially satisfied since each vector space is the sum of simple (one dimensional) vector spaces, one simply has to form $V = \sum_{v \in V \setminus \{0\}} Kv$ or $V = \sum_{v \in E} Kv$ for an arbitrary set of generating elements E of V. Thus each vector space V is semisimple. So condition (3) holds. It says that each set of generating elements E contains a basis. (4) is the important statement that each subspace of a vector space has a direct complement. Lemma 8.17 also contains claims about the dimension of vector spaces, subspaces and quotient spaces.

Theorem 8.22. (Wedderburn) *The following are equivalent for R:*

(1) ${}_RR$ *is semisimple (as a ring).*
(2) *Each R-module is projective.*
(3) *Each R-module is injective.*
(4) *Each R-module is semisimple.*
(5) ${}_RR$ *is semisimple (as an R-module).*
(6) *R is a direct sum of simple left ideals.*
(7) $R \cong R_1 \times \ldots \times R_n$ *with simple rings R_i* $(i = 1, \ldots, n)$.
(8) $R \cong B_1 \oplus \ldots \oplus B_n$*, where the B_i are minimal two sided ideals and ${}_RR$ is Artinian.*
(9) R_R *is semisimple (as a ring).*

Proof. (1) $\Longrightarrow$ (3): Corollary 8.11.
(3) $\Longrightarrow$ (4): Theorem 8.18 (4) and Theorem 8.10 (3).
(4) $\Longrightarrow$ (5): Specialization.
(5) $\Longrightarrow$ (6): Theorem 8.18 (3).
(6) $\Longrightarrow$ (3): Theorem 8.18 (4) and 8.11.
(6) $\Longrightarrow$ (2): Theorem 8.18 (4) and 8.11.
(2) $\Longrightarrow$ (4): Let $N \subseteq M$ be a submodule. Then M/N is projective, so there is $f : M/N \longrightarrow M$ with $(M/N \longrightarrow M \longrightarrow M/N) = \mathrm{id}$ or $(M \longrightarrow M/N \longrightarrow M) = p$ with $p^2 = p$. Hence $M = \mathrm{Ke}(p) \oplus \mathrm{Im}(p)$ and $\mathrm{Ke}(p) = N$.
(6) $\Longrightarrow$ (8): Let $R = I_{11} \oplus \ldots \oplus I_{1i_1} \oplus I_{21} \oplus \ldots \oplus I_{2i_2} \oplus \ldots \oplus I_{n1} \oplus \ldots \oplus I_{ni_n}$ be a direct sum of simple ideals, finitely many, since R is finitely generated, and let $I_{ij} \cong I_{ik}$ for all i, j, k and $I_{i1} \not\cong I_{j1}$ for $i \neq j$. Let $B_k := \bigoplus_{j=1}^{i_k} I_{kj}$.
Let $I \subseteq R$ be simple. Let $p_k : R \longrightarrow B_k$ be the projection onto B_k w.r.t. $R = B_1 \oplus \ldots \oplus B_n$. Then there is at least one k with $p_k(I) \neq 0$. Then $I \cong p_k(I) = J \subseteq B_k$ is a simple ideal. Because of 8.17 we get $I \oplus (\bigoplus_{j=r+1}^{m} I_{kj}) = B_k = I_{k1} \oplus \ldots \oplus I_{kr} \oplus (\bigoplus_{j=r+1}^{m} I_{kj})$ using a suitable numbering. Hence $J \cong I_{k1} \oplus \ldots \oplus I_{kr}$ and thus $r = 1$ and $I \cong J \cong I_{k1}$. So there is a unique k with $p_k(I) \neq 0$. In particular we have $I \subseteq B_k$. If $f : {}_RR \longrightarrow {}_RR$ with $f(I) \neq 0$ is given, then $f(I) \cong I$ is simple and $f(I) \subseteq B_k$ for one k. So $f(B_k) \subseteq B_k$ holds for all $f \in \mathrm{Hom}_R(.R, .R) \cong R$, and B_k is a two sided ideal.
Observe that $B_iB_j \subseteq B_i \cap B_j = 0$. For $1 \in R = B_1 \oplus \ldots \oplus B_n$ let $1 = e_1 + \ldots + e_n$ with $e_i \in B_i$. For $b \in B_i$ we get $e_ib = (e_1 + \ldots + e_n)(0 + \ldots + b + \ldots + 0) = b = be_i$. Thus B_i can be considered as ring with unit e_i. (B_i is not a subring of R but a quotient ring of R.) Since $B_iB_j = 0$ we have that $L \subseteq B_i$ is a (one sided resp. two sided) B_i-ideal of B_i iff L is an R-ideal. Since $B_i = I_1 \oplus \ldots \oplus I_n$ is a direct sum of simple R-ideals resp. B_i-ideals and since $I_j \cong I_k$ holds, B_i is a simple ring by Theorem 8.7. In particular B_i has no two sided nontrivial ideals, i.e. the two sided ideals $B_i \subseteq R$ are minimal. 8.12 implies that R is Artinian.
(8) $\Longrightarrow$ (7): Since $B_iB_j \subseteq B_i \cap B_j = 0$ the B_i are simple rings as above, hence $R = R_1 \times \ldots \times R_n$ with $R_i = B_i$, because addition and multiplication are performed in the B_i (componentwise).
(7) $\Longrightarrow$ (1): Lemma 8.12.
(7) $\Longrightarrow$ (9): In order to have condition (7) symmetric in the sides, it suffices to show that a simple ring R is right Artinian. But $R \cong M_n(D) \cong \mathrm{Hom}_D(V^*., V.^*)$ is left and right Artinian. $\square$

8.4. **Noetherian Modules.**

Definition 8.23. A module ${}_FM$ is called *Noetherian* (Emmy Noether 1882-1935), if each nonempty set of submodules of M has a maximal element.

Theorem 8.24. *For ${}_RM$ the following are equivalent:*

(1) *M is Noetherian.*
(2) *Each ascending chain $M_i \subseteq M_{i+1}, i \in \mathbb{N}$ of submodules of M becomes stationary, i.e. there is an $n \in \mathbb{N}$ with $M_n = M_{n+i}$ for all $i \in \mathbb{N}$.*
(3) *Each submodule of M is finitely generated.*

Proof. (2) $\Longrightarrow$ (1): Let $\mathcal{M}$ be a nonempty set of submodules without a maximal element. Using the axiom of choice we choose for each $N \in \mathcal{M}$ an $N' \in \mathcal{M}$ with $N \subsetneqq N'$. For $N \in \mathcal{M}$ we then have an ascending chain $M_1 = N, M_{i+1} = M_i'$ with

$$M_1 \subsetneqq M_2 \subsetneqq \ldots \subsetneqq M_i \subsetneqq M_{i+1} \subsetneqq \ldots$$

This is impossible by (2).
(1) $\Longrightarrow$ (3): Let $M' \subseteq M$. Then $\{N | N \subseteq M', N \text{ finitely generated}\} \neq \emptyset$ has a maximal element N'. If $N' \neq M'$, then there is an $m \in M' \setminus N'$. So $N' + Rm \subseteq M'$ is finitely generated and $N' \subsetneqq N' + Rm$ in contradiction to the maximality of N'. Hence $N' = M'$, i.e. M' is finitely generated.
(3) $\Longrightarrow$ (2): Let $M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots \subseteq M$ be an ascending chain of submodules of M. Let $N := \bigcup_{i \in \mathbb{N}} M_i$. N is a finitely generated submodule of M, i.e. $N = Ra_1 + \ldots + Ra_n$. Then there is an M_r with $a_1, \ldots, a_n \in M_r$. This implies $M_r = N = M_{r+i}$ for all $i \in \mathbb{N}$, i.e. the chain becomes stationary. $\square$

Lemma 8.25. *Let $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ be a short exact sequence. M and P are Noetherian iff N is Noetherian. In particular if M and N are Noetherian then so is $M \oplus N$.*

Proof. Let N be Noetherian. Then it is clear that M Noetherian. If $\{L_i\}$ is a set of submodules of P then $\{g^{-1}(L_i)\}$ is a set of submodules of N. Let $g^{-1}(L_0)$ be maximal in this set. With $gg^{-1}(L_i) = L_i$ we get that L_0 is maximal in $\{L_i\}$.
Let M and P be Noetherian. Let $\{L_i\}$ be a set of submodules of N. Let L_0 be chosen such that $g(L_0)$ is maximal in the set $\{g(L_i)\}$. Let L be chosen such that $f^{-1}(L)$ is maximal in the set $\{f^{-1}(L_j) | L_j \in \{L_i\} \text{ and } g(L_j) = g(L_0)\}$. We show that L is maximal in $\{L_i\}$. Let $L' \in \{L_i\}$ with $L \subseteq L'$. Then $g(L_0) = g(L) \subseteq g(L')$ hence $g(L') = g(L_0)$. Furthermore we have $f^{-1}(L) \subseteq f^{-1}(L')$ hence $L = L'$. $\square$

Corollary 8.26. *${}_RR$ is Noetherian as a left R-module iff all finitely generated left R-modules are Noetherian.*

Proof. $\Longleftarrow$: trivial.
$\Longrightarrow$: If M is finitely generated then there is a short exact sequence $0 \longrightarrow K \longrightarrow R \oplus \ldots \oplus R \longrightarrow M \longrightarrow 0$. Since R is Noetherian $R \oplus \ldots \oplus R$ Noetherian, too, so that M is Noetherian. $\square$

Theorem 8.27. (Hilbert Basis Theorem) *If R is left Noetherian then $R[x]$ is left Noetherian.*

Proof. Let $J \subseteq R[x]$ be an ideal. We have to show that J finitely generated. Let $J_0 := \{r \in R | \exists p(x) \in J \text{ with highest coefficient } r\}$. (The highest coefficient of the zero polynomial is 0 by definition.) $J_0 \subseteq R$ is an ideal, hence $J_0 = \langle r_1, \ldots, r_n \rangle$. For the r_i choose $p_i(x) \in J$ with highest coefficients r_i. Let $m \geq \deg(p_i(x))$ for $i = 1, \ldots, n$. Let $g \in J$ with $\deg(g) \geq m$. Then $g = sx^t + \sum_{i \leq t} s_i x^i$. Since $s \in J_0$ we have $s = \sum_{j=1}^n \lambda_j r_j$. This implies $g_1 := g - \sum_{j=1}^n \lambda_j p_j(x) x^{t - \deg(p_j(x))} \in J$ and $\deg(g_1) \leq t - 1$. By induction we have $g = g_0 + \overline{g}$

with $g_0 \in \sum_{j=1}^n R[x]p_j(x)$ and $\deg(\overline{g}) < m$. This implies $\overline{g} \in J \cap (R + Rx + \ldots + Rx^{m-1}) \subseteq R + Rx + \ldots + Rx^{m-1}$. Both R-modules are finitely generated hence $\overline{g} = \sum_{i=1}^k \mu_i q_i(X)$ with $\langle q_1(x), \ldots, q_k(x)\rangle = J \cap (R + Rx + \ldots + Rx^{m-1})$. Thus $\{p_1(x), \ldots, p_n(x), q_1(x), \ldots, q_k(x)\}$ form a set of generating elements of J. $\square$

Corollary 8.28. *Let R be a commutative Noetherian ring and let S be a commutative R-algebra. Let S be finitely generated as an R-algebra (i.e. there are $s_1, \ldots, s_n \in S$ such that for all $s \in S$ there are representations $s = \sum r_{i_1,\ldots,i_n} s_1^{i_1} \ldots s_n^{i_n}$). Then S is Noetherian.*

Proof. 1. By induction we have $R[x_1, \ldots, x_n]$ Noetherian.
2. There is an epimorphism $R[x_1, \ldots, x_n] \longrightarrow S$. Thus S is a Noetherian $R[x_1, \ldots, x_n]$-module hence it is also a Noetherian S-module. $\square$

Proposition 8.29. *Let R be commutative or M be Noetherian. Let M be finitely generated. Let $f : N \longrightarrow M$ be an epimorphism where $N \subseteq M$ is a submodule. Then f is an isomorphism.*

Proof. 1. Let M be Noetherian. We construct an ascending chain $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ by $K_0 := \mathrm{Ke}(f) = f^{-1}(0)$, $K_i := f^{-1}(K_{i-1})$. We have $K_0 = f^{-1}(0) \subseteq f^{-1}(K_0) = K_1$. If $K_{i-2} \subseteq K_{i-1}$ then we have $K_{i-1} = f^{-1}(K_{i-2}) \subseteq f^{-1}(K_{i-1}) = K_i$. Since M is Noetherian the chain becomes stationary $K_n = K_{n+1} = \ldots$. Let $x_0 \in K_0$. We want to show $x_0 = 0$. There is $x_1 \in K_1$ with $f(x_1) = x_0$, since f is an epimorphism. Similarly there are $x_0, x_1, x_2, \ldots$ with $f(x_i) = x_{i-1}$ and $f^{n+1}(x_{n+1}) = f^n(x_n) = \ldots = f(x_1) = x_0$. Since the chain becomes stationary we get $x_{n+1} \in K_n$, which implies $f(x_{n+1}) \in K_{n-1}$ and thus $f^n(x_{n+1}) \in K_0$. Hence $x_0 = f^{n+1}(x_{n+1}) = 0$. This proves that f is a monomorphism.
2. Let R commutative. Let $M = Ry_1 + \ldots + Ry_n$. Let $x_i \in N_i$ with $f(x_i) = y_i$. Let $x_0 \in N$ with $f(x_0) = 0$. Then there are coefficients $r_{ij} \in R$ with $x_i = \sum_{j=1}^n r_{ij} y_j, i = 0, \ldots, n$. We consider $R' := \mathbb{Z}[r_{ij}] \subseteq R$, the subring of R generated by the r_{ij}. Since $\mathbb{Z}$ is Noetherian and R' is finitely generated as a $\mathbb{Z}$-algebra R' is Noetherian. Let $M' := \sum_{i=1}^n R'y_i \subseteq M$ and $N' = \sum_{i=0}^n R'x_i \subseteq N$. Then $N' \subseteq M'$ is an R'-submodule, M' as an R'-module is finitely generated, hence Noetherian, and the $f(x_i) = y_i, f(x_0) = 0$ generate a homomorphism of R'-modules $f' : N' \longrightarrow M'$. Since f' is surjective f' is injective and thus $x_0 = 0$ so that f is injective. $\square$

Problem 8.1. Where does the commutativity of R enter the second part of the proof of Proposition 8.29?

Corollary 8.30. *Let R be commutative or ${}_RM$ be Noetherian. Let $M = Ry_1 + \ldots + Ry_m$. Let $N \subseteq M$ be a free submodule with the free generating elements $x_1, \ldots, x_n$. Then $n \leq m$. If $n = m$ then M is free over $y_1, \ldots, y_m$.*

Proof. Since N is free there is a homomorphism $f : N \longrightarrow M$ with $f(x_i) = y_i$ for $i = 1, \ldots, \min(m, n)$ and $f(x_i) = 0$ else. If $n \geq m$ then f is surjective, hence bijective. Thus we have $n \leq m$. If $n = m$ then f is bijective and M free with the generating elements $y_1, \ldots, y_n$. $\square$

Corollary 8.31. *Let R be commutative or Noetherian. Let M be free over $x_1, \ldots, x_n$ and free over $y_1, \ldots, y_m$. Then we have $m = n$.*

Proof. If R is Noetherian then M is also Noetherian. Thus the claim follows from 8.30. $\square$

Definition 8.32. Let R be commutative or Noetherian. The *rank* of a finitely generated free module ${}_RM$ is the number of free generating elements uniquely determined by 8.31.

Example 8.33. The endomorphism ring of a countably infinite dimensional vector space is neither left nor right Noetherian.

Proof. From $ap + bq = 1, pa = 1, qb = 1, pb = 0, qa = 0$ we get (as in the exercise 1.4) ${}_RR = {}_RRp \oplus {}_RRq$ free and $R_R = aR_R \oplus bR_R$ free. □

Definition 8.34. An element $r \in R$ in a ring R is called a *left unit* (*right unit*), if $rR = R$ ($Rr = R$). $r \in R$ is called a *unit*, if $Rr = R = rR$.

Lemma 8.35. *If $r \in R$ is a unit, then there is a unique $s \in R$ with $sr = 1$. Furthermore we have $rs = 1$ and s is a unit.*

Proof. Let $sr = s'r = 1$ and let $rt = 1$. Then $s = s1 = srt = 1t = t$ and analogously $s' = t$. □

Corollary 8.36. *In each left Noetherian ring R each right unit $x \in R$ (i.e. $Rx = R$) is also a left unit and conversely.*

Proof. Let $Rx = R$. Then $\cdot x : R \longrightarrow R$ is an epimorphism, hence an isomorphism. So there is an inverse isomorphism $g : R \longrightarrow R$ with $g \in \mathrm{Hom}_R(.R, .R) \cong R$, hence $g = \cdot y$. This implies $1 \cdot x \cdot y = 1$ and $1 \cdot y \cdot x = 1$, i.e. $x^{-1} = y$ and x is a unit. If $xR = R$ then there is a $y \in R$ with $xy = 1$. So y is a right unit hence y is a unit. By 8.36 x is the unique inverse of y, hence x is a unit. □

9. Radical and Socle

Definition 9.1. (1) $N \subseteq M$ is called *large* (*essential*) iff

$$\forall U \subseteq M : N \cap U = 0 \Longrightarrow U = 0.$$

(2) $N \subseteq M$ is called *small* (*superfluous*) iff

$$\forall U \subseteq M : N + U = M \Longrightarrow U = M.$$

Lemma 9.2. *Let $N \subseteq M \subseteq P$, $U \subseteq P$ be submodules. Then the* modular law *holds:*

$$N + (U \cap M) = (N + U) \cap M.$$

Proof. $\subseteq$: From $n + u \in N + U$ with $n \in N$ and $u \in U \cap M \subseteq M$ it follows that $n + u \in M$ and hence $n + u \in (N + U) \cap M$.
$\supseteq$: From $n + u = m \in (N + U) \cap M$ it follows that $u = m - n \in M \cap U$ and hence $n + u \in N + (U \cap M)$. $\square$

Lemma 9.3. (1) *Let $N \subseteq N' \subseteq M' \subseteq M$ be submodules and let N be large in M. Then N' is large in M'.*
(2) *Let $N \subseteq N' \subseteq M' \subseteq M$ be submodules and let N' be small in M'. Then N is small in M.*
(3) *Let $N, N' \subseteq M$ be large submodules in M. Then $N \cap N'$ is large in M.*
(4) *Let $N, N' \subseteq M$ be small submodules in M. Then $N + N'$ is small in M.*

Proof. (1) Let $U \subseteq M'$ with $N' \cap U = 0$. Then $N \cap U = 0$ hence $U = 0$.
(2) Let $U \subseteq M$ with $N + U = M$, then $N' + U = M$. From $N' + (U \cap M') = (N' + U) \cap M' = M \cap M' = M'$ we get $U \cap M' = M'$ and thus $M' \subseteq U$ which implies $N \subseteq U$. Now from $N + U = M$ we get $U = M$.
(3) Let $(N \cap N') \cap U = 0$. Then $N \cap (N' \cap U) = 0$ hence $N' \cap U = 0$ and thus $U = 0$.
(4) Let $(N + N') + U = M$. Then $N + (N' + U) = M$ hence $N' + U = M$ and thus $U = M$. $\square$

Lemma 9.4. *Let $N, U \subseteq M$ be submodules.*
(1) *If N is maximal w.r.t. the condition $N \cap U = 0$ then $N + U \subseteq M$ is a large submodule.*
(2) *If N is minimal w.r.t. the condition $N + U = M$ then $N \cap U \subseteq M$ is a small submodule.*
(3) *There is a submodule N that is maximal w.r.t. $N \cap U = 0$.*

Proof. (1) Let $V \subseteq M$ with $(N + U) \cap V = 0$ be given. We have $N \cap U = 0$. Let $n + v = u \in (N + V) \cap U$. This implies $v = u - n \in (N + U) \cap V = 0$ hence $n = u \in N \cap U = 0$ and $(N + V) \cap U = 0$. Thus $N + V = N$, since N is maximal w.r.t. $N \cap U = 0$. This implies $V \subseteq N$ hence $V \subseteq (N + U) \cap V = 0$ and $V = 0$. So we get that $N + U \subseteq M$ is large.
(2) Let $V \subseteq M$ with $(N \cap U) + V = M$. We have $N + U = M$. Let $m \in M$ with $m = n + u \in N + U$. Furthermore let $n = n' + v$ with $n' \in N \cap U$ and $v \in V$ (since $n \in M$). This implies $v \in V \cap N$ and $m = (n' + u) + v \in U + (V \cap N)$ and thus $(N \cap V) + U = M$. Since N is minimal w.r.t. $N + U = M$ we have $N = N \cap V$ hence $N \subseteq V$. From this and from $(N \cap U) + V = M$ we get $V = M$. Thus $N \cap U \subseteq M$ is small.
(3) The set $\mathcal{V} := \{V \subseteq M | V \cap U = 0\}$ is inductively ordered, for let $(V_i)_{i \in I}$ be a chain in $\mathcal{V}$ and let $x \in (\cup V_i) \cap U$. Then there is an $i \in I$ with $x \in V_i \cap U$ hence $x = 0$. Thus $\cup V_i$ in $\mathcal{V}$ is an upper bound of the V_i. Consequently there is a submodule N of M that is maximal w.r.t. $N \cap U = 0$. $\square$

Lemma 9.5. *$N \subseteq M$ is large if and only if the following holds*

$$\forall m \in M \setminus \{0\} \exists r \in R : rm \in N \setminus \{0\}.$$

Proof. $N \subseteq M$ large $\iff [\forall U \subseteq M : N \cap U = 0 \Longrightarrow U = 0] \iff [\forall U \subseteq M : U \neq 0 \Longrightarrow N \cap U \neq 0] \overset{(*)}{\iff} [\forall Rm \subseteq M : Rm \neq 0 \Longrightarrow N \cap Rm \neq 0] \iff [\forall m \in M \setminus \{0\} \exists r \in R : rm \in N \setminus \{0\}]$. Only one direction $(*)$ needs an additional argument. If $U \neq 0$ and the right hand side of $(*)$ holds, then there exists an $m \in U$ with $Rm \neq 0$. Hence we get $0 \neq N \cap Rm \subseteq N \cap U$. □

Lemma 9.6. *Let $Rm \subseteq M$ be not small. Then there exists a submodule $N \subseteq M$ that is a maximal submodule and that does not contain m.*

Proof. The set $\mathcal{S} := \{U \subsetneqq M | Rm + U = M\}$ is not empty since Rm is not small in M. $\mathcal{S}$ is inductively ordered. In fact let $(U_i | i \in I)$ be a chain in $\mathcal{S}$. Then we have $m \notin U_i$ for all $i \in I$. Hence $\cup U_i \subsetneqq M$ and obviously $Rm + (\cup U_i) = M$. Then there is a maximal element N in $\mathcal{S}$. Let $N \subsetneqq N' \subseteq M$. Then $Rm + N' = M$. Since $N' \notin \mathcal{S}$ we get $N' = M$ hence N is a maximal submodule. Furthermore we have obviously $m \notin N$. □

Definition 9.7. (1) $\mathrm{Radical}(M) = \mathrm{Rad}(M) := \cap\{U \subsetneqq M | U$ maximal submodule$\}$,
(2) $\mathrm{Socle}(M) = \mathrm{Soc}(M) := \sum\{U \subseteq M | U$ simple submodule$\}$.

Proposition 9.8. (1) $\mathrm{Rad}(M) = \sum\{V \subseteq M$ *small*$\}$.
(2) $\mathrm{Soc}(M) = \cap\{V \subseteq M$ *large*$\}$.

Proof. (1) $\supseteq$: Let $V \subseteq M$ small. For all maximal submodules $U \subseteq M$ we have $U \subseteq U + V \subsetneqq M$ since V is small and $U \neq M$. This implies $U = U + V$ and $V \subseteq U$. Thus $V \subseteq \cap U$ and thus $\sum V \subseteq \cap U$.
$\subseteq$: If Rm is not small in M then by 9.6 there is a maximal submodule N in M with $m \notin N$. So we have $m \notin \cap U = \mathrm{Rad}(M) \subseteq N$. If also $m \in \mathrm{Rad}(M)$ holds then Rm is small in M. So we get $m \in \sum\{V \subseteq M$ small$\}$.
(2) $\subseteq$: Let V be large in M and let U be simple. Then we have $V \cap U \neq 0$ so that $V \cap U = U$ and thus $U \subseteq V$. This implies $\sum U \subseteq \cap V$.
$\supseteq$: First we show that each submodule of $\cap V_i$ is a direct summand of $\cap V_i$. Let $N \subseteq \cap V_i$ be given. Let X be maximal in M with $N \cap X = 0$ (Lemma 9.4 (3)). Then $N + X = V \subseteq M$ is large by Lemma 9.4 (1). This implies $N + (X \cap (\cap V_i)) = (N + X) \cap (\cap V_i)$ (Lemma 9.2) $= V \cap (\cap V_i) = \cap V_i$ and $N \cap (X \cap (\cap V_i)) = 0$. So we have $N \oplus (X \cap (\cap V_i)) = \cap V_i$.
Theorem 8.16 implies that $\cap V_i$ is a sum of simple submodules of $\cap V_i$. Thus $\cap V_i$ is contained in the sum of all simple submodules of M, i.e. in the socle of M. □

Remark 9.9. A module M is semisimple if and only if it coincides with its socle.

Corollary 9.10. *$m \in \mathrm{Rad}(M)$ iff $Rm \subseteq M$ is small.*

Proof. $\Longleftarrow$: by Proposition 9.8.
$\Longrightarrow$: was explicitly noted in the proof of Proposition 9.8. □

Corollary 9.11. *Each finitely generated submodule of* $\mathrm{Rad}(M)$ *is small in* M.

Proof. By 9.10 the modules $Rm_1, \ldots, Rm_n \subseteq M$ are small, if $m_1, \ldots, m_n \in \mathrm{Rad}(M)$. By 9.3 (4) we then get that $\sum_{i=1}^n Rm_i$ is small in M. □

Proposition 9.12. *Let M be finitely generated. Then* $\mathrm{Rad}(M)$ *is small in* M.

Proof. Since M is finitely generated each proper submodule of M is contained in a maximal submodule (8.14). Let $N \subsetneqq M$ and let U be a maximal submodule with $N \subseteq U \subsetneqq M$. Then $\mathrm{Rad}(M) \subseteq U$ thus $\mathrm{Rad}(M) + N \subseteq U \subsetneqq M$. So $\mathrm{Rad}(M)$ is small in M. □

Proposition 9.13. *Let $f \in \mathrm{Hom}_R(M, N)$. Then we have*

(1) $f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(N)$.
(2) $f(\mathrm{Soc}(M)) \subseteq \mathrm{Soc}(N)$.

Proof. (1) Let $U \subseteq M$ be small. Let $V \subseteq N$ with $f(U) + V = N$. This implies $f^{-1}(f(U) + V) = f^{-1}(N) = M = U + f^{-1}(V)$, because $f(x) = f(u) + v$ implies $f(x - u) = v, x - u \in f^{-1}(V)$ and thus $x \in U + f^{-1}(V)$, so $f^{-1}(f(U) + V) \subseteq U + f^{-1}(V)$. Since U is small we get $f^{-1}(V) = M$. This implies $f(f^{-1}(V)) = f(M) \subseteq V$, hence $f(U) \subseteq V$ and $V = N$. So we have $f(U)$ small in M. This shows $f(\mathrm{Rad}(M)) = \sum_{U \text{ small}} f(U) \subseteq \sum_{V \text{ small}} V = \mathrm{Rad}(N)$.
(2) Let $U \subseteq M$ be simple. Then $f(U) \subseteq N$ is simple or 0. So we have $f(\sum U_i) \subseteq \mathrm{Soc}(N)$. □

Corollary 9.14. Rad *and* Soc *are covariant subfunctors of* $\mathrm{Id} : R\text{-Mod} \longrightarrow R\text{-Mod}$.

Corollary 9.15. (1) *Let $U \subseteq M$ be small and $f \in \mathrm{Hom}_R(M, N)$. Then $f(U) \subseteq N$ is small.*
(2) *Let $U \subseteq N$ be large and $f \in \mathrm{Hom}_R(M, N)$. Then $f^{-1}(U) \subseteq M$ is large.*

Proof. (1) was proved in Proposition 9.13 (1).
(2) Let $V \subseteq M$ and $f^{-1}(U) \cap V = 0$. Then $f(f^{-1}(U) \cap V) = 0 = ff^{-1}(U) \cap f(V)$, because if $x \in ff^{-1}(U) \cap f(V)$ with $x = f(v)$, then $f(v) \in U$ by $ff^{-1}(U) \subseteq U$. This implies $v \in f^{-1}(U) \cap V$, so $x \in f(f^{-1}(U) \cap V) = 0$. Now this implies $0 = ff^{-1}(U) \cap f(V) = U \cap \mathrm{Im}(f) \cap f(V) = U \cap f(V)$ and thus $f(V) = 0$, because U is large in N. So we have $V \subseteq \mathrm{Ke}(f) \subseteq f^{-1}(U)$. From $f^{-1}(U) \cap V = 0$ we get $V = 0$. Thus $f^{-1}(U)$ is large in M. □

Corollary 9.16. (1) $\mathrm{Rad}({}_RR)M \subseteq \mathrm{Rad}(M)$.
(2) $\mathrm{Soc}({}_RR)M \subseteq \mathrm{Soc}(M)$.

Proof. Let $m \in M$. Then $(R \ni r \mapsto rm \in M) \in \mathrm{Hom}_R(R, M)$. This implies $\mathrm{Rad}({}_RR)m \subseteq \mathrm{Rad}(M)$, $\mathrm{Soc}({}_RR)m \subseteq \mathrm{Soc}(M)$ and that implies the claim. □

Corollary 9.17. $\mathrm{Rad}({}_RR)$ *and* $\mathrm{Soc}({}_RR)$ *are two sided ideals.*

Proposition 9.18. *Let $f \in \mathrm{Hom}_R(M, N)$ and $\mathrm{Ke}(f) \subseteq \mathrm{Rad}(M)$. Then we have*

$$f(\mathrm{Rad}(M)) = \mathrm{Rad}(f(M)).$$

Proof. $\subseteq$: follows from 9.13.
$\supseteq$: Let $f(m) \in \mathrm{Rad}(f(M))$. If $Rm \subseteq M$ is small then $m \in \mathrm{Rad}(M)$ and $f(m) \in f(\mathrm{Rad}(M))$. If $Rm \subseteq M$ is not small then by 9.6 there is a maximal submodule $U \subsetneq M$ with $m \notin U$. We have $Rm + U = M$ and thus $f(U) + Rf(m) = f(M)$. From $f(m) \in \mathrm{Rad}(f(M))$ we get that $Rf(m) \subseteq f(M)$ is small. This implies $f(U) = f(M)$ and thus $U + \mathrm{Ke}(f) = M$. From the assumption $\mathrm{Ke}(f) \subseteq \mathrm{Rad}(M) \subseteq U$ we get $U = M$, a contradiction. □

Corollary 9.19. *Let $N \subseteq M$ be a submodule. Then the following hold*
(1) $(\mathrm{Rad}(M) + N)/N \subseteq \mathrm{Rad}(M/N)$.
(2) $N \subseteq \mathrm{Rad}(M) \Longrightarrow \mathrm{Rad}(M)/N = \mathrm{Rad}(M/N)$.

Proof. (1) $f : M \longrightarrow M/N$ implies $f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(M/N)$ and $f(\mathrm{Rad}(M)) = (\mathrm{Rad}(M) + N)/N$.
(2) From $N = \mathrm{Ke}(f) \subseteq \mathrm{Rad}(M)$ the claim follows. □

Corollary 9.20. $\mathrm{Rad}(M)$ *is the smallest submodule $U \subseteq M$ with* $\mathrm{Rad}(M/U) = 0$.

Proof. We have $\mathrm{Rad}(M/\mathrm{Rad}(M)) = \mathrm{Rad}(M)/\mathrm{Rad}(M) = 0$. If $\mathrm{Rad}(M/U) = 0$ then $\mathrm{Rad}(M) + U/U = 0$ and thus $\mathrm{Rad}(M) + U = U$ so that $\mathrm{Rad}(M) \subseteq U$. □

Lemma 9.21. *If* $\mathrm{Soc}(M) = M$ *then* $\mathrm{Rad}(M) = 0$.

Proof. If $\mathrm{Soc}(M) = M$ holds then M is semisimple. So no submodule is small and thus $\mathrm{Rad}(M) = 0$. □

Lemma 9.22. *Let M be Artinian. Then we have*

$$\mathrm{Rad}(M) = 0 \Longleftrightarrow \mathrm{Soc}(M) = M.$$

Proof. Let M be Artinian and $\mathrm{Rad}(M) = 0$. Let $U \subseteq M$ and N be minimal with $N+U = M$. By 9.4 (2) we have $N \cap U \subseteq M$ small so that $N \cap U = 0$. Thus U is a direct summand of M, M is semisimple and $M = \mathrm{Soc}(M)$. □

Proposition 9.23. *The following are equivalent for M:*

(1) *M is finitely generated and semisimple.*
(2) *M is Artinian and* $\mathrm{Rad}(M) = 0$*.*

Proof. It suffices to show the following: If M is semisimple, then M is finitely generated iff M is Artinian. Let M be semisimple. Then $M = \oplus U_i$ with simple modules U_i. M is finitely generated if and only if the direct sum has only finitely many summands ($\neq 0$). If M is Artinian then the direct sum has only finitely many summands. If the direct sum has only finitely many summands, then each descending chain $N_1 \supseteq N_2 \supseteq \ldots$ in M can only have finitely many direct complements by 8.17. Thus such a chain must become stationary, i.e. M is Artinian. □

Proposition 9.24. (Lemma of Nakayama) *For ${}_RI \subseteq {}_RR$ the following are equivalent:*

(1) $I \subseteq \mathrm{Rad}({}_RR)$*.*
(2) $1 + I$ *contains only right units.*
(3) $1 + I$ *contains only units.*
(4) $1 + IR$ *contains only units.*
(5) $IM = M \Longrightarrow M = 0$ *for all finitely generated modules ${}_RM$.*
(6) $IM + U = M \Longrightarrow U = M$ *for all finitely generated modules ${}_RM$.*
(7) $IM \subseteq \mathrm{Rad}({}_RM)$ *for all finitely generated modules ${}_RM$.*

Proof. (1) $\Longrightarrow$ (2): $\mathrm{Rad}(R) \subseteq R$ is small. Thus $I \subseteq R$ is small. From $R(1 + i) + I = R$ it follows $R(1 + i) = R$. Thus $1 + i$ is a right unit.
(2) $\Longrightarrow$ (3): Let $k(1 + i) = 1$. This implies $ki = 1 - k \in I$ and thus $k - 1 \in I$. So $k = 1 + (k - 1)$ is a right unit. Since k is also a left unit, we get $(1 + i)k = 1$, so that $1 + i$ is a unit.
(3) $\Longrightarrow$ (4): Given $i \in I$ and $r \in R$. Then $1 + ri$ is a unit with inverse $(1 + ri)^{-1}$. Since $(1 - i(1 + ri)^{-1}r)(1 + ir) = 1 + ir - i(1 + ri)^{-1}(r + rir) = 1 + ir - i(1 + ri)^{-1}(1 + ri)r = 1 + ir - ir = 1$ and symmetrically $(1 + ir)(1 - i(1 + ri)^{-1}r) = 1$ we get that $1 + ir$ is a unit. If a is a unit and $i \in I$, $r \in R$ then $a + ir$ is a unit, since $a(1 + a^{-1}ir) = (a + ir)$ is a product of two units by $a^{-1}i \in I$.
If $\sum_{k=1}^{n} i_k r_k \in IR$ then $1 + \sum i_k r_k$ is a unit, since $1 + \sum i_k r_k = (((1 + i_1 r_1) + i_2 r_2) \ldots + i_n r_n)$ and each of the bracketed terms is a unit.
(4) $\Longrightarrow$ (5): Let M be finitely generated and $IM = M$. Let t be the minimal length of a system of generators of $M = Rm_1 + \ldots + Rm_t$. By $IM = M$ each element in M can be represented as a finite sum of the form $\sum i'_j m'_j$; the m'_j can be represented as a linear combination of the m_i. So there are coefficients $i_k r_k \in I$ with $m_1 = \sum_{k=1}^{t} i_k r_k m_k$. This implies $(1 - i_1 r_1)m_1 = \sum_{k=2}^{t} i_k r_k m_k$. Since also $1 - i_1 r_1$ is a unit, we get $m_1 = \sum_{k=2}^{t} (1 - i_1 r_1)^{-1} i_k r_k m_k \in Rm_2 + \ldots + Rm_t$ a contradiction to the minimality of t. So we have $M = 0$.
(5) $\Longrightarrow$ (6): $IM + U = M \Longrightarrow I(M/U) = (IM + U)/U = M/U \Longrightarrow M/U = 0 \Longrightarrow M = U$.

(6) $\Longrightarrow$ (7): IM small in $M \Longrightarrow IM \subseteq \mathrm{Rad}(M)$.
(7) $\Longrightarrow$ (1): $M = R \Longrightarrow IR \subseteq \mathrm{Rad}(_RR)$. $\square$

Corollary 9.25. $\mathrm{Rad}(_RR) = \mathrm{Rad}(R_R)$.

Proof. Let $I = \mathrm{Rad}(_RR)$. Then $1 + I$ consists of units. Since I is a right ideal, we get $I \subseteq \mathrm{Rad}(R_R)$. By symmetry we get $\mathrm{Rad}(_RR) = \mathrm{Rad}(R_R)$. $\square$

Lemma 9.26. *R left Artinian $\Longrightarrow R/\mathrm{Rad}(R)$ semisimple.*

Proof. By 8.12 $R/\mathrm{Rad}(R)$ is Artinian. By 9.20 $\mathrm{Rad}(R/\mathrm{Rad}(R)) = 0$ and by 9.23 $R/\mathrm{Rad}(R)$ is semisimple. $\square$

Lemma 9.27. *R Artinian $\Longrightarrow \mathrm{Rad}(R)$ nilpotent.*

Proof. Let $I := \mathrm{Rad}(R)$. Since R is Artinian, the chain $I \supseteq I^2 \supseteq I^3 \supseteq \ldots \supseteq I^{t+1} = \ldots$ becomes stationary. Assume $I^t \neq 0$. Since also $I^tI \neq 0$ there is a minimal module $K \subseteq I$ w.r.t. $I^tK \neq 0$. So there exists an $x \in K$ with $I^tx \neq 0$, i.e. we have $K = Rx$. Because of $I^tK = I^{t+1}K = I^t(IK) \neq 0$ and $IK \subseteq K$ we get $IK = K$. By the Lemma of Nakayama we get $K = 0$, a contradiction, so $I^t = 0$. $\square$

Theorem 9.28. (Hopkins) *Let $_RR$ be Artinian. Then $_RR$ is Noetherian.*

Proof. Let $I := \mathrm{Rad}(R)$ and $I^{n+1} = 0$. Then I^i/I^{i+1} is an R/I-module and it is Artinian as an R-module. So I^i/I^{i+1} is also Artinian as R/I-module. By 9.26 R/I is semisimple hence I^i/I^{i+1} is also semisimple, i.e. $I^i/I^{i+1} = \oplus_{k\in X} E_k$ with simple R/I-modules E_k. Since I^i/I^{i+1} is Artinian the direct sum is finite hence I^i/I^{i+1} are Noetherian (as R/I-module and as R-module). With the exact sequences $0 \longrightarrow I^{i+1} \longrightarrow I^i \longrightarrow I^i/I^{i+1} \longrightarrow 0$, with $I^{n+1} = 0, I^0 = R$ and with 8.25 we get by induction that R is Noetherian. $\square$

Corollary 9.29. *If $_RI \subseteq {_RR}$ is nilpotent then $I \subseteq \mathrm{Rad}(R)$.*

Proof. Let $I^n = 0$ and $i \in I$. Then $(1 + i) \cdot (1 - i + i^2 - \ldots \pm i^{n+1}) = 1$ hence $(1 + i)$ is a unit. By the Lemma of Nakayama we get $I \subseteq \mathrm{Rad}(R)$. $\square$

Proposition 9.30. *$_RM$ is finitely generated if and only if*

(1) *$\mathrm{Rad}(M) \subseteq M$ is small, and*
(2) *$M/\mathrm{Rad}(M)$ is finitely generated.*

Proof. $\Longrightarrow$: trivial by 9.12 .
$\Longleftarrow$: Let $\{\overline{x}_i = x_i + \mathrm{Rad}(M) | i = 1, \ldots, n\}$ be a set of generating elements of $M/\mathrm{Rad}(M)$. Then $M = Rx_1 + \ldots + Rx_n + \mathrm{Rad}(M)$ which implies by (1) that $M = Rx_1 + \ldots + Rx_n$. $\square$

Corollary 9.31. *M is Noetherian if and only if for all submodules $U \subseteq M$ the following hold:*

(1) *$\mathrm{Rad}(U) \subseteq U$ is small.*
(2) *$U/\mathrm{Rad}(U)$ is finitely generated*

10. Localization

10.1. **Local rings.**

Definition 10.1. Let R be a ring. An element $r \in R$ is called a *non unit*, if r is not a unit. The element r is called *invertible*, if r is a left or a right unit.
R is called a *local ring*, if the sum of any two non invertible elements is a non unit.

Lemma 10.2. *Let r be an idempotent ($r^2 = r$) in a local ring R. Then $r = 0$ or $r = 1$.*

Proof. We have $(1-r)^2 = 1 - 2r + r^2 = 1 - r$. Since $1 = (1-r) + r$ is a unit, r or $1-r$ is invertible. If r is invertible, e.g. by $sr = 1$, then we have $r = sr^2 = sr = 1$. If $1-r$ is invertible e.g. by $s(1-r) = 1$, then we have $1 - r = 1$, thus $r = 0$. □

Lemma 10.3. *Let R be a ring with the unique idempotents 0 and 1. Then each invertible element in R is a unit.*

Proof. Let r be invertible e.g. by $sr = 1$. Then $(rs)^2 = rsrs = rs$, so $rs \in \{0, 1\}$. If $rs = 0$, then we have $1 = (sr)^2 = srsr = 0$, a contradiction. So we have $rs = 1$, i.e. r is a unit. □

Corollary 10.4. *In a local ring R all non units are not invertible.*

Proposition 10.5. *Let R be a local ring. Then the following hold:*

(1) *All non units are not invertible and form a two sided ideal N.*
(2) *N is the only maximal (one sided and two sided) and largest ideal of R.*

Proof. (1) Let N be the set of the non units of R. Since R is local, so non units are not invertible, N is closed w.r.t. to the addition. Given $s \in N$ and $r \in R$. We show that also $rs \in N$ holds. In fact if $rs \notin N$ then rs is a unit, so there is a $t \in R$ with $trs = 1$. Because of 10.3 s is also a unit in contradiction to $s \in N$. Thus N is a two sided ideal.
(2) Obviously we have $N \subsetneqq R$. If $I \subsetneqq R$ and $s \in I$, then $Rs \subsetneqq R$, so s is a non unit and thus $s \in N$. So $I \subseteq N$ holds. □

Proposition 10.6. *R is local, if and only if R possesses a unique maximal (largest) left ideal.*

Proof. $\Longrightarrow$: follows from 10.5.
$\Longleftarrow$: Let N be the only maximal ideal of R. Then $N = \mathrm{Rad}(R)$ is a two sided ideal. Let $r \in R \setminus N$. Then $N + Rr = R$. Since $N = \mathrm{Rad}(R)$ is small in R, we have $Rr = R$, so there is a t with $tr = 1$. If t is a right unit, then also r is a unit by Lemma 8.35. But if t is not a right unit, then $Rt \neq R$, so $Rt \subseteq N$ and thus $t \in N$. Since N is a two sided ideal we have also $1 = tr \in N$, a contradiction. Thus each $r \in R \setminus N$ is a unit. So each non unit lies in N. If x, y are non units, then it follows from $x, y \in N$ that $x + y \in N$ hence $x + y$ is a non unit and thus R is local. □

Lemma 10.7. *Let R be a local ring with maximal ideal $\mathfrak{m} \subsetneqq R$. Let M be a finitely generated module. If $M/\mathfrak{m}M = 0$ then $M = 0$.*

Proof. From $\mathfrak{m} = \mathrm{Rad}(R)$ and $\mathfrak{m}M = M$ it follows that $M = 0$ by the Lemma of Nakayama. □

10.2. **Localization.**

In this section let R be always a commutative ring.
Recall from Basic Algebra: A set S with $\emptyset \subsetneqq S \subset R$ is called *multiplicatively closed*, if

$$\forall s, s' \in S : ss' \in S \qquad \text{and} \qquad 0 \notin S.$$

On $R \times S$ define an equivalence relation by

$$(r, s) \sim (r', s') :\Longleftrightarrow \exists t \in S : tsr' = ts'r.$$

$R[S^{-1}] = S^{-1}R := R \times S/\sim$ is a commutative ring with unit element. The elements are denoted by

$$\frac{r}{s} := \overline{(r,s)}.$$

The map

$$\varphi : R \ni r \mapsto \frac{sr}{s} \in R[S^{-1}]$$

is a homomorphism of rings. It is independent of the choice of $s \in S$. If R has no zero divisors, then φ is injective.

Proposition 10.8. *Let $S \subseteq R$ be a multiplicatively closed set. Let ${}_RM$ be an R-module. Then the relation*

$$(m,s) \sim (m',s') :\Longleftrightarrow \exists t \in S : tsm' = ts'm$$

on $M \times S$ is an equivalence relation. Furthermore

$$S^{-1}M := M \times S/\sim \qquad \textit{with the elements} \qquad \frac{m}{s} := \overline{(m,s)}$$

is an $S^{-1}R$-module with the operations

$$\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + sm'}{ss'} \qquad \textit{and} \qquad \frac{r}{s}\frac{m'}{s} = \frac{rm}{ss'}.$$

Proof. as in Basic Algebra for $S^{-1}R$. □

Problem 10.1. Give a complete proof of Proposition 10.8.

Lemma 10.9. *$\frac{m}{s} = 0$ holds in $S^{-1}M$ if and only if there is a $t \in S$ with $tm = 0$.*

Proof. $(m,s) \sim (0,s') \iff \exists t' \in S : t's'm = 0 \iff \exists t's' \in S : t's'm = 0$. □

Lemma 10.10. (1) *$\varphi_M : M \ni m \mapsto \frac{sm}{s} \in S^{-1}M$ is a homomorphism of groups independent of $s \in S$.*
(2) *φ_M is injective if and only if S contains no zero divisors for M, i.e. $sm = 0 \Longrightarrow m = 0$.*
(3) *φ_M is bijective if and only if the map $M \ni m \mapsto sm \in M$ is bijective for all $s \in S$.*
(4) *φ_R is a homomorphism of rings.*
(5) *$\varphi_M : M \longrightarrow S^{-1}M$ is φ_R-semilinear, i.e. $\varphi_M(rm) = \varphi_R(r)\varphi_M(m)$.*

Proof. (1) $t'(tsm - stm) = 0$ implies $\frac{sm}{s} = \frac{tm}{t}$.
(2) $\varphi_M(m) = 0 \iff \frac{sm}{s} = 0 \iff \exists t \in S : tm = 0$ by 10.9.
(3) φ_M surjective $\iff \forall \frac{m}{s} \in S^{-1}M\ \exists m' \in M : \frac{sm'}{s} = \frac{m}{s} \iff \forall m \in M, s \in S\ \exists m' \in M : sm' = m \iff \forall s \in S : (s\cdot : M \longrightarrow M)$ surjective.
(4) + (5) $\varphi_M(rm) = \frac{s^2rm}{s^2} = \frac{sr}{s}\frac{sm}{s} = \varphi_R(r)\varphi_M(m)$. □

Corollary 10.11. *$S^{-1} : R$-Mod $\longrightarrow S^{-1}R$-Mod is an additive functor.*

Proof. For $f \in \mathrm{Hom}_R(M,N)$ we form $S^{-1}f \in \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ by $S^{-1}f(\frac{m}{s}) := \frac{f(m)}{s}$. In order to show that $S^{-1}f$ is a well defined map assume $(m,s) \sim (m',s')$. Then $ts'm = tsm'$ for a $t \in S$ and thus $ts'f(m) = tsf(m')$. This implies $\frac{f(m)}{s} = \frac{f(m')}{s'}$.
With the usual rules for calculations with fractions one proves that $S^{-1}f$ is an $S^{-1}R$-homomorphism and that $S^{-1}\mathrm{id}_M = \mathrm{id}_{S^{-1}M}$, $S^{-1}(fg) = S^{-1}(f)S^{-1}(g)$ and $S^{-1}(f+g) = S^{-1}(f) + S^{-1}(g)$ hold. □

Proposition 10.12. *The map*

$$\alpha(M) : S^{-1}R \otimes_R M \ni \frac{r}{s} \otimes m \mapsto \frac{rm}{s} \in S^{-1}M$$

defines a functorial isomorphism

$$\alpha : S^{-1}R \otimes_R M \cong S^{-1}M$$

of functors $S^{-1}R \otimes_R$ -, S^{-1}- : R-Mod $\longrightarrow S^{-1}R$-Mod.

Proof. $\alpha(M)$ is a well defined map, for $\widetilde{\alpha}(M) : S^{-1}R \times M \ni (\frac{r}{s}, m) \mapsto \frac{rm}{s} \in S^{-1}M$ is well defined: $(\frac{r}{s}, m) = (\frac{r'}{s'}, m) \Longrightarrow \exists t \in S : ts'r = tsr' \Longrightarrow ts'rm = tsr'm \Longrightarrow \frac{rm}{s} = \frac{r'm}{s'}$. Furthermore $\widetilde{\alpha}(M)$ is obviously additive in both arguments. Finally we have $\widetilde{\alpha}(M)(\frac{r}{s}t, m) = \frac{rtm}{s} = \widetilde{\alpha}(M)(\frac{r}{s}, tm)$, i.e. $\widetilde{\alpha}(M)$ is R-bilinear.
We define an inverse map $\beta(M) : S^{-1}M \ni \frac{m}{s} \mapsto \frac{t}{st} \otimes m \in S^{-1}R \otimes_R M$. The map $\beta(M)$ is well defined, since $\frac{m}{s} = \frac{m'}{s'} \Longrightarrow \exists t' \in S : t's'm = t'sm' \Longrightarrow \frac{t}{st} \otimes m = \frac{ts't'}{sts't'} \otimes m = \frac{t}{sts't'} \otimes s't'm = \frac{t}{sts't'} \otimes st'm = \frac{tst'}{sts't'} \otimes m' = \frac{t}{s't} \otimes m'$.
We have $\beta\alpha = \mathrm{id}$, since $\beta(M)\alpha(M)(\frac{r}{s} \otimes m) = \beta(M)(\frac{rm}{s}) = \frac{t}{st} \otimes rm = \frac{rt}{st} \otimes m = \frac{r}{s} \otimes m$.
Similarly we have $\alpha\beta = \mathrm{id}$, since $\alpha(M)\beta(M)(\frac{m}{s}) = \alpha(M)(\frac{t}{st} \otimes m) = \frac{tm}{st} = \frac{m}{s}$.
α is an $S^{-1}R$-homomorphism, since $\alpha(M)(\frac{r'}{s'}\frac{r}{s} \otimes m) = \alpha(M)(\frac{r'r}{s's} \otimes m) = \frac{r'rm}{s's} = \frac{r'}{s'}\frac{rm}{s} = \frac{r'}{s'}\alpha(M)(\frac{r}{s} \otimes m)$.
α is a functorial homomorphism. In fact the diagram

$$\begin{tikzcd}
S^{-1}R \otimes_R M \arrow[r, "\alpha(M)"] \arrow[d, "S^{-1}R \otimes_R f"'] & S^{-1}M \arrow[d, "S^{-1}f"] \\
S^{-1}R \otimes_R N \arrow[r, "\alpha(N)"'] & S^{-1}N
\end{tikzcd}$$

commutes since we have $S^{-1}f \circ \alpha(M)(\frac{r}{s} \otimes m) = S^{-1}f(\frac{rm}{s}) = \frac{f(rm)}{s} = \frac{rf(m)}{s} = \alpha(N)(\frac{r}{s} \otimes f(m)) = \alpha(N) \circ S^{-1}R \otimes_R f(\frac{r}{s} \otimes m)$. □

Definition 10.13. An additive functor $T : R$-Mod $\longrightarrow S$-Mod is called *exact*, if for each exact sequence

$$\ldots \longrightarrow M_{i-1} \stackrel{f_{i-1}}{\longrightarrow} M_i \stackrel{f_i}{\longrightarrow} M_{i+1} \longrightarrow \ldots$$

the sequence

$$\ldots \longrightarrow T(M_{i-1}) \stackrel{T(f_{i-1})}{\longrightarrow} T(M_i) \stackrel{T(f_i)}{\longrightarrow} T(M_{i+1}) \longrightarrow \ldots$$

is also exact.

Lemma 10.14. *Let* $P \in$ Mod-R. *Then the functor* $P \otimes_R$ - : R-Mod $\longrightarrow$ Ab *preserves exact sequences of the form*

$$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0,$$

i.e. the sequences

$$P \otimes_R M_1 \longrightarrow P \otimes_R M_2 \longrightarrow P \otimes_R M_3 \longrightarrow 0$$

are exact. (The functor $P \otimes_R$ - *is* right exact.*)*

Proof. This follows from Corollary 6.13, Exercise 5.2 (1) and Exercise 6.2. We give a direct proof. Let

$$M_1 \stackrel{f}{\longrightarrow} M_2 \stackrel{g}{\longrightarrow} M_3 \longrightarrow 0$$

be exact. This is equivalent to g surjective, $gf = 0$ and $\mathrm{Ke}(g) \subseteq \mathrm{Im}(f)$. The map $P \otimes_R g$ is surjective, for $\sum p_i \otimes m_{i3} = \sum p_i \otimes g(m_{i2})$ for arbitrary $m_{i3} \in M_3$ and suitable $m_{i2} \in M_2$. Furthermore $(P \otimes_R g)(P \otimes_R f) = P \otimes_R gf = 0$. It remains to show $\mathrm{Ke}(P \otimes_R g) \subseteq \mathrm{Im}(P \otimes_R f)$. Since $\mathrm{Im}(P \otimes_R f) \subseteq \mathrm{Ke}(P \otimes_R g)$, we obtain a homomorphism by the homomorphism theorem

$$\psi : (P \otimes_R M_2)/\mathrm{Im}(P \otimes_R f) \longrightarrow P \otimes_R M_3$$

with $\psi(\overline{p \otimes m_2}) = p \otimes g(m_2)$. Furthermore we define a homomorphism

$$\varphi : P \otimes_R M_3 \longrightarrow (P \otimes_R M_2)/\mathrm{Im}(P \otimes_R f)$$

with $\varphi(p \otimes m_3) := \overline{p \otimes m_2}$ for an $m_2 \in M_2$ with $g(m_2) = m_3$. For this purpose we first define $\widetilde{\varphi} : P \times M_3 \longrightarrow P \otimes_R M_2/\mathrm{Im}(P \otimes_R f)$ by $\widetilde{\varphi}(p, m_3) := \overline{p \otimes m_2}$ for an $m_2 \in M_2$ with $g(m_2) = m_3$. If also $g(m_2') = m_3$ holds then we have $g(m_2 - m_2') = 0$, so there is an $m_1 \in M_1$ with $m_2 - m_2' = f(m_1)$. This implies $\overline{p \otimes m_2} = \overline{p \otimes (m_2' + f(m_1))} = \overline{p \otimes m_2'} + \overline{p \otimes f(m_1)} = \overline{p \otimes m_2'}$, i.e. $\widetilde{\varphi}$ is well defined. It is easy to verify that $\widetilde{\varphi}$ is R-bilinear and thus φ is a well defined homomorphism.
Now $\varphi\psi = \mathrm{id}$ and $\psi\varphi = \mathrm{id}$ hold since $\varphi\psi(\overline{p \otimes m_2}) = \varphi(p \otimes g(m_2)) = \overline{p \otimes m_2}$ and $\psi\varphi(p \otimes m_3) = \psi(\overline{p \otimes m_2}) = p \otimes g(m_2) = p \otimes m_3$. So we get $\mathrm{Ke}(P \otimes_R g) = \mathrm{Ke}(\varphi(P \otimes_R g)) = \mathrm{Ke}(\nu : P \otimes_R M_2 \longrightarrow P \otimes_R M_2/\mathrm{Im}(P \otimes_R f)) = \mathrm{Im}(P \otimes_R f)$. Thus $P \otimes_R M_1 \longrightarrow P \otimes_R M_2 \longrightarrow P \otimes_R M_3 \longrightarrow 0$ is exact. □

Definition 10.15. A module P_R is called *R-flat*, if $P \otimes_R$ - is an exact functor.

Proposition 10.16. *A module P_R is flat if and only if $P \otimes_R$ - preserves monomorphisms, i.e. if for each monomorphism $f : M \longrightarrow N$ the map $P \otimes_R f : P \otimes_R M \longrightarrow P \otimes_R N$ is a monomorphism.*

Proof. If P_R is flat and if $f : M \longrightarrow N$ is a monomorphism then $0 \longrightarrow M \xrightarrow{f} N$ is exact. Consequently $0 \longrightarrow P \otimes_R M \xrightarrow{P \otimes_R f} P \otimes_R N$ is exact and thus $P \otimes_R f : P \otimes_R M \longrightarrow P \otimes_R N$ is a monomorphism.
Assume that $P \otimes_R$ - preserves monomorphisms and that the sequence

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \ldots$$

is exact. Then the sequences

$$0 \longrightarrow \mathrm{Im}(f_{i-1}) \longrightarrow M_i \longrightarrow \mathrm{Im}(f_i) \longrightarrow 0$$

are exact. Since $P \otimes_R$ - preserves monomorphisms, the sequences

$$0 \longrightarrow P \otimes_R \mathrm{Im}(f_{i-1}) \longrightarrow P \otimes_R M_i \longrightarrow P \otimes_R \mathrm{Im}(f_i) \longrightarrow 0$$

are exact. The canonical map $P \otimes_R \mathrm{Im}(f) \longrightarrow \mathrm{Im}(P \otimes_R f)$ is surjective, since each element $\sum p_i \otimes f(m_i) \in \mathrm{Im}(P \otimes_R f)$ is in the image of this map. Observe, however, that this map is in general not injective. The maps $\mathrm{Im}(f) \longrightarrow N$ and thus also $P \otimes_R \mathrm{Im}(f) \longrightarrow P \otimes_R N$ are, however, by hypothesis injective hence $P \otimes_R \mathrm{Im}(f) \longrightarrow \mathrm{Im}(P \otimes_R f)$ is injective and thus bijective.
From the isomorphism $P \otimes_R \mathrm{Im}(f) \cong \mathrm{Im}(P \otimes_R f)$ we thus get the exactness of

$$0 \longrightarrow \mathrm{Im}(P \otimes_R f_{i-1}) \longrightarrow P \otimes_R M_i \longrightarrow \mathrm{Im}(P \otimes_R f_i) \longrightarrow 0.$$

So the sequence

$$\ldots \longrightarrow P \otimes_R M_{i-1} \xrightarrow{P \otimes_R f_{i-1}} P \otimes_R M_i \xrightarrow{P \otimes_R f_i} P \otimes_R M_{i+1} \longrightarrow \ldots$$

is also exact. □

Proposition 10.17. *$S^{-1}R$ is a flat R-module.*

Proof. Let $f : M \longrightarrow N$ be a monomorphism and let $S^{-1}f(\frac{m}{s}) = 0 = \frac{f(m)}{s}$. Then there is a $t \in S$ with $tf(m) = 0 = f(tm)$, so with $tm = 0$. Then $\frac{m}{s} = 0$, hence $S^{-1}f$ is a monomorphism. □

Recall from Basic Algebra:

(1) An ideal $\mathfrak{p} \subseteq R$ is called a *prime ideal* if and only if $\mathfrak{p} \neq R$ and $(rs \in \mathfrak{p} \Longrightarrow r \in \mathfrak{p} \vee s \in \mathfrak{p})$.
(2) If $\mathfrak{m} \subseteq R$ is a maximal ideal, then $\mathfrak{m}$ is a prime ideal.
(3) $\mathfrak{p} \in R$ is a prime ideal if and only if the residue class ring $R/\mathfrak{p}$ is an integral domain.

Lemma 10.18. *Let $\mathfrak{p} \subseteq R$ be an ideal. The following are equivalent*

(1) $\mathfrak{p}$ *is a prime ideal.*
(2) $R \setminus \mathfrak{p}$ *is a multiplicatively closed set.*

Proof. follows immediately from the definition. □

Definition 10.19. Let $\mathfrak{p} \subseteq R$ be a prime ideal and M be an R-module. Then $M_\mathfrak{p} := S^{-1}M$ with $S = R \setminus \mathfrak{p}$ is called the *localization* of the module M at $\mathfrak{p}$.
The set $\mathrm{Spec}(R) := \{\mathfrak{p} \subseteq R | \mathfrak{p} \text{ prime ideal}\}$ is called the *spectrum* of the ring R. The set $\mathrm{Specm}(R) := \{\mathfrak{m} \subseteq R | \mathfrak{m} \text{ maximal ideal}\}$ is called the *maximal spectrum* of the ring R.

Proposition 10.20. *Let M be an R-module, such that $M_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \mathrm{Spec}(R)$. Then $M = 0$.*

Proof. Assume there is an $m \in M$ with $m \neq 0$. Then $I := \mathrm{Ke}(R \ni r \mapsto rm \in M) \subsetneqq R$ is an ideal. Since R is finitely generated there is a maximal ideal $\mathfrak{m}$ with $I \subseteq \mathfrak{m} \subsetneqq R$. Since $M_\mathfrak{m} = 0$, we have $\frac{m}{s} = 0$ in $M_\mathfrak{m}$, hence there is a $t \in R \setminus \mathfrak{m}$ with $tm = 0$. This, however, gives $t \in I \subseteq \mathfrak{m}$, a contradiction. □

Corollary 10.21. *Let $f : M \longrightarrow N$ be given. The following are equivalent*

(1) f *is a mono-(epi- resp. iso-)morphism.*
(2) *For all $\mathfrak{m} \in \mathrm{Spec}(R)$ the localization $f_\mathfrak{m}$ is a mono-(epi- resp. iso-)morphism.*

Proof. (1) $\Longrightarrow$ (2): follows from 10.17 and 10.12.
(2) $\Longrightarrow$ (1): The sequence $0 \longrightarrow \mathrm{Ke}(f) \longrightarrow M \xrightarrow{f} N \longrightarrow \mathrm{Cok}(f) \longrightarrow 0$ is exact. Consequently

$$0 \longrightarrow \mathrm{Ke}(f)_\mathfrak{m} \longrightarrow M_\mathfrak{m} \xrightarrow{f_\mathfrak{m}} N_\mathfrak{m} \longrightarrow \mathrm{Cok}(f)_\mathfrak{m} \longrightarrow 0$$

is exact. Thus we get in particular $\mathrm{Ke}(f)_\mathfrak{m} \cong \mathrm{Ke}(f_\mathfrak{m})$ and $\mathrm{Cok}(f)_\mathfrak{m} \cong \mathrm{Cok}(f_\mathfrak{m})$. Now if $f_\mathfrak{m}$ is a monomorphism for all $\mathfrak{m} \in \mathrm{Specm}(R)$, then we have $\mathrm{Ke}(f)_\mathfrak{m} = 0$ for all $\mathfrak{m}$, hence $\mathrm{Ke}(f) = 0$ and f is a monomorphism. An analogous argument can be used for epimorphisms with $\mathrm{Cok}(f)$. Taken together these two results give the claim for isomorphisms. □

Proposition 10.22. *Let R be a commutative ring and $\mathfrak{p} \subseteq R$ be a prime ideal. Then $R_\mathfrak{p}$ is a local ring.*

Proof. Since $0 \longrightarrow \mathfrak{p} \longrightarrow R \longrightarrow R/\mathfrak{p} \longrightarrow 0$ is exact and $R/\mathfrak{p} \neq 0$, the sequence $0 \longrightarrow \mathfrak{p}_\mathfrak{p} \longrightarrow R_\mathfrak{p} \longrightarrow (R/\mathfrak{p})_\mathfrak{p} \longrightarrow 0$ is exact and $(R/\mathfrak{p})_\mathfrak{p} \neq 0$ since there is no $t \in R \setminus \mathfrak{p}$ with $t \cdot \overline{r} = 0$ for any $\overline{r} \neq 0$ (10.9). So $\mathfrak{p}_\mathfrak{p} \subsetneqq R_\mathfrak{p}$ is a proper ideal. If $\frac{r}{s} \notin \mathfrak{p}_\mathfrak{p}$, then $r \notin \mathfrak{p}$ and $s \notin \mathfrak{p}$ hence $\frac{s}{r}\frac{r}{s} = 1$ and thus $\frac{r}{s}$ is a unit. So the non units of $R_\mathfrak{p}$ form an ideal $\mathfrak{p}_\mathfrak{p}$, i.e. $R_\mathfrak{p}$ is local and $\mathfrak{p}_\mathfrak{p}$ is the maximal ideal. □

Corollary 10.23. *Let $\mathfrak{p} \subseteq R$ be a prime ideal. Then the quotient field $Q(R/\mathfrak{p})$ is isomorphic to $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$.*

Proof. As in the preceding proof $(R/\mathfrak{p})_\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. Furthermore $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is a field, because $\mathfrak{p}_\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$. Furthermore we have

$$(R/\mathfrak{p})_\mathfrak{p} = S^{-1}(R/\mathfrak{p}) = \{\frac{\overline{r}}{s}|\overline{r} \in R/\mathfrak{p}, s \notin \mathfrak{p}\} \cong \{\frac{\overline{r}}{\overline{s}}|\overline{r} \in R/\mathfrak{p}, \overline{s} \in R/\mathfrak{p}, \overline{s} \neq 0\} = Q(R/\mathfrak{p}).$$

□

Proposition 10.24. *Let $_RM$ be a finitely generated module. Let $M/\mathfrak{m}M = 0$ for all maximal ideals $\mathfrak{m} \subseteq R$. Then $M = 0$.*

Proof. $M/\mathfrak{m}M \cong R/\mathfrak{m} \otimes_R M \cong R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m} \otimes_R M \cong M_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} M_\mathfrak{m}$. Since $R_\mathfrak{m}$ is local and $M_\mathfrak{m}$ is finitely generated, it follows that $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \subseteq R$. So we get $M = 0$. □

Corollary 10.25. *Let $f : M \longrightarrow N$ be an R-homomorphism and let N be finitely generated. Let $f/\mathfrak{m}f : M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N$ be an epimorphism for all maximal ideals $\mathfrak{m} \subseteq R$. Then f is an epimorphism.*

Proof. $M \xrightarrow{f} N \longrightarrow Q \longrightarrow 0$ is exact and thus Q is finitely generated. We apply the functor $R/\mathfrak{m} \otimes_R$ - and get the exact sequence $M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N \longrightarrow Q/\mathfrak{m}Q \longrightarrow 0$. Since $f/\mathfrak{m}f$ is an epimorphism, we get $Q/\mathfrak{m}Q = 0$, hence $Q = 0$. So f is an epimorphism. □

11. Monoidal Categories

For our further investigations it is useful to introduce a generalized version of a tensor product. This shall be done in this section. With this generalized notion of a tensor product we also obtain generalizations of the notion of an algebra and of a representation.

Definition 11.1. A *monoidal category* (or *tensor category*) consists of
a category $\mathcal{C}$,
a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called *tensor product*,
an object $I \in \mathcal{C}$, called *unit*,
natural isomorphisms

$$\begin{aligned} &\alpha(A, B, C) : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C),\\ &\lambda(A) : I \otimes A \longrightarrow A,\\ &\rho(A) : A \otimes I \longrightarrow A, \end{aligned}$$

called *associativity, left unit* and *right unit*, such that the following diagrams, called *coherence diagrams* or *constraints*, commute:

$$\begin{tikzcd} ((A\otimes B)\otimes C)\otimes D \arrow[r, "\alpha(A,B,C)\otimes 1"] \arrow[d, "\alpha(A\otimes B,C,D)"] & (A\otimes(B\otimes C))\otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A\otimes((B\otimes C)\otimes D) \arrow[d, "1\otimes\alpha(B,C,D)"'] \\ (A\otimes B)\otimes(C\otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"] & & A\otimes(B\otimes(C\otimes D)) \end{tikzcd}$$

$$\begin{tikzcd} (A\otimes I)\otimes B \arrow[rr, "\alpha(A,I,B)"] \arrow[dr, "\rho(A)\otimes 1"'] & & A\otimes(I\otimes B) \arrow[dl, "1\otimes\lambda(B)"] \\ & A\otimes B & \end{tikzcd}$$

A monoidal category is called *strict*, if the morphisms α, λ, ρ are identities.

Remark 11.2. We define $A_1 \otimes \ldots \otimes A_n := (\ldots(A_1 \otimes A_2) \otimes \ldots) \otimes A_n$.
The coherence theorem of S. MacLane says that all diagrams whose morphisms are formed using α, λ, ρ, identities, inverses, tensor products, and compositions thereof commute. We will not prove this theorem. It implies that each monoidal category can be replaced by (is monoidally equivalent to) a strict monoidal category, that is in all diagrams we may omit the morphisms α, λ, ρ, i. e. replace them by identities. In particular on $A_1 \otimes \ldots \otimes A_n$ there is only one automorphism formed with coherence morphisms, the identity.

Remark 11.3. For each monoidal category $\mathcal{C}$ one can construct the monoidal category $\mathcal{C}^{symm}$ symmetric to $\mathcal{C}$ which coincides with $\mathcal{C}$ as a category, which has the tensor product $A \boxtimes B := B \otimes A$, and coherence morphisms

$$\begin{aligned} &\alpha(C, B, A)^{-1} : (A \boxtimes B) \boxtimes C \longrightarrow A \boxtimes (B \boxtimes C),\\ &\rho(A) : I \boxtimes A \longrightarrow A,\\ &\lambda(A) : A \boxtimes I \longrightarrow A. \end{aligned}$$

Then the coherence diagrams commute again, so that $\mathcal{C}^{symm}$ becomes a monoidal category.

Example 11.4. (1) Let R be an arbitrary ring. The category ${}_R\mathcal{M}_R$ of R-R- bimodules with the tensor product $M \otimes_R N$ is a monoidal category. In particular the $\mathbb{K}$-modules form a monoidal category.
(2a) Let G be a monoid. A vector space V together with a family of subspaces $(V_g | g \in G)$ is called *G-graded*, if $V = \oplus_{g\in G} V_g$ holds.
Let V and W be G-graded vector spaces. A linear map $f : V \longrightarrow W$ is called *of degree* $e \in G$, if for all $g \in G$ $f(V_g) \subseteq W_g$ holds.

The G-graded vector spaces and linear maps of degree $e \in G$ form the category $\mathcal{M}^G$ of *G-graded vector spaces.*
$\mathcal{M}^G$ carries a monoidal structure with the tensor product $V \otimes W$ where the subspaces $(V \otimes W)_g$ are defined by

$$(V \otimes W)_g := \oplus_{h,k \in G, hk=g} V_h \otimes W_k.$$

If G is a group, this can also be written as $(V \otimes W)_g := \oplus_{h \in G} V_h \otimes W_{h^{-1}g}$.
(2b) Let G be a monoid. A family of vector spaces $(V_g | g \in G)$ is called *a G-family of vector spaces.*
Let (V_g) and (W_g) be G-families of vector spaces. A family of linear maps $(f_g : V_g \longrightarrow W_g$ is called *a G-family of linear maps.*
The G-families of vector spaces and G-families of linear maps form the category $(\mathcal{M})^G$ of *G-families of vector spaces.*
$(\mathcal{M})^G$ carries a monoidal structure with the tensor product $(V_g) \otimes (W_g)$ where the subspaces $(V \otimes W)_g$ are defined by

$$(V_g) \otimes (W_g) := ((\oplus_{h,k \in G, hk=g} V_h \otimes W_k)_g).$$

(3) A *(chain) complex of R-modules* over a ring R

$$M = (\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0)$$

consists of a family of R-modules M_i and a family of homomorphisms $\partial_n : M_n \longrightarrow M_{n-1}$ with $\partial_{n-1}\partial_n = 0$. (This chain complex is indexed with $\mathbb{N}_0$. One can also consider chain complexes, that are indexed with $\mathbb{Z}$. See also Section 1.6.)
Let M and N be two chain complexes. A *homomorphism $f : M \longrightarrow N$ of chain complexes* consists of a family of homomorphisms of R-modules $f_n : M_n \longrightarrow N_n$, such that $f_n \partial_{n+1} = \partial_{n+1} f_{n+1}$ for all $n \in \mathbb{N}_0$.
The chain complexes of R-modules with these homomorphisms form the category Comp-R of chain complexes.

Lemma 11.5. *The following diagrams in a monoidal category commute*

$$\begin{tikzcd}
(I \otimes A) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\lambda(A)\otimes 1_B"'] & & I \otimes (A \otimes B) \arrow[dl, "\lambda(A \otimes B)"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "\alpha"] \arrow[dr, "\rho(A \otimes B)"'] & & A \otimes (B \otimes I) \arrow[dl, "1_A \otimes \rho(B)"] \\
& A \otimes B &
\end{tikzcd}$$

and $\lambda(I) = \rho(I)$ holds.

Proof. We first observe that the identity functor $\mathrm{Id}_{\mathcal{C}}$ and the functor $I \otimes$ - are isomorphic by the natural isomorphism λ. In particular we have $I \otimes f = I \otimes g \Longrightarrow f = g$. In the diagram

$$\begin{tikzcd}
((I \otimes I) \otimes A) \otimes B \arrow[rr, "\alpha\otimes 1"] \arrow[dr, "(\rho\otimes 1)\otimes 1"] \arrow[ddd, "\alpha"'] & & (I \otimes (I \otimes A)) \otimes B \arrow[rr, "\alpha"] \arrow[dl, "(1\otimes\lambda)\otimes 1"'] & & I \otimes ((I \otimes A) \otimes B) \arrow[dl, "1\otimes(\lambda\otimes 1)"'] \arrow[ddd, "1\otimes\alpha"] \\
& (I \otimes A) \otimes B \arrow[rr, "\alpha"] \arrow[d, "\alpha"] & & I \otimes (A \otimes B) \arrow[d, "1"] & \\
& I \otimes (A \otimes B) \arrow[rr, "1"] & & I \otimes (A \otimes B) \arrow[dr, "1\otimes\lambda"'] & \\
(I \otimes I) \otimes (A \otimes B) \arrow[ur, "\rho\otimes(1\otimes 1)"'] \arrow[rrrr, "\alpha"] & & & & I \otimes (I \otimes (A \otimes B))
\end{tikzcd}$$

all subdiagrams commute, except for the right hand trapezoid. Since the morphisms are isomorphisms, also the right hand trapezoid commutes, hence the whole diagram commutes. The commutativity of the second diagram follows by analogous conclusions.
Furthermore the following diagram commutes

$$\begin{tikzcd}
I \otimes (I \otimes I) \arrow[dr, "1\otimes\rho"'] & & (I \otimes I) \otimes I \arrow[ll, "\alpha"'] \arrow[rr, "\alpha"] \arrow[dl, "\rho"] \arrow[dr, "\rho\otimes 1"'] & & I \otimes (I \otimes I) \arrow[dl, "1\otimes\lambda"] \\
& I \otimes I \arrow[dr, "\rho"'] & & I \otimes I \arrow[dl, "\rho"] & \\
& & I & &
\end{tikzcd}$$

Here the left hand triangle commutes because of the property shown before, the right hand triangle is given through the axiom. Finally the lower square commutes, since ρ is a natural transformation. In particular we get $\rho(1 \otimes \rho) = \rho(1 \otimes \lambda)$. Since ρ is an isomorphism and $I \otimes \text{-} \cong \mathrm{Id}_{\mathcal{C}}$, it follows $\rho = \lambda$. $\square$

Problem 11.1. For morphisms $f : I \longrightarrow M$ and $g : I \longrightarrow N$ in a monoidal category $\mathcal{C}$ we define $(f \otimes 1 : N \longrightarrow M \otimes N) := (f \otimes 1_I)\rho(I)^{-1}$ and $(1 \otimes g : M \longrightarrow M \otimes N) := (1 \otimes g)\lambda(I)^{-1}$. Show that the diagram

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "g"'] & I \arrow[d, "1 \otimes g"] \\
N \arrow[r, "f \otimes 1"'] & M \otimes N
\end{tikzcd}$$

commutes.

Definition 11.6. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ be monoidal categories. A functor

$$\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$$

together with a natural transformation

$$\xi(M, N) : \mathcal{F}(M) \otimes \mathcal{F}(N) \longrightarrow \mathcal{F}(M \otimes N)$$

and a morphism

$$\xi_0 : I_{\mathcal{D}} \longrightarrow \mathcal{F}(I_{\mathcal{C}})$$

is called *weakly monoidal*, if the following diagrams commute:

$$\begin{tikzcd}
(\mathcal{F}(M) \otimes \mathcal{F}(N)) \otimes \mathcal{F}(P) \arrow[r, "\xi\otimes 1"] \arrow[d, "\alpha"'] & \mathcal{F}(M \otimes N) \otimes \mathcal{F}(P) \arrow[r, "\xi"] & \mathcal{F}((M \otimes N) \otimes P) \arrow[d, "\mathcal{F}(\alpha)"] \\
\mathcal{F}(M) \otimes (\mathcal{F}(N) \otimes \mathcal{F}(P)) \arrow[r, "1\otimes\xi"] & \mathcal{F}(M) \otimes \mathcal{F}(N \otimes P) \arrow[r, "\xi"] & \mathcal{F}(M \otimes (N \otimes P))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes \mathcal{F}(M) \arrow[r, "\xi_0\otimes 1"] \arrow[dr, "\mathcal{F}(\lambda)"'] & \mathcal{F}(I) \otimes \mathcal{F}(M) \arrow[r, "\xi"] & \mathcal{F}(I \otimes M) \arrow[dl, "\lambda"] \\
& \mathcal{F}(M) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(M)\otimes I \arrow[r, "1\otimes\xi_0"] \arrow[dr, "\mathcal{F}(\rho)"'] & \mathcal{F}(M)\otimes\mathcal{F}(I) \arrow[r, "\xi"] & \mathcal{F}(M\otimes I) \arrow[dl, "\rho"] \\
& \mathcal{F}(M). &
\end{tikzcd}$$

In addition if ξ and ξ_0 are isomorphisms then the functor is called *monoidal.* The functor is called *strict monoidal*, if ξ and ξ_0 are identity morphisms.
A natural transformation $\zeta : \mathcal{F} \longrightarrow \mathcal{F}'$ between weakly monoidal functors is called *monoidal*, if the diagrams

$$\begin{tikzcd}
& \mathcal{F}(I) \arrow[dd, "\zeta"] \\
I \arrow[ur, "\xi_0"] \arrow[dr, "\xi_0'"'] & \\
& \mathcal{F}'(I)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(M)\otimes\mathcal{F}(N) \arrow[r, "\xi"] \arrow[d, "\zeta\otimes\zeta"] & \mathcal{F}(M\otimes N) \arrow[d, "\zeta"] \\
\mathcal{F}'(M)\otimes\mathcal{F}'(N) \arrow[r, "\xi'"'] & \mathcal{F}'(M\otimes N)
\end{tikzcd}$$

commute.

In monoidal categories one can generalize notions like algebra and coalgebra. For this purpose we define

Definition 11.7. Let $\mathcal{C}$ be a monoidal category. An *algebra* or a *monoid* in $\mathcal{C}$ is an object A together with a multiplication $\nabla : A \otimes A \longrightarrow A$, that is associative:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "\mathrm{id}\otimes\nabla"] \arrow[d, "\nabla\otimes\mathrm{id}"'] & A\otimes A \arrow[d, "\nabla"] \\
A\otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}$$

and a unit element $\eta : I \longrightarrow A$, for which the following diagram commutes

$$\begin{tikzcd}
I\otimes A \cong A \cong A\otimes I \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, "\eta\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A\otimes A \arrow[d, "\nabla"] \\
A\otimes A \arrow[r, "\nabla"'] & A.
\end{tikzcd}$$

Let A and B algebras in $\mathcal{C}$. A *morphism of algebras* $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$, such that the following diagrams commute:

$$\begin{tikzcd}
A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "\nabla_A"'] & B\otimes B \arrow[d, "\nabla_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
& I \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}$$

Remark 11.8. Obviously the composition of two morphisms of algebras is again a morphism of algebras. Also the identity morphism is a morphism of algebras. Thus we obtain the category $\mathrm{Alg}(\mathcal{C})$ of algebras in $\mathcal{C}$.

Definition 11.9. A *coalgebra* or a *comonoid* in a monoidal category $\mathcal{C}$ is an object C together with a comultiplication $\Delta : C \longrightarrow C \otimes C$, that is coassociative:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes\mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id}\otimes\Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

and a counit $\epsilon : C \longrightarrow I$, for which the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ C \otimes C \arrow[r, "\epsilon\otimes\mathrm{id}"'] & I \otimes C \cong C \cong C \otimes I \end{tikzcd}$$

commutes.
Let C and D be coalgebras. A *morphism of coalgebras* $f : C \longrightarrow D$ is a morphism in $\mathcal{C}$, such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_{\mathcal{C}}"'] & D \arrow[d, "\Delta_{\mathcal{D}}"] \\ C \otimes C \arrow[r, "f\otimes f"'] & D \otimes D \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\epsilon_{\mathcal{C}}"'] & & D \arrow[dl, "\epsilon_{\mathcal{D}}"] \\ & I & \end{tikzcd}$$

commute.

Remark 11.10. Obviously the composition of two morphisms of coalgebras is again a morphism of coalgebras. Also the identity morphism is a morphism of coalgebras. Thus we obtain the category $\mathrm{Coalg}(\mathcal{C})$ of coalgebras in $\mathcal{C}$.

12. Bialgebras and Hopf Algebras

12.1. **Bialgebras.**

Definition 12.1. (1) A *bialgebra* $(B, \nabla, \eta, \Delta, \epsilon)$ consists of an algebra (B, ∇, η) and a coalgebra (B, Δ, ϵ) such that the diagrams

$$\begin{tikzcd}
B \otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "\nabla"'] & B \otimes B \otimes B \otimes B \arrow[dr, "1\otimes\tau\otimes 1"] & \\
& & B \otimes B \otimes B \otimes B \arrow[d, "\nabla\otimes\nabla"] \\
B \arrow[rr, "\Delta"] & & B \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{K} \arrow[dl, "\eta"'] \arrow[dr, "\eta\otimes\eta"] & \\
B \arrow[rr, "\Delta"] & & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\epsilon\otimes\epsilon"'] & & B \arrow[dl, "\epsilon"] \\
& \mathbb{K} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & \mathbb{K} \\
& B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

commute, i.e. Δ and ϵ are homomorphisms of algebras resp. ∇ and η are homomorphisms of coalgebras.
(2) Given bialgebras A and B. A map $f : A \longrightarrow B$ is called a *homomorphism of bialgebras* if it is a homomorphism of algebras and a homomorphism of coalgebras.
(3) The category of bialgebras is denoted by $\mathbb{K}$- Bialg.

Problem 12.1. (1) Let (B, ∇, η) be an algebra and (B, Δ, ε) be a coalgebra. The following are equivalent:

a) $(B, \nabla, \eta, \Delta, \varepsilon)$ is a bialgebra.
b) $\Delta : B \longrightarrow B \otimes B$ and $\varepsilon : B \longrightarrow \mathbb{K}$ are homomorphisms of $\mathbb{K}$-algebras.
c) $\nabla : B \otimes B \longrightarrow B$ and $\eta : \mathbb{K} \longrightarrow B$ are homomorphisms of $\mathbb{K}$-coalgebras.

(2) Let B be a finite dimensional bialgebra over field $\mathbb{K}$. Show that the dual space B^* is a bialgebra.

One of the most important properties of bialgebras B is that the tensor product over $\mathbb{K}$ of two B-modules or two B-comodules is again a B-module.

Proposition 12.2. (1) *Let B be a bialgebra. Let M and N be left B-modules. Then $M \otimes_{\mathbb{K}} N$ is a B-module by the map*

$$B \otimes M \otimes N \xrightarrow{\Delta\otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{1\otimes\tau\otimes 1} B \otimes M \otimes B \otimes N \xrightarrow{\mu\otimes\mu} M \otimes N.$$

(2) *Let B be a bialgebra. Let M and N be left B-comodules. Then $M \otimes_{\mathbb{K}} N$ is a B-comodule by the map*

$$M \otimes N \xrightarrow{\delta\otimes\delta} B \otimes M \otimes B \otimes N \xrightarrow{1\otimes\tau\otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{\nabla\otimes 1} B \otimes M \otimes N.$$

(3) $\mathbb{K}$ *is a B-module by the map* $B \otimes \mathbb{K} \cong B \xrightarrow{\varepsilon} \mathbb{K}$.
(4) $\mathbb{K}$ *is a B-comodule by the map* $\mathbb{K} \xrightarrow{\eta} B \cong B \otimes \mathbb{K}$.

Proof. We give a diagrammatic proof for (1). The associativity law is given by

$$\begin{tikzcd}
B\otimes B\otimes M\otimes N \arrow[r, "1\otimes\Delta\otimes1\otimes1"] \arrow[ddd, "\nabla\otimes1\otimes1"'] & B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes1\otimes\tau\otimes1"] \arrow[d, "\Delta\otimes1\otimes1\otimes1\otimes1"] & B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes\mu\otimes\mu"] \arrow[d, "\Delta\otimes1\otimes1\otimes1\otimes1"] & B\otimes M\otimes N \arrow[d, "\Delta\otimes1\otimes1"] \\
 & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes1\otimes1\otimes\tau\otimes1"] \arrow[d, "1\otimes\tau\otimes1\otimes1\otimes1"] & B\otimes B\otimes B\otimes M\otimes B\otimes N \arrow[r, "1\otimes1\otimes\mu\otimes\mu"] \arrow[d, "1\otimes\tau(B,B\otimes M)\otimes1\otimes1"] & B\otimes B\otimes M\otimes N \arrow[d, "1\otimes\tau\otimes1"] \\
 & B\otimes B\otimes B\otimes B\otimes M\otimes N \arrow[r, "1\otimes1\otimes\tau(B\otimes B,M)\otimes1"] \arrow[d, "\nabla\otimes\nabla\otimes1\otimes1"] & B\otimes B\otimes M\otimes B\otimes B\otimes N \arrow[r, "1\otimes\mu\otimes1\otimes\mu"] \arrow[d, "\nabla\otimes1\otimes\nabla\otimes1"] & B\otimes M\otimes B\otimes N \arrow[d, "\mu\otimes\mu"] \\
B\otimes M\otimes N \arrow[r, "\Delta\otimes1\otimes1"] & B\otimes B\otimes M\otimes N \arrow[r, "1\otimes\tau\otimes1"] & B\otimes M\otimes B\otimes N \arrow[r, "\mu\otimes\mu"] & M\otimes N
\end{tikzcd}$$

The unit law is the commutativity of

$$\begin{tikzcd}
M\otimes N\cong \arrow[dd, "="'] & \mathbb{K}\otimes M\otimes N \arrow[r, "\eta\otimes1\otimes1"] \arrow[d, "\cong"] & B\otimes M\otimes N \arrow[d, "\Delta\otimes1\otimes1"] \\
 & \mathbb{K}\otimes\mathbb{K}\otimes M\otimes N \arrow[r, "\eta\otimes\eta\otimes1\otimes1"] \arrow[d, "1\otimes\tau\otimes1"] & B\otimes B\otimes M\otimes N \arrow[d, "1\otimes\tau\otimes1"] \\
M\otimes N\cong & \mathbb{K}\otimes M\otimes\mathbb{K}\otimes N \arrow[r, "\eta\otimes1\otimes\eta\otimes1"] \arrow[dr, "1"'] & B\otimes M\otimes B\otimes N \arrow[d, "\mu\otimes\mu"] \\
 & & M\otimes N
\end{tikzcd}$$

The corresponding properties for comodules follows from the dualized diagrams. The module and comodule properties of $\mathbb{K}$ are easily checked. ☐

Problem 12.2. (1) Let B be a bialgebra and $\mathcal{M}_B$ be the category of right B- modules. Show that $\mathcal{M}_B$ is a monoidal category.
(2) Let B a bialgebra and $\mathcal{M}^B$ be the category of right B- comodules. Show that $\mathcal{M}^B$ is a monoidal category.

Definition 12.3. (1) Let $(B,\nabla,\eta,\Delta,\epsilon)$ be a bialgebra. Let A be a left B-module with structure map $\mu : B\otimes A \longrightarrow A$. Let furthermore (A,∇_A,η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-modules. Then (A,∇_A,η_A,μ) is called a *B-module algebra.*
(2) Let $(B,\nabla,\eta,\Delta,\epsilon)$ be a bialgebra. Let C be a left B-module with structure map $\mu : B\otimes C \longrightarrow C$. Let furthermore $(C,\Delta_C,\varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B-modules. Then $(C,\Delta_C,\varepsilon_C,\mu)$ is called a *B-module coalgebra.*
(3) Let $(B,\nabla,\eta,\Delta,\epsilon)$ be a bialgebra. Let A be a left B-comodule with structure map $\delta : A \longrightarrow B\otimes A$. Let furthermore (A,∇_A,η_A) be an algebra such that ∇_A and η_A are homomorphisms of B-comodules. Then $(A,\nabla_A,\eta_A,\delta)$ is called a *B-comodule algebra.*
(4) Let $(B,\nabla,\eta,\Delta,\epsilon)$ be a bialgebra. Let C be a left B-comodule with structure map $\delta : C \longrightarrow B\otimes C$. Let furthermore $(C,\Delta_C,\varepsilon_C)$ be a coalgebra such that Δ_C and ε_C are homomorphisms of B-comodules. Then $(C,\Delta_C,\varepsilon_C,\delta)$ is called a *B-comodule coalgebra.*

Remark 12.4. If $(C,\Delta_C,\varepsilon_C)$ is a $\mathbb{K}$-coalgebra and (C,μ) is a B-module, then $(C,\Delta_C,\varepsilon_C,\mu)$ is a B-module coalgebra iff μ is a homomorphism of $\mathbb{K}$-coalgebras.
If (A,∇_A,η_A) is a $\mathbb{K}$-algebra and (A,δ) is a B-comodule, then $(A,\nabla_A,\eta_A,\delta)$ is a B-comodule algebra iff δ is a homomorphism of $\mathbb{K}$-algebras.

Similar statements for module algebras or comodule coalgebras do *not* hold.

Problem 12.3. (1) Let B be a bialgebra. Describe what an algebra A and a coalgebra C are in the monoidal category $\mathcal{M}_B$ (in the sense of section 11).
(2) Let B be a bialgebra. Describe what an algebra A and a coalgebra C are in the monoidal category $\mathcal{M}^B$ (in the sense of section 11).

Remark 12.5. The notions of a bialgebra, a comodule algebra, and a Hopf algebra cannot be generalized in the usual way to an arbitrary monoidal category, since we need the multiplication on the tensor product of two algebras. To define this we need the commutation, exchange morphism, or flip of two tensor factors. Such exchange morphisms are known under the name of *symmetry* or *quasisymmetry* (*braiding*). They will be discussed later on.

12.2. **Hopf Algebras.** The difference between a monoid and a group lies in the existence of an additional map $S : G \ni g \mapsto g^{-1} \in G$ for a group G that allows forming inverses. This map satisfies the equation $S(g)g = 1$ or in a diagrammatic form

$$\begin{tikzcd} G \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \{1\} \arrow[r, "1"] & G \\ G \times G \arrow[rr, "S\times\mathrm{id}"] & & G \times G \arrow[u, "\mathrm{mult}"'] \end{tikzcd}$$

We want to carry this property over to bialgebras B instead of monoids. An "inverse map" shall be a morphism $S : B \longrightarrow B$ with a similar property. This will be called a Hopf algebra.

Definition 12.6. A *left Hopf algebra* H is a bialgebra H together with a *left antipode* $S : H \longrightarrow H$, i.e. a $\mathbb{K}$-module homomorphism S such that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

Symmetrically we define a *right Hopf algebra* H. A *Hopf algebra* is a left and right Hopf algebra. The map S is called a (left, right, two-sided) *antipode*.

Using the Sweedler notation (2.20) the commutative diagram above can also be expressed by the equation

$$\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$$

for all $a \in H$. Observe that we do not require that $S : H \longrightarrow H$ is an algebra homomorphism.

Problem 12.4. (1) Let H be a bialgebra and $S \in \mathrm{Hom}(H, H)$. Then S is an antipode for H (and H is a Hopf algebra) iff S is a two sided inverse for id in the algebra $(\mathrm{Hom}(H, H), *, \eta\varepsilon)$ (see 2.21). In particular S is uniquely determined.
(2) Let H be a Hopf algebra. Then S is an antihomomorphism of algebras and coalgebras i.e. S "inverts the order of the multiplication and the comultiplication".
(3) Let H and K be Hopf algebras and let $f : H \longrightarrow K$ be a homomorphism of bialgebras. Then $fS_H = S_K f$, i.e. f is compatible with the antipode.

Definition 12.7. Because of Problem 12.4 (3) every homomorphism of bialgebras between Hopf algebras is compatible with the antipodes. So we define a *homomorphism of Hopf algebras* to be a homomorphism of bialgebras. The category of Hopf algebras will be denoted by $\mathbb{K}$-Hopf.

Proposition 12.8. *Let H be a bialgebra with an algebra generating set X. Let $S : H \longrightarrow H^{op}$ be an algebra homomorphism such that $\sum S(x_{(1)})x_{(2)} = \eta\varepsilon(x)$ for all $x \in X$. Then S is a left antipode of H.*

Proof. Assume $a, b \in H$ such that $\sum S(a_{(1)})a_{(2)} = \eta\varepsilon(a)$ and $\sum S(b_{(1)})b_{(2)} = \eta\varepsilon(b)$. Then

$$\begin{aligned}\sum S((ab)_{(1)})(ab)_{(2)} &= \sum S(a_{(1)}b_{(1)})a_{(2)}b_{(2)} = \sum S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)} \\ &= \sum S(b_{(1)})\eta\varepsilon(a)b_{(2)} = \eta\varepsilon(a)\eta\varepsilon(b) = \eta\varepsilon(ab).\end{aligned}$$

Since every element of H is a finite sum of finite products of elements in X, for which the equality holds, this equality extends to all of H by induction. $\square$

Example 12.9. (1) Let V be a vector space and $T(V)$ the tensor algebra over V. We have seen in Problem 2.2 that $T(V)$ is a bialgebra and that V generates $T(V)$ as an algebra. Define $S : V \longrightarrow T(V)^{op}$ by $S(v) := -v$ for all $v \in V$. By the universal property of the tensor algebra this map extends to an algebra homomorphism $S : T(V) \longrightarrow T(V)^{op}$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $T(V)$ is a Hopf algebra by the preceding proposition.
(2) Let V be a vector space and $S(V)$ the symmetric algebra over V (that is commutative). We have seen in Problem 2.3 that $S(V)$ is a bialgebra and that V generates $S(V)$ as an algebra. Define $S : V \longrightarrow S(V)$ by $S(v) := -v$ for all $v \in V$. S extends to an algebra homomorphism $S : S(V) \longrightarrow S(V)$. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ we have $\sum S(v_{(1)})v_{(2)} = \nabla(S \otimes 1)\Delta(v) = -v + v = 0 = \eta\varepsilon(v)$ for all $v \in V$, hence $S(V)$ is a Hopf algebra by the preceding proposition.

Example 12.10. (**Group Algebras**) For each algebra A we can form the *group of units* $U(A) := \{a \in A | \exists a^{-1} \in A\}$ with the multiplication of A as composition of the group. Then U is a covariant functor $U : \mathbb{K}\text{-Alg} \longrightarrow \text{Gr}$. This functor leads to the following universal problem.
Let G be a group. An algebra $\mathbb{K}G$ together with a group homomorphism $\iota : G \longrightarrow U(\mathbb{K}G)$ is called a (the) *group algebra of* G, if for every algebra A and for every group homomorphism $f : G \longrightarrow U(A)$ there exists a unique homomorphism of algebras $g : \mathbb{K}G \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathbb{K}G) \arrow[d, "g"] \\ & U(A). \end{tikzcd}$$

The group algebra $\mathbb{K}G$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of G. The map $\iota : G \longrightarrow U(\mathbb{K}G) \subseteq \mathbb{K}G$ is injective and the image of G in $\mathbb{K}G$ is a basis.
The group algebra can be constructed as the free vector space $\mathbb{K}G$ with basis G and the algebra structure of $\mathbb{K}G$ is given by $\mathbb{K}G \otimes \mathbb{K}G \ni g \otimes h \mapsto gh \in \mathbb{K}G$ and the unit $\eta : \mathbb{K} \ni \alpha \mapsto \alpha e \in \mathbb{K}G$.
The group algebra $\mathbb{K}G$ is a Hopf algebra. The comultiplication is given by the diagram

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "\Delta"] \\ & \mathbb{K}G \otimes \mathbb{K}G \end{tikzcd}$$

with $f(g) := g \otimes g$ which defines a group homomorphism $f : G \longrightarrow U(\mathbb{K}G \otimes \mathbb{K}G)$. The counit is given by

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "\varepsilon"] \\ & \mathbb{K} \end{tikzcd}$$

where $f(g) = 1$ for all $g \in G$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : \mathbb{K}G \longrightarrow (\mathbb{K}G)^{op}$ by

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{K}G \arrow[d, "S"] \\ & (\mathbb{K}G)^{op} \end{tikzcd}$$

with $f(g) := g^{-1}$ which is a group homomorphism $f : G \longrightarrow U((\mathbb{K}G)^{op})$. Then one shows with Proposition 12.8 that $\mathbb{K}G$ is a Hopf algebra.

Proposition 12.11. *The following three monoidal categories are monoidally equivalent*
(1) *the category $\mathcal{M}^G$ of G-graded vector spaces $\mathcal{M}^G$,*
(2) *the category of G-families of vector spaces $(\mathcal{M})^G$,*
(3) *the monoidal category of $\mathbb{K}G$-comodules $\mathcal{M}^{\mathbb{K}G}$.*

Proof. We only indicate the construction for the equvalence between (1) and (3).
For a G-graded vector space V one constructs the $\mathbb{K}G$-comodule V with the structure map $\delta : V \longrightarrow V \otimes \mathbb{K}G$, $\delta(v) := v \otimes g$ for all $v \in V_g$ and for all $g \in G$. Conversely let $V, \delta : V \longrightarrow V \otimes \mathbb{K}G$ be a $\mathbb{K}G$-comodule. Then one constructs the graded vector space V with graded (homogenous) components $V_g := \{v \in V | \delta(v) = v \otimes g\}$. It is easy to verify, that this is an equivalence of categories.
Since $\mathbb{K}G$ is a bialgebra, the category of $\mathbb{K}G$-comodules is a monoidal category by Exercise 12.2 (2). One checks that under the equivalence between $\mathcal{M}^G$ and $\mathcal{M}^{\mathbb{K}G}$ tensor products are mapped into corresponding tensor products so that we have a monoidal equivalence. □

Example 12.12. The following is a bialgebra $B = \mathbb{K}\langle x, y\rangle/I$, where I is generated by $x^2, xy + yx$. The diagonal is $\Delta(y) = y \otimes y$, $\Delta(x) = x \otimes y + 1 \otimes x$ and the counit is $\epsilon(y) = 1, \epsilon(x) = 0$.

Proposition 12.13. *The monoidal category Comp-$\mathbb{K}$ of chain complexes over $\mathbb{K}$ is monoidally equivalent to the category of B-comodules $\mathcal{M}^B$ with B as in the preceding example.*

Proof. We use the following construction. A chain complex M is mapped to the B-comodule $M = \oplus_{i \in \mathbb{N}} M_i$ with the structure map $\delta : M \longrightarrow M \otimes B$, $\delta(m) := \sum m \otimes y^i + \partial_i(m) \otimes xy^{i-1}$ for all $m \in M_i$ and for all $i \in \mathbb{N}$ resp. $\delta(m) := m \otimes 1$ for $m \in M_0$. Conversely if $M, \delta : M \longrightarrow M \otimes B$ is a B-comodule, then one associates with it the vector spaces $M_i := \{m \in M | \exists m' \in M[\delta(m) = m \otimes y^i + m' \otimes xy^{i-1}\}$ and the linear maps $\partial_i : M_i \longrightarrow M_{i-1}$ with $\partial_i(m) := m'$ for $\delta(m) = m \otimes y^i + m' \otimes xy^{i-1}$. One checks that this is an equivalence of categories. By Exercise 12.5 this is a monoidal equivalence. □

Problem 12.5. (1) Give a detailed proof that $\mathcal{M}^G$ and $\mathcal{M}^{\mathbb{K}G}$ are equivalent as monoidal categories.
(2) Give a detailed proof that Comp-$\mathbb{K}$ and $\mathcal{M}^B$ with B as in the preceding Proposition 12.13 are equivalent categories. Since $\mathcal{M}^B$ is a monoidal category, the tensor product can be transported to Comp-$\mathbb{K}$. Describe the tensor product in the category Comp-B.

You may use the following arguments:
Let $m \in M \in \mathcal{M}^B$. Since y^i, xy^i form a basis of B we have $\delta(m) = \sum_i m_i \otimes y^i + \sum_i m'_i \otimes xy^i$. Apply $(\delta \otimes 1)\delta = (1 \otimes \Delta)\delta$ to this equation and compare coefficients then $\delta(m_i) = m_i \otimes y^i + m'_{i-1} \otimes xy^{i-1}$, $\delta(m'_i) = m'_i \otimes y^i$. Hence for each $m_i \in M_i$ there is exactly one $\partial(m_i) \in M_{i-1}$, so that

$$\delta(m_i) = m_i \otimes y^i + \partial(m_i) \otimes xy^{i-1}, \quad \delta(m'_i) = m'_i \otimes y^i.$$

Apply furthermore $(\epsilon \otimes 1)\delta(m) = m$ then you get $m = \sum m_i$ with $m_i \in M_i$, so $M = \bigoplus_{i \in \mathbb{N}} M_i$. Thus define $\partial : M_i \longrightarrow M_{i-1}$ by the above equation. Furthermore one has $\partial^2 = 0$. The converse construction can be found in the proof of the proposition.
(3) A cochain complex over $\mathbb{K}$ has the form

$$M = (M^0 \xrightarrow{\partial_0} M^1 \xrightarrow{\partial_1} M^2 \xrightarrow{\partial_2} \ldots)$$

with $\partial_{i+1}\partial_i = 0$. Show that the category $\mathbb{K}$-Cocomp of cochain complexes is equivalent to ${}^B\mathcal{M}$, where B is chosen as in Example 12.12.
(4) Show that the bialgebra B from Example 12.12 is not a Hopf algebra.
(5) Find a bialgebra B' such that the category of complexes $\ldots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \ldots$ and $\mathcal{M}^{B'}$ are monoidally equivalent. Show that B' is a Hopf algebra.

The example $\mathbb{K}G$ of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a group. We will use and study these elements later on in the course on Non Commutative Geometry and Quantum Groups.

Definition 12.14. Let H be a Hopf algebra. An element $g \in H, g \neq 0$ is called a *group-like element* if

$$\Delta(g) = g \otimes g.$$

Observe that $\varepsilon(g) = 1$ for each group-like element g in a Hopf algebra H. In fact we have $g = \nabla(\varepsilon \otimes 1)\Delta(g) = \varepsilon(g)g \neq 0$ hence $\varepsilon(g) = 1$. If the base ring is not a field then one adds this property to the definition of a group-like element.

Problem 12.6. (1) Let $\mathbb{K}$ be a field. Show that an element $x \in \mathbb{K}G$ satisfies $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ if and only if $x = g \in G$.
(2) Show that the group-like elements of a Hopf algebra form a group under multiplication of the Hopf algebra.

Example 12.15. (**Universal Enveloping Algebras**) A *Lie algebra* consists of a vector space $\mathfrak{g}$ together with a (linear) multiplication $\mathfrak{g} \otimes \mathfrak{g} \ni x \otimes y \mapsto [x, y] \in \mathfrak{g}$ such that the following laws hold:

$$\begin{aligned} &[x, y] = -[y, x], \\ &[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{(Jacobi identity).} \end{aligned}$$

A *homomorphism of Lie algebras* $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear map f such that $f([x, y]) = [f(x), f(y)]$. Thus Lie algebras form a category $\mathbb{K}$-Lie.
An important example is the Lie algebra associated with an associative algebra (with unit). If A is an algebra then the vector space A with the Lie multiplication

$$[x, y] := xy - yx$$

is a Lie algebra denoted by A^L. This construction of a Lie algebra defines a covariant functor $\text{-}^L : \mathbb{K}\text{-Alg} \longrightarrow \mathbb{K}\text{-Lie}$. This functor leads to the following universal problem.
Let $\mathfrak{g}$ be a Lie algebra. An algebra $U(\mathfrak{g})$ together with a Lie algebra homomorphism $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is called a (the) *universal enveloping algebra of* $\mathfrak{g}$, if for every algebra A and

for every Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow A^L$ there exists a unique homomorphism of algebras $g : U(\mathfrak{g}) \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g})^L \arrow[d, "g"] \\ & A^L. \end{tikzcd}$$

The universal enveloping algebra $U(\mathfrak{g})$ is (if it exists) unique up to isomorphism. It is generated as an algebra by the image of $\mathfrak{g}$.
The universal enveloping algebra can be constructed as $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$ where $T(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \ldots$ is the tensor algebra. The map $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^L$ is injective.
The universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra. The comultiplication is given by the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "\Delta"] \\ & U(\mathfrak{g}) \otimes U(\mathfrak{g}) \end{tikzcd}$$

with $f(x) := x \otimes 1 + 1 \otimes x$ which defines a Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^L$.
The counit is given by

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "\varepsilon"] \\ & \mathbb{K} \end{tikzcd}$$

with $f(x) = 0$ for all $x \in \mathfrak{g}$. One shows easily by using the universal property, that Δ is coassociative and has counit ε. Define an algebra homomorphism $S : U(\mathfrak{g}) \longrightarrow (U(\mathfrak{g}))^{op}$ by

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, "S"] \\ & (U(\mathfrak{g}))^{op} \end{tikzcd}$$

with $f(x) := -x$ which is a Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow (U(\mathfrak{g})^{op})^L$. Then one shows with Proposition 12.8 that $U(\mathfrak{g})$ is a Hopf algebra.
(Observe, that the meaning of U in this example and the previous example (group of units, universal enveloping algebra) is totally different, in the first case U can be applied to an algebra and gives a group, in the second case U can be applied to a Lie algebra and gives an algebra.)

The preceding example of a Hopf algebra gives rise to the definition of particular elements in arbitrary Hopf algebras, that share certain properties with elements of a Lie algebra.
We will use and study these elements later on in the course on Non Commutative Geometry and Quantum Groups.

Definition 12.16. Let H be a Hopf algebra. An element $x \in H$ is called a *primitive element* if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Let $g \in H$ be a group-like element. An element $x \in H$ is called a *skew primitive or g-primitive element* if

$$\Delta(x) = x \otimes 1 + g \otimes x.$$

Problem 12.7. Show that the set of primitive elements $P(H) = \{x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x\}$ of a Hopf algebra H is a Lie subalgebra of H^L.

Proposition 12.17. *Let H be a Hopf algebra with antipode S. The following are equivalent:*
(1) $S^2 =$id.
(2) $\sum S(a_{(2)})a_{(1)} = \eta\varepsilon(a)$ for all $a \in H$.
(3) $\sum a_{(2)}S(a_{(1)}) = \eta\varepsilon(a)$ for all $a \in H$.

Proof. Let $S^2 =$id. Then

$$\begin{aligned} \sum S(a_{(2)})a_{(1)} &= S^2(\sum S(a_{(2)})a_{(1)}) = S(\sum S(a_{(1)})S^2(a_{(2)})) \\ &= S(\sum S(a_{(1)})a_{(2)}) = S(\eta\varepsilon(a)) = \eta\varepsilon(a) \end{aligned}$$

by using Problem 12.4.
Conversely assume that (2) holds. Then

$$\begin{aligned} S * S^2(a) &= \sum S(a_{(1)}S^2(a_{(2)}) = S(\sum S(a_{(2)})a_{(1)} \\ &= S(\eta\varepsilon(a)) = \eta\varepsilon(a). \end{aligned}$$

Thus S^2 and id are inverses of S in the convolution algebra $\mathrm{Hom}(H, H)$, hence $S^2 = \mathrm{id}$. Analogously one shows that (1) and (3) are equivalent. □

Corollary 12.18. *If H is a commutative Hopf algebra or a cocommutative Hopf algebra with antipode S, then $S^2 =$id.*

Remark 12.19. Kaplansky: Ten conjectures on Hopf algebras
In a set of lecture notes on bialgebras based on a course given at Chicago university in 1973, made public in 1975, I. Kaplansky formulated ten conjectures on Hopf algebras that have been the aim of intensive research.

(1) If C is a Hopf subalgebra of the Hopf algebra B then B is a free left C-module.
(Yes, if H is finite dimensional [Nichols-Zoeller]; No for infinite dimensional Hopf algebras [Oberst-Schneider]; $B : C$ is not necessarily faithfully flat [Schauenburg])

(2) Call a coalgebra C *admissible* if it admits an algebra structure making it a Hopf algebra. The conjecture states that C is admissible if and only if every finite subset of C lies in a finite-dimensional admissible subcoalgebra.
(Remarks.
(a) Both implications seem hard.
(b) There is a corresponding conjecture where "Hopf algebra" is replaced by "bialgebra".
(c) There is a dual conjecture for locally finite algebras.)
(No results known.)

(3) A Hopf algebra of characteristic 0 has no non-zero central nilpotent elements.
(First counter example given by [Schmidt-Samoa]. If H is unimodular and not semisimple, e.g. a Drinfel'd double of a not semisimple finite dimensional Hopf algebra, then the integral Λ satisfies $\Lambda \neq 0$, $\Lambda^2 = \varepsilon(\Lambda)\Lambda = 0$ since $D(H)$ is not semisimple, and $a\Lambda = \varepsilon(a)\Lambda = \Lambda\varepsilon(a) = \Lambda a$ since $D(H)$ is unimodular [Sommerhäuser].)

(4) (Nichols). Let x be an element in a Hopf algebra H with antipode S. Assume that for any a in H we have

$$\sum b_i x S(c_i) = \varepsilon(a)x$$

where $\Delta a = \sum b_i \otimes c_i$. Conjecture: x is in the center of H.

(Yes, since $ax = \sum a_{(1)} x \varepsilon(a_{(2)}) = \sum a_{(1)} x S(a_{(2)}) a_{(3)}) = \sum \varepsilon(a_{(1)}) x a_{(2)} = xa$.)

In the remaining six conjectures H is a finite-dimensional Hopf algebra over an algebraically closed field.

(5) If H is semisimple on either side (i.e. either H or the dual H^* is semisimple as an algebra) the square of the antipode is the identity.

(Yes if $\mathrm{char}(\mathbb{K}) = 0$ [Larson-Radford], yes if $\mathrm{char}(\mathbb{K})$ is large [Sommerhäuser])

(6) The size of the matrices occurring in any full matrix constituent of H divides the dimension of H.

(Yes if Hopf algebra is defined over $\mathbb{Z}$ [Larson]; in general not known; work by [Montgomery-Witherspoon], [Zhu], [Gelaki])

(7) If H is semisimple on both sides the characteristic does not divide the dimension.

(Larson-Radford)

(8) If the dimension of H is prime then H is commutative and cocommutative.

(Yes in characteristic 0 [Zhu: 1994])

Remark. Kac, Larson, and Sweedler have partial results on 5 – 8.

(Was also proved by [Kac])

In the two final conjectures assume that the characteristic does not divide the dimension of H.

(9) The dimension of the radical is the same on both sides.

(Counterexample by [Nichols]; counterexample in Frobenius-Lusztig kernel of $U_q(sl(2))$ [Schneider])

(10) There are only a finite number (up to isomorphism) of Hopf algebras of a given dimension.

(Yes for semisimple, cosemisimple Hopf algebras: Stefan 1997)

(Counterexamples: [Andruskiewitsch, Schneider], [Beattie, others] 1997)

13. Quickies in Advanced Algebra

I. Allgemeine Modultheorie.

(1) Sei R ein Ring. Dann ist ${}_RR$ ein R-Links-Modul.

(2) Sei M eine abelsche Gruppe und $\mathrm{End}(M)$ der Endomorphismenring von M. Dann ist M ein $\mathrm{End}(M)$-Modul.

(3) $\{(\bar{1},\bar{0}),(\bar{0},\bar{1})\}$ ist eine Erzeugendenmenge für den $\mathbb{Z}$-Modul $\mathbb{Z}/(2)\times\mathbb{Z}/(3)$.

(4) $\{(\bar{1},\bar{1})\}$ ist eine Erzeugendenmenge für den $\mathbb{Z}$-Modul $\mathbb{Z}/(2)\times\mathbb{Z}/(3)$.

(5) ${}_{\mathbb{Z}}\mathbb{Z}/(n)$ besitzt als Modul keine Basis, d.h. dieser Modul ist nicht frei.

(6) Sei $V=\bigoplus_{i=0}^{\infty}Kb_i$ ein abzählbar unendlich dimensionaler Vektorraum über dem Körper K. Seien $p,q,a,b\in\mathrm{Hom}(V,V)$ definiert durch

$$\begin{aligned}
p(b_i) &:= b_{2i},\\
q(b_i) &:= b_{2i+1},\\
a(b_i) &:= \begin{cases} b_{i/2}, & \text{wenn } i \text{ gerade ist, und}\\ 0, & \text{wenn } i \text{ ungerade ist.}\end{cases}\\
b(b_i) &:= \begin{cases} b_{i-1/2}, & \text{wenn } i \text{ ungerade ist, und}\\ 0, & \text{wenn } i \text{ gerade ist.}\end{cases}
\end{aligned}$$

Zeige $pa+qb=\mathrm{id}_V$, $ap=bq=\mathrm{id}$, $aq=bp=0$.
Zeige, daß für $R=\mathrm{End}_K(V)$ gilt ${}_RR=Ra\oplus Rb$ und $R_R=pR\oplus qR$.

(7) Sind $\{(0,\ldots,a,\ldots,0)|a\in K_n\}$ und $\{(a,0,\ldots,0)|a\in K_n\}$ isomorph als $M_n(K)$-Moduln?

(8) Zu jedem Modul P gibt es einen Modul Q mit $P\oplus Q\cong Q$.

(9) Welche der folgenden Aussagen ist wahr?
  (a) $P_1\oplus Q=P_2\oplus Q\Longrightarrow P_1=P_2$?
  (b) $P_1\oplus Q=P_2\oplus Q\Longrightarrow P_1\cong P_2$?
  (c) $P_1\oplus Q\cong P_2\oplus Q\Longrightarrow P_1\cong P_2$?

(10) $\mathbb{Z}/(2)\oplus\mathbb{Z}/(6)\oplus\mathbb{Z}/(6)\oplus\ldots\cong\mathbb{Z}/(6)\oplus\mathbb{Z}/(6)\oplus\mathbb{Z}/(6)\oplus\ldots$.

(11) $\mathbb{Z}/(2)\oplus\mathbb{Z}/(4)\oplus\mathbb{Z}/(4)\oplus\ldots\not\cong\mathbb{Z}/(4)\oplus\mathbb{Z}/(4)\oplus\mathbb{Z}/(4)\oplus\ldots$.

(12) Man finde zwei abelsche Gruppen P und Q, so daß P isomorph zu einer Untergruppe von Q ist und Q isomorph zu einer Untergruppe von P ist und $P\not\cong Q$ gilt.

II. Tensorprodukte

(1) In $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}$ gilt $1\otimes i-i\otimes 1=0$.
In $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ gilt $1\otimes i-i\otimes 1\neq 0$.

(2) Für jeden R-Modul gilt $R\otimes_R M\cong M$.

(3) Sei der $\mathbb{Q}$-Vektorraum $V=\mathbb{Q}^n$ gegeben.
  (a) Bestimme $\dim_{\mathbb{R}}(\mathbb{R}\otimes_{\mathbb{Q}}V)$.
  (b) Gib explizit einen Isomorphismus $\mathbb{R}\otimes_{\mathbb{Q}}V\cong\mathbb{R}^n$ an.

(4) Sei V ein $\mathbb{Q}$-Vektorraum und W ein $\mathbb{R}$-Vektorraum.
  (a) $\mathrm{Hom}_{\mathbb{R}}(.\mathbb{R}_{\mathbb{Q}},.W)\cong W$ in $\mathbb{Q}$-Mod.
  (b) $\mathrm{Hom}_{\mathbb{Q}}(.V,.W)\cong\mathrm{Hom}_{\mathbb{R}}(.\mathbb{R}\otimes_{\mathbb{Q}}V,.W)$.
  (c) Sei $\dim_{\mathbb{Q}}V<\infty$ und $\dim_{\mathbb{R}}W<\infty$. Wie kann man verstehen, daß in 4b links unendliche Matrizen und rechts endliche Matrizen stehen?

(d) $\mathrm{Hom}_{\mathbb{Q}}(.V, \mathrm{Hom}_{\mathbb{R}}(.\mathbb{R}, .W) \cong \mathrm{Hom}_{\mathbb{R}}(.\mathbb{R} \otimes_{\mathbb{Q}} V, .W)$.

(5) $\mathbb{Z}/(18) \otimes_{\mathbb{Z}} \mathbb{Z}/(30) \neq 0$.

(6) $m : \mathbb{Z}/(18) \otimes_{\mathbb{Z}} \mathbb{Z}/(30) \ni \overline{x} \otimes \overline{y} \mapsto \overline{xy} \in \mathbb{Z}/(6)$ ist ein Homomorphismus und m ist bijektiv.

(7) Für $\mathbb{Q}$-Vektorräume V und W gilt $V \otimes_{\mathbb{Z}} W \cong V \otimes_{\mathbb{Q}} W$.

(8) Für jede endliche abelsche Gruppe M gilt $\mathbb{Q} \otimes_{\mathbb{Z}} M = 0$.

(9) $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathbb{Z}/(\mathrm{ggT}(m, n))$.

(10) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(n) = 0$.

(11) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/(n)) = 0$.

(12) Gib explizit Isomorphismen an für

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q},$$
$$3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}.$$

Zeige, daß das Diagramm kommutiert:

$$\begin{array}{ccc} 3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{Q} & \xrightarrow[3\cdot]{} & \mathbb{Q} \end{array}$$

(13) Der Homomorphismus $2\mathbb{Z} \otimes_Z \mathbb{Z}/(2) \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(2)$ ist der Nullhomomorphismus, beide Moduln sind aber von Null verschieden.

III. Projektive Moduln

(1) Bestimme die Dual-Basis von $\mathbb{R}^2$ im Sinne der Vorlesung.

(2) Zeige, daß die Spur eines Homomorphismus $f : V \longrightarrow V$ gegeben ist durch

$$\mathrm{End}_K(V) \cong V \otimes V^* \xrightarrow{\mathrm{ev}} K.$$

(3) Bestimme die Dual-Basis von ${}_{R\times S}R \times 0 \subseteq R \times S$.

(4) K_n ist ein projektiver $M_n(K)$-Modul.

(5) Sei $R := K \times K$ mit einem Körper K.

(a) Zeige: $P := \{(a, 0) | a \in K\}$ ist ein endlich erzeugter projektiver R-Modul.

(b) Sind die R-Moduln P und $Q := \{(0, a) | a \in K\}$ isomorph?

(c) Man finde eine Dual-Basis für P.

(6) Zeige für $R := M_n(K)$, daß $P = K_n$ endlich erzeugt projektiv ist, und finde eine Dual-Basis.

(7) Zu jedem projektiven Modul P gibt es einen freien Modul F mit $P \oplus F \cong F$.

IV. Kategorien und Funktoren

(1) In R-Mod gilt:
$f : M \longrightarrow N$ Monomorphismus $\iff$ f injektiver Homomorphismus.

(2) (a) Wenn $f : M \longrightarrow N$ surjektiv ist, dann ist $\mathrm{Hom}_R(f, P) : \mathrm{Hom}_R(N, P) \longrightarrow \mathrm{Hom}_R(M, P)$ injektiv.

(b) $\mathbb{Z} \longrightarrow \mathbb{Z}/(n)$ induziert eine injektive Abbildung

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}.M) \cong M.$$

Warum kann man $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), M)$ mit $\{x \in M | nx = 0\} \subseteq M$ identifizieren?

(c) $T_n(M) := \{x \in M | nx = 0\}$ ist ein Funktor Ab $\longrightarrow$ Ab.

(d) Die Einbettung $T_n(M) \longrightarrow M$ ist ein funktorieller Homomorphismus.

(3) In R-Mod gilt:
$f : M \longrightarrow N$ Epimorphismus $\iff$ f surjektiv.

(4) Wenn $\mathcal{F}$ ein kovarianter darstellbarer Funktor ist und $f : M \longrightarrow N$ ein Monomorphismus ist, dann ist $\mathcal{F}(f) : \mathcal{F}(M) \longrightarrow \mathcal{F}(N)$ ebenfalls ein Monomorphismus.

(5) Der Funktor $\mathcal{F} : M \mapsto \mathbb{Z}/(n) \otimes_{\mathbb{Z}} M$ ist nicht darstellbar.

(6) Der Funktor $\mathcal{F} : V \mapsto \mathbb{Q}^n \otimes_{\mathbb{Q}} V$ ist darstellbar.

(7) Der Funktor T_n : Ab $\longrightarrow$ Ab mit $T_n(M) := \{x \in M | nx = 0\}$ ist darstellbar.

(8) Jeder additive Funktor $F : R$-Mod $\longrightarrow S$-Mod erhält endliche direkte Summen, d.h. $F(M \oplus N) \cong F(M) \oplus F(N)$.

V. Morita-Äquivalenz

(1) Zeige, daß $(K \times K)$-Mod nicht äquivalent zu K-Mod ist.

(2) Sei K ein Körper, $B := M_n(K)$, ${}_KP_B := K^n$ die Menge der Zeilenvektoren, ${}_BQ_K$ die Menge der Spaltenvektoren. Finde $f : P \otimes_B Q \longrightarrow K$ und $g : Q \otimes_K P \longrightarrow B$, so daß (K, B, P, Q, f, g) einen Morita- Kontext bildet. Ist dieser Morita-Kontext strikt?

(3) Zeige $\mathbb{R}$-Mod $\ncong$ $\mathbb{C}$-Mod.

(4) Bestimme das Bild der Abbildungen f und g im kanonischen Morita-Kontext (A, B, P, Q, f, g) für

(a) $A := \mathbb{Z}/(6)$ und $P := \mathbb{Z}/(2)$,

(b) $A := \mathbb{Z}/(4)$ und $P := \mathbb{Z}/(4) \oplus \mathbb{Z}/(2)$,

(c) $A := \mathbb{Z}/(6)$ und $P := \mathbb{Z}/(6) \oplus \mathbb{Z}/(2)$.

VI. Halbeinfache Moduln

(1) Finde alle einfachen Moduln über K, $\mathbb{Z}$, $K[x]$.

(2) Finde alle einfachen Moduln über $\mathbb{C}[x]$, $M_2(K)$, $\mathbb{Q}[x]/(x^2)$.

(3) Finde alle einfachen Moduln über

$$\begin{pmatrix} K & K \\ 0 & K \end{pmatrix}.$$

(4) Stelle $\mathrm{End}_{K[x]}(K[x]/(x) \oplus K[x]/(x-1))$ als Ring von Matrizen dar.

VII. Radikal und Sockel

(1) Radikal und Sockel endlich erzeugter abelscher Gruppen. Bestimme

(a) $\mathrm{Rad}({}_{\mathbb{Z}}\mathbb{Z}/(p))$, $\mathrm{Soc}({}_{\mathbb{Z}}\mathbb{Z}/(p))$.

(b) $\mathrm{Rad}(\mathbb{Z}/(p^n))$, $\mathrm{Soc}(\mathbb{Z}/(p^n))$.

(c) $\mathrm{Rad}(\mathbb{Z}/(p^n) \oplus \mathbb{Z}/(p^m))$, $\mathrm{Soc}(\mathbb{Z}/(p^n) \oplus \mathbb{Z}/(p^m))$.

(d) Für welche $n \in \mathbb{N}$ ist $\mathrm{Rad}({}_{\mathbb{Z}}\mathbb{Z}/(n)) = 0$?

(2) Bestimme Radikal und Sockel der abelschen Gruppen

(a) $\mathbb{Z}$,

(b) $\mathbb{Q}$,

(c) $\mathbb{Q}/\mathbb{Z}$.

VIII. Lokale Ringe

(1) Sei R ein lokaler Ring. Dann ist $R/\mathfrak{m}$ ein Schiefkörper.

(2) Der Ring der formalen Potenzreihen $K[[x]]$ ist ein lokaler Ring.
(3) Der Polynomring $K[x]$ ist kein lokaler Ring.

IX. Lokalization
(1) $S := 2\mathbb{Z} \setminus \{0\}$ ist multiplikativ abgeschlossen. $S^{-1}\mathbb{Z} \subsetneqq \mathbb{Q}$.
(2) (a) Wenn $S \subseteq T$ multiplikativ abgeschlossene Mengen sind, dann wird dadurch ein Homomorphismus $\psi : S^{-1}M \longrightarrow T^{-1}M$ induziert.
(b) Finde eine hinreichende Bedingung dafür, daß ψ injektiv ist.
(c) Für $S := \mathbb{Z} \setminus (p)$ und $T := \mathbb{Z} \setminus \{0\}$ beschreibe man den Homomorphismus ψ.
(d) Für $S \subset T$ zeige man $S^{-1}T^{-1}M = T^{-1}S^{-1}M = T^{-1}M$.
(e) Wenn S, T multiplikativ abgeschlossen sind, dann ist auch $S \cap T$ multiplikativ abgeschlossen. Wie drückt sich das für $(S \cap T)^{-1}M$ aus?
(f) Sei $T := (\mathbb{Z} \setminus (2)) \cap (\mathbb{Z} \setminus (3))$. Bestimme $T^{-1}\mathbb{Z}$.
(g) Ist $\mathbb{Z}/(6) \longrightarrow T^{-1}(\mathbb{Z}/(6))$ injektiv? surjektiv?

XII. Bialgebras and Hopf algebras
(1) Let H be a bialgebra and A be an H-left-module algebra. On $A \otimes H$ define a multiplication

$$(a \otimes h)(a' \otimes h') := a(h_{(1)} \cdot a') \otimes h_{(2)}h'.$$

Show that this defines a structure of an algebra on $A \otimes H$. This algebra is usually denoted by $A\#H$ and the elements are denoted by $a\#h := a \otimes h$.
(2) Let A be a G-Galois extension field of the base field $\mathbb{K}$. Define an homomorphism $\varphi : A\#\mathbb{K}G \longrightarrow \mathrm{End}_{\mathbb{K}}(A)$ by $\varphi(a\#g)(b) := ag(b)$. Show that φ is a homomorphism of algebras.
(3) Let $G := C_2$ be the cyclic group with two elements., $A := \mathbb{C}$, and $\mathbb{K} := \mathbb{R}$. Show that $\varphi : \mathbb{C}\#\mathbb{R}C_2 \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ is an isomorphism of algebras.