Twisted Group Rings

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1. Introduction

This paper deals with (twisted) Hopf algebra forms of group rings, i.e. with Hopf algebras H over a base ring K, such that for a suitable extension L of K the Hopf algebra $L \otimes_K H$ is isomorphic to a group ring LG over L. These Hopf algebras arise from an adjoint situation as a solution of a universal problem in universal algebra.

One of the well known examples of a pair of adjoint functors is the units functor $U: K\text{-Alg} \longrightarrow \text{Gr}$ associating the group of units or invertible elements $U(A)$ with each K-algebra A, and its left adjoint functor $K\text{-}: \text{Gr} \longrightarrow K\text{-Alg}$ associating with each group G its group ring KG. The group ring KG actually turns out to be a cocommutative Hopf algebra. We are going to generalize this situation.

Let F be a finite group. We consider the category F-Gr of groups on which F acts by automorphisms and group homomorphisms compatible with the action of F. Furthermore let L be a Galois extension of the commutative base ring K with Galois group F. Then there is a functor

$$\Gamma: K\text{-Alg} \longrightarrow F\text{-Gr}$$

defined by $\Gamma(A) := U(L \otimes_K A)$. The action of F on $\Gamma(A)$ is given by the Galois action of F on L. In this paper the Galois extension L of K

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is assumed to be a commutative ring and free as a K- module. Our first result will be that Γ possesses a left adjoint functor

$$K_\Gamma\colon F\text{-Gr} \longrightarrow K\text{-Alg}.$$

The K-algebras $K_\Gamma(G)$ or simply $K_\Gamma G$ will be called twisted group rings. They should not be confused with crossed product group rings, skew group rings, smash products, or similar constructions. These twisted group rings do in general not even allow a canonical map from G into $K_\Gamma G$. As we will see, there will, however, be a canonical map $G \longrightarrow (K_\Gamma G)^n$ where n is the order of F.

The twisted group rings will turn out to be cocommutative Hopf algebras. They even have the property to coincide after base ring extension to L with the ordinary group ring $L \otimes K_\Gamma G \cong L \otimes KG$. This is an isomorphism of Hopf algebras. Thus $K_\Gamma G$ is a (twisted) L-form of the group ring KG.

In [2] we studied the construction and general theory of (twisted) forms of group rings KG for a fixed group G. We showed that the forms are in one to one correspondence with the F-Galois extensions of K where $F = \text{Aut}(G)$. Using the specific knowledge of all quadratic extensions of K (under minor restrictions), we were able to describe all Hopf algebra forms of $K\mathbf{Z}$, KC_3, KC_4, and KC_6 by generators and relations. A form H of KG which corresponds to the F-Galois extension L under [2 Thm.5] is certainly split by L, i.e. $L \otimes_K H \cong L \otimes_K KG$ as Hopf algebras, but it may also be split by a different, even smaller Galois extension L' of K. So we will not make use of the correspondence between the forms and Galois extensions. We will fix a Galois extension L of K and then study the Hopf algebra forms H of KG for all groups G which are split by the Galois extension L.

The group elements g in KG are connected with certain elements in $K_\Gamma G$ by formulas which generalize the Euler formulas and functional equations for the trigonometric functions

$$\begin{aligned}
&\exp = \cos + i \cdot \sin, \quad \cos = \frac{1}{2}(\exp + \exp^{-1}), \quad \sin = \frac{1}{2i}(\exp - \exp^{-1}),\\
&\exp(x+y) = \exp(x) + \exp(y),\\
&\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)\\
&\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y).
\end{aligned}$$

In fact these elements will be used in the construction of twisted group rings as canonical generators. These generators and the additional relations will be derived from certain generators and relations of the F-group G. A specific example is the group ring $K\mathbf{Z} = K\langle g\rangle$ and the twisted group ring $K_\Gamma\mathbf{Z} = K[c,s]/(c^2+s^2-1)$. If 2 is invertible in K then these two Hopf algebras are isomorphic iff K contains a square root i of -1. Then the isomorphism is given by

$$g = c + is, \quad c = \frac{1}{2}(g+g^{-1}), \quad s = \frac{1}{2i}(g-g^{-1}).$$

The diagonal on g, c, and s satisfies the 'functional equations'

$$\Delta(g) = g\otimes g, \quad \Delta(c) = c\otimes c - s\otimes s, \quad \Delta(s) = c\otimes s + s\otimes c.$$

These generators and relations descibing $K_\Gamma G$ will even be generators and relations of the group ring KG in case K contains a subring K' over which it is F-Galois. Thus we obtain very interesting new generators and relations even for the classical case of group rings.

We will show that all L-forms of group rings for a connected Galois extension L are twisted group rings as described above. So, many new Hopf algebras can be constructed in this way. A number of questions in the representation theory of groups are answered by base field extension, i.e. the splitting of group rings. One can now try to find twisted group rings defined already over the base field, which also have the required properties. Furthermore the structure of twisted group rings seems to be quite unknown yet, e.g. their centers, radicals, representation type, etc. Many of our calculations could be simplified thanks to helpful comments in [0].

2. The Multiplication Coefficients of a Free Galois Extension.

Throughout this paper we shall work with a free Galois extension of commutative rings $L : K$ with finite Galois group F in the sense of [1], i.e. an extension of commutative rings such that L is a finitely generated free K-module, K is the fixed ring under the operation of a finite group F of automorphisms of L and there is an isomorphism $h : L\otimes_K L \longrightarrow L\otimes_K (KF)^*$, $h(l'\otimes l) = \sum_{\pi\in F} l'\pi(l)\otimes\pi^*$, where $\{\pi^*\}$ is a dual basis to the basis $\{\pi\in F\}$ of KF.

The choice of a free generating system $\{x_i\}$ for L over K defines an isomorphism of free K-modules $L\cong K^n, x_i \mapsto e_i = (0,\ldots,1,\ldots,0)$. This isomorphism defines a K-algebra structure on K^n, whose multiplication coefficients are determined by

$$x_i x_j = \sum_{k=1}^{n} \alpha_{ij}^k x_k.$$

We determine these coefficients and those of the action of F and L solely in terms of a matrix $C := (\pi(x_i)|i = 1, \ldots, n; \pi \in F)$. We avoid enumerating the group elements and thus the columns of C, but the reader can use any enumeration for F.

LEMMA 1. *Let L be a free F-Galois extension of K with basis $\{x_i|i = 1, \ldots, n\}$. Let $C = (\gamma_{i,\pi}|i = 1, \ldots, n; \pi \in F)$ be the matrix with coefficients $\gamma_{i,\pi} := \pi(x_i)$ in L. Then C is invertible with inverse matrix $B = (\beta_{\pi,i})$. The unit coefficients, the multiplication coefficients, and the F-action coefficients of L with respect to the basis $\{x_i\}$ lie in K and are given by*

$$\varepsilon_i = \sum_{\pi \in F} \beta_{\pi,i} \qquad \text{for} \qquad 1 = \sum_{i=1}^{n} \varepsilon_i x_i, \tag{1}$$

$$\alpha_{ij}^k = \sum_{\pi \in F} \gamma_{i,\pi}\gamma_{j,\pi}\beta_{\pi,k} \qquad \text{for} \qquad x_i x_j = \sum_{k=1}^{n} \alpha_{ij}^k x_k, \tag{2}$$

$$\xi_{i,\sigma}^k = \sum_{\pi \in F} \gamma_{i,\pi\sigma}\beta_{\pi,k} \qquad \text{for} \qquad \sigma(x_i) = \sum_{k=1}^{n} \xi_{i,\sigma}^k x_k. \tag{3}$$

PROOF: For the "norm" (or "integral") $e^* \in (KF)^*$ defined by $e^*(\pi) := \delta_{e,\pi}$, where e is the unit element of the group F, there is a uniquely determined element $\sum x_i \otimes y_i \in L \otimes L$ such that $h(\sum_i x_i \otimes y_i) = \sum_{\pi,i} x_i \pi(y_i) \otimes \pi^* = 1 \otimes e^*$. So by definition this element satisfies $h(\sum_i x_i \otimes y_i)(\sigma) = \sum_i x_i \sigma(y_i) = \delta_{e,\sigma}$, hence for all $s \in L$

$$s = \sum_{\sigma} \sigma(s)\delta_{e,\sigma} = \sum_{i,\sigma} x_i \sigma(y_i s) = \sum_i x_i \mathrm{tr}(y_i s). \tag{4}$$

We first show that the matrix $B = (\beta_{\pi,i}) := (\pi(y_i)|\pi \in F; i = 1, \ldots, n)$ is the inverse matrix of C. We observe that $\sum_i \sigma(x_i)\pi(y_i) = \sigma(\sum_i x_i \sigma^{-1}\pi(y_i)) = \sigma(\delta_{e,\sigma^{-1}\pi}) = \delta_{e,\sigma^{-1}\pi} = \delta_{\sigma,\pi}$. Furthermore if we substitute $s = x_j$ in (4) we get $x_j = \sum_i x_i \sum_\pi \pi(y_i)\pi(x_j)$. This implies $\sum_\pi \pi(y_i)\pi(x_j) = \delta_{ij}$.

If we substitute $s = 1$ in (4), we get the coefficients of the unit element of L from $1 = \sum_i \mathrm{tr}(y_i)x_i$ as

$$\varepsilon_i = \mathrm{tr}(y_i) = \sum_{\pi \in F} \beta_{\pi,i}.$$

If we substitute $s = x_i x_j$ in (4), we get the multiplication coefficients of L from $x_i x_j = \sum_k x_k \mathrm{tr}(y_k x_i x_j) = \sum_k x_k \sum_\pi \pi(x_i)\pi(x_j)\pi(y_k)$ as

$$\alpha_{ij}^k = \sum_{\pi \in F} \gamma_{i,\pi}\gamma_{j,\pi}\beta_{\pi,k}.$$

Finally if we substitute $s = \sigma(x_i)$ in (4), we get the coefficients for the F-action from $\sigma(x_i) = \sum_k x_k \sum_\pi \pi(y_k)\pi\sigma(x_i)$ as

$$\xi_{i,\sigma}^k = \sum_{\pi \in F} \gamma_{i,\pi\sigma}\beta_{\pi,k}. \qquad \square$$

So we have described the algebra structure of L solely in terms of the matrix C (and its inverse B). The description of the F-action uses the right regular action of F on itself. The action is thus given by the induced permutations of the columns of C. Observe that the coefficients of C and B are in L, the coefficients of the unit, the multiplication and the F-action, however, are all elements in K (since they are obtained from the trace).

We consider the isomorphism

$$L^n \cong L \otimes K^n \cong L \otimes L \xrightarrow{h} L \otimes (KF)^* \cong L \otimes K^F \cong L^F$$

where L^F is the set of n-tuples of elements from L, indexed by the set F, or the set of maps from F to L. This isomorphism sends $(l_i | i = 1, \ldots, n)$ to $(\sum_i l_i \pi(x_i) | \pi \in F) = (\sum_i l_i \gamma_{i,\pi})$, i.e. it is multiplication by C on the right. The inverse map is multiplication by B. For $a, b \in L^F$ we get $(a \cdot b)B = aB \star bB$, where $\star$ denotes the multiplication on L^n with multiplication coefficients (α_{ij}^k) and L^F has componentwise multiplication. Similarly the map is also compatible with the F-action, on L^F componentwise and on L^n with coefficients $\xi_{i,\sigma}^k$.

The coefficients as calculated in Lemma 1 can also be used to describe a K-algebra structure on S^n for any (non-commutative) K-algebra S and an action of F on S^n just by applying the isomorphism $L \otimes_K S \cong K^n \otimes_K S \cong S^n$. If S is an algebra of the form $S \cong L \otimes T$, then we can even use the isomorphism of F-algebras

$$(L \otimes T)^n \cong (L \otimes T)^F$$

where $(L \otimes T)^F$ has componentwise operations and $(L \otimes T)^n$ has operations with coefficients from Lemma 1. The isomorphism is still given by the matrices B and C.

3. The Construction of a Twisted Group Ring

We want to construct the K-algebra $K_\Gamma G$ for an F-group G by using defining generators and relations of G. We first have to look at free algebras.

Let X be a set and G_f be the free monoid on X. Then the monoid ring LG_f and the free (non-commutative) L-algebra $L\langle X \rangle$ coincide. This is due to the fact that the underlying functor from the category Alg of non-commutative L-algebras to the category Set of sets factors through the category Mon of monoids. The free constructions are the corresponding left adjoint functors.

A similar argument can be used in a slightly more complicated setting. Let F be a finite group. Any of the categories Set, Mon, Alg named above can be used to construct a new category F-Set, F-Mon, F-Alg of objects on which F acts as automorphisms (actually pairs consisting of an object plus an action) and morphisms which are compatible with the F-action. It is easy to see that the left adjoint functors of the underlying functors Alg $\longrightarrow$ Set, Alg $\longrightarrow$ Mon, and Mon $\longrightarrow$ Set can be used as left adjoint functors for the underlying functors F-Alg $\longrightarrow$ F-Set, F-Alg $\longrightarrow$ F-Mon, and F-Mon $\longrightarrow$ F-Set, i.e. the diagrams

$$\begin{array}{ccc} F\text{-Cat1} & \xrightarrow{\textit{free}} & F\text{-Cat2} \\ U \downarrow & & U \downarrow \\ \text{Cat1} & \xrightarrow{\textit{free}} & \text{Cat2} \end{array}$$

commute, since F acts by automorphisms. In particular we get that the monoid algebra LG_f of the free F-monoid G_f with generating set Y is the free F-algebra on the set Y and also the free algebra on the free F-set defined over Y. Now the free F-set on Y is just $Y \times F$, so the algebra $L\langle Y \times F \rangle$ is the monoid algebra LG_f over the free F-monoid generated by Y.

For a set $Y = \{g_j | j \in I\}$ consider $Y \times F$ with elements $\pi(g_j) := (g_j, \pi)$ and a set of "variables" $\{S_{ij} | i = 1, \ldots, n; j \in I\}$. Then the map $\Phi : LG_f \longrightarrow L\langle S_{ij} \rangle$ defined by

$$\Phi(\pi(g_j)) := \sum_{i=1}^{n} \gamma_{i,\pi} S_{ij}$$

is an L-algebra isomorphism with inverse map

$$\Psi(S_{ij}) := \sum_{\pi \in F} \beta_{\pi,i} \pi(g_j).$$

This follows immediately from the fact that B and C are inverse matrices of each other and that L-algebra homomorphisms are uniquely determined by the action on the free generating set. The algebra isomorphism Φ, however, is not an F-isomorphism, since no F-structure has been defined on $L\langle S_{ij} \rangle$, but it could be used to define such a structure.

We want to use the isomorphism Φ to give a different representation of a group ring LG of an F-group G by generators and relations. For that purpose we first construct the monoid algebra LG_f of a free F-monoid G_f which has G as a quotient group. This algebra is isomorphic to $L\langle S_{ij} \rangle$ by the isomorphism Φ. Then we import the relations of G written as pairs of elements of G_f. Their differences generate the ideal J of LG_f to be factored out to obtain (a representation of) LG. This ideal will be transferred to the ideal M of $L\langle S_{ij} \rangle$ via the given isomorphism Φ. The central point of the calculation will be that the generators of M are already defined over K (instead of L), i.e. they are elements of $K\langle S_{ij} \rangle$ which generate an ideal $M' \subseteq K\langle S_{ij} \rangle$. Thus we obtain a K-algebra $K_\Gamma G := K\langle S_{ij} \rangle / M'$ with the property $L \otimes_K K_\Gamma G \cong LG$.

More precisely let G be an F-group generated as an F-*monoid* by the set $Y = \{g_i | i \in I_1\}$. Then there is a surjective F-homomorphism $\phi: G_f \longrightarrow G$, where G_f is the free F-monoid on Y. A relation for G is a pair of elements $(r_1, r_2) \in G_f \times G_f$ with $\phi(r_1) = \phi(r_2)$. The set of all relations for G is an F-submonoid of $G_f \times G_f$. Let $\{r_j = (r_{j1}, r_{j2}) | j \in I_2\}$ be a generating set of the F-submonoid of relations. Then G can be considered as a quotient of G_f modulo the congruence relation generated by $\{r_j\}$.

We consider two elements r and s of G_f (written with non-negative powers and F-multiples of the generators in Y). Let J be the ideal generated by $\{\pi(r) - \pi(s) | \pi \in F\}$ in LG_f. Then $M := \Phi(J)$ is generated by

$\{\Phi(\pi(r) - \pi(s)) | \pi \in F\}$. We want to change this generating set to a set of K-linear combinations of the S_{ij}. For this purpose we form

$$\{\sum_{\pi \in F} \beta_{\pi,i} \Phi(\pi(r) - \pi(s)) | i = 1, \ldots, n\}.$$

LEMMA 2. *The ideal M in $L\langle S_{ij}\rangle$ generated by $\{\Phi(\pi(r) - \pi(s)) | \pi \in F\}$ is also generated by the set $\{\sum_{\pi \in F} \beta_{\pi,i} \Phi(\pi(r) - \pi(s)) | i = 1, \ldots, n\} \subseteq K\langle S_{ij}\rangle$.*

PROOF: Since the matrix $(\beta_{\pi,i})$ is invertible, this set generates the same ideal M in $L\langle S_{ij}\rangle$. We show that the coefficients of the products of the S_{ij}'s are all in K. Let $r = \tau_1(g_1) \cdot \ldots \cdot \tau_p(g_p)$. Then we obtain

$$\begin{aligned}
&\sum_{\pi \in F} \beta_{\pi,i} \Phi(\pi(\tau_1(g_1) \cdot \ldots \cdot \tau_p(g_p)) \\
&\qquad = \sum_{\pi \in F} \beta_{\pi,i} \Phi(\pi\tau_1(g_1)) \cdot \ldots \cdot \Phi(\pi\tau_p(g_p)) \\
&\qquad = \sum_{\pi \in F} \sum_{i_1,\ldots,i_p} \beta_{\pi,i} \gamma_{i_1,\pi\tau_1} S_{i_1,1} \cdot \ldots \cdot \gamma_{i_p,\pi\tau_p} S_{i_p,p} \\
&\qquad = \sum_{i_1,\ldots,i_p} (\sum_{\pi \in F} \beta_{\pi,i} \gamma_{i_1,\pi\tau_1} \cdot \ldots \cdot \gamma_{i_p,\pi\tau_p}) S_{i_1,1} \cdot \ldots \cdot S_{i_p,p}.
\end{aligned}$$

Since the coefficients

$$\sum_{\pi \in F} \beta_{\pi,i} \gamma_{i_1,\pi\tau_1} \cdot \ldots \cdot \gamma_{i_p,\pi\tau_p} = \sum_{\pi \in F} \pi(y_i) \pi\tau_1(x_{i_1}) \ldots \pi\tau_p(x_{i_p})$$

are the trace of certain elements they are in K. $\square$

THEOREM 3. *Let G be an F-group with F-monoid generators $\{g_i | i \in I_1\}$ and F-monoid relations $\{r_i | i \in I_2\}$ generating all F-monoid relations of G. Let M' be the ideal of the K-algebra $K\langle S_{ij} | i = 1, \ldots, n; j \in I_1\rangle$ generated by the set*

$$\{\sum_{\pi \in F} \beta_{\pi,i} \Phi(\pi(r_{j1}) - \pi(r_{j2})) | i = 1, \ldots, n; j \in I_2\}$$

where $\Phi(\pi(g_j)) := \sum_i \gamma_{i,\pi} S_{ij}$. Then $K_\Gamma G := K\langle S_{ij}\rangle / M'$ is a Hopf algebra and an L-form of the Hopf algebra KG, i.e. $\Phi\colon L \otimes_K KG \cong L \otimes_K K_\Gamma G$.

PROOF: By hypothesis the F-group G is represented as a quotient of a free F-monoid by certain relations. If $r = (r_1, r_2)$ is a relation in G then

$r_1 = r_2$ in G hence $\pi(r_1) = \pi(r_2)$ for all $\pi \in F$. It is easy to see that the group algebra KG is the quotient KG_f/J' where G_f is the free F-monoid generated by $\{g_i | i \in I_1\}$ and J' is the ideal of KG_f generated by $\{\pi(r_{i1}) - \pi(r_{i2}) | i \in I_2, \pi \in F\}$. If we identify $L \otimes_K KG$ and LG, $L \otimes_K K\langle S_{ij}\rangle$ and $L\langle S_{ij}\rangle$, more generally the tensor product with L with the multiplication with L, then we get an algebra isomorphism $\Phi: L \otimes KG \cong L \otimes K\langle S_{ij}\rangle/M'$, since L is faithfully flat.

The diagonal on $L\langle S_{ij}\rangle/L \cdot M'$ is induced by $\Phi: LG \longrightarrow L\langle S_{ij}\rangle/L \cdot M'$. We have

$$\begin{aligned}
\Delta(s_{kl}) &= \Delta(\sum_{\pi \in F} \beta_{\pi,k} \Phi(\pi(g_l))) \\
&= \sum_{\pi \in F} \beta_{\pi,k} \Phi(\pi(g_l)) \otimes \Phi(\pi(g_l)) = \sum_{i,j,\pi} \gamma_{i,\pi}\gamma_{j,\pi}\beta_{\pi,k} s_{il} \otimes s_{jl} \\
&= \sum_{i,j} \alpha_{ij}^k s_{il} \otimes s_{jl}
\end{aligned}$$

where s_{ij} is the residue class of S_{ij}. Since the coefficients α_{ij}^k are in K the diagonal is already defined in $K\langle s_{ij}\rangle = K\langle S_{ij}\rangle/M'$. In a similar way we obtain $\varepsilon(s_{ij}) = \sum_\pi \beta_{\pi,i} = \varepsilon_i$ (independent of j!) also defined on $K\langle s_{ij}\rangle$. Since $L\langle s_{ij}\rangle$ is a bialgebra (isomorphic to LG) and L/K is faithfully flat, we get that $K\langle s_{ij}\rangle$ is also a bialgebra. It is a bialgebra form of KG. By [2 Remark following Thm.1] a bialgebra form of a Hopf algebra is already a Hopf algebra. $\square$

The Hopf algebra $K_\Gamma G := K\langle s_{ij}\rangle$ is called a Γ-*twisted group ring*. We have constructed the Γ-twisted group ring by generators s_{ij} and relations using the formula

$$s_{ij} = \sum_{\pi \in F} \pi(y_i)\pi(g_j)$$

in LG. Each relation r for G induces n relations in $K_\Gamma G$

$$\sum_{\pi \in F} \beta_{\pi,i} \Phi(\pi(r_1) - \pi(r_2)) = 0.$$

These generators and relations are obtained from the defining F-generators and relations for G. Each generator g_j (in fact each element) of G and its F-orbit in $G \subseteq KG$ corresponds to a family $(s_{ij} | i = 1, \ldots, n)$ in $K_\Gamma G$ by the given formula. The diagonal of $K_\Gamma G$ applied to an element of such a family can be expressed solely by the elements of the same family (and coefficients in K) so we define a Γ-*group-like sequence* of a Hopf algebra H to be an n-tuple (s_i) in H such that

$$\Delta(s_k) = \sum_{i,j} \alpha_{ij}^k s_i \otimes s_j \quad \text{and} \quad \varepsilon(s_i) = \varepsilon_i$$

holds.

COROLLARY 4. *Let G be an F-group. Let L be a commutative ring which contains subrings $K \subseteq L' \subseteq L$ such that $L' : K$ is an F-Galois extension. Then the group algebra LG has generators $\{s_{il} | i = 1, \ldots, n; l \in I\}$ such that for all $l \in I$*

$$\Delta(s_{kl}) = \sum \alpha_{ij}^k s_{il} \otimes s_{jl} \qquad \text{and} \qquad \varepsilon(s_{il}) = \varepsilon_i.$$

The relations for the $\{s_{ij}\}$ are those of Theorem 3.

PROOF: follows immediately from the isomorphism $LG \cong L \otimes_{L'} L' \otimes_K KG \cong L \otimes_{L'} L' \otimes_K K_\Gamma G$. $\square$

4. Twisted Group Rings as Adjoint Functors

In this paragraph we shall identify LG and $L \otimes K_\Gamma G$ via the isomorphism Φ as constructed above. Furthermore we view KG and $K_\Gamma G$ as subalgebras in LG. In particular we have the equations

$$\pi(g_j) = \sum_{i=1}^{n} \gamma_{i,\pi} s_{ij}, \qquad s_{ij} = \sum_{\pi \in F} \beta_{\pi,i} \pi(g_j),$$

where $\{g_i | i \in I_1\}$ is a generating set of G as an F-monoid.

THEOREM 5. *Let G be an F-group. Then there exists a cocommutative Hopf algebra $K_\Gamma G$ and a homomorphism of F-groups $\iota: G \longrightarrow \Gamma(K_\Gamma G)$ such that for each K-algebra S and homomorphism of F-groups $f: G \longrightarrow \Gamma(S)$ there exists precisely one K-algebra homomorphism $\widetilde{f}: K_\Gamma G \longrightarrow S$, such that the diagram*

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & \Gamma(K_\Gamma G) \arrow[d, "\Gamma(\widetilde{f})"] \\ & \Gamma(S) \end{tikzcd}$$

commutes.

PROOF: We define $\iota: G \longrightarrow \Gamma(K_\Gamma G) = U(LG)$ by $\iota(g) := g \in LG$. This is obviously an F-homomorphism.

We choose a set of F-monoid generators $\{g_i | i \in I_1\}$ of G and a generating set of relations $\{r_i | i \in I_2\}$. Construct $K_\Gamma G$ as in Theorem 3. For the generating elements of G let $f(g_j) = \sum_i x_i \otimes t_{ij} \in U(L \otimes S)$. The coefficients $t_{ij} \in S$ are uniquely determined. Observe that

$$f(\pi(g_j)) = \pi f(g_j) = \sum_{i=1}^{n} \pi(x_i) \otimes t_{ij}.$$

Define $\widetilde{f}(s_{ij}) := t_{ij}$. In a first step this is only defined on $K\langle S_{ij}\rangle$. In order to show that this gives a well-defined algebra homomorphism $\tilde{f}\colon K_\Gamma G \longrightarrow S$ we have to check, that it preserves the relations in $K_\Gamma G$. Let $r = (r_1, r_2)$ be a generating relation for G and $\sum_{\pi\in F} \beta_{\pi,i}(\pi(r_1) - \pi(r_2))$ for some $i \in \{1, \dots, n\}$ be one of the defining relations for the ideal M'. We calculate the action of $\widetilde{f}$ only on the first part of this term

$$\begin{aligned}
\widetilde{f}(\sum_{\pi\in F} \beta_{\pi,i}\pi(r_1)) &= \widetilde{f}(\sum_{\pi\in F} \beta_{\pi,i}\pi(\tau_1(g_1)\dots\tau_p(g_p))) \\
&= \widetilde{f}(\sum_{i_1,\dots,i_p}\sum_{\pi\in F} \beta_{\pi,i}\gamma_{i_1,\pi\tau_1}\dots\gamma_{i_p,\pi\tau_p} s_{i_1 1}\dots s_{i_p p}) \\
&= \widetilde{f}(\sum_{i_1,\dots,i_p}(\sum_{\pi\in F}\pi(y_i)\pi(\tau_1(x_{i_1})\dots\tau_p(x_{i_p})))s_{i_1 1}\dots s_{i_p p}) \\
&= \sum_{i_1,\dots,i_p} \mathrm{tr}(y_i\tau_1(x_{i_1})\dots\tau_p(x_{i_p}))\widetilde{f}(s_{i_1 1}\dots s_{i_p p}) \\
&= \sum_{i_1,\dots,i_p}(\sum_{\pi\in F}\pi(y_i\tau_1(x_{i_1})\dots\tau_p(x_{i_p}))\otimes t_{i_1 1}\dots t_{i_p p}) \\
&= \sum_{\pi\in F}\beta_{\pi,i}(\sum_{i_1}\pi\tau_1(x_{i_1})\otimes t_{i_1 1})\dots(\sum_{i_p}\pi\tau_p(x_{i_p})\otimes t_{i_p p}) \\
&= \sum_{\pi\in F}\beta_{\pi,i}\pi\tau_1 f(g_1)\dots\pi\tau_p f(g_p) \\
&= \sum_{\pi\in F}\beta_{\pi,i} f(\pi\tau_1(g_1)\dots\pi\tau_p(g_p)) \\
&= \sum_{\pi\in F}\beta_{\pi,i} f(\pi(r_1)).
\end{aligned}$$

Now $f(r_1) = f(r_2)$ implies $\widetilde{f}(\sum_{\pi\in F}\beta_{\pi,i}(\pi(r_1) - \pi(r_2))) = 0$. Thus we get a well-defined K-algebra homomorphism $\widetilde{f}\colon K_\Gamma G \longrightarrow S$. For $g_j \in G$ we then get $\Gamma(\widetilde{f})\iota(g_j) = \Gamma(\widetilde{f})(\sum_i \gamma_{i,e}s_{ij}) = \sum_i x_i \otimes (\widetilde{f}(s_{ij})) = \sum_i x_i \otimes t_{ij} = f(g_j)$.

To show the uniqueness of $\widetilde{f}$ let $\widehat{f}$ be another extension of f so that $\Gamma(\widetilde{f})\iota = \Gamma(\widehat{f})\iota = f$. $\widehat{f}$ induces an L-algebra homomorphism from LG to $L \otimes S$ which we also denote by $\widehat{f}$. Then $\widetilde{f}(s_{ij}) = \widetilde{f}(\sum_\pi \beta_{\pi,i}\pi(g_j)) = \sum_\pi \beta_{\pi,i}\widetilde{f}(\pi(g_j)) = \sum_\pi \beta_{\pi,i}\Gamma(\widetilde{f})\iota(\pi(g_j)) = \sum_\pi \beta_{\pi,i} f(\pi(g_j)) = \widehat{f}(\sum_\pi \beta_{\pi,i}\pi(g_j)) = \widehat{f}(s_{ij})$ so that $\widetilde{f} = \widehat{f}$. □

The universal property shown above implies immediately

COROLLARY 6. *The functor* $K_\Gamma : F\text{-Gr} \longrightarrow K\text{-Alg}$ *is left adjoint to* $\Gamma : K\text{-Alg} \longrightarrow F\text{-Gr}$. □

One could also have shown that Γ is an algebraic functor in the sense of [4]. Then the adjointness property would have resulted from the general theory, but the explicit construction as K_Γ would not have followed. Furthermore the Hopf algebra property of $K_\Gamma G$ would have required additional investigations.

5. The Structure of Forms of Group Rings

Up to now we have constructed a specific version of (twisted) L-forms of group rings (Theorem 3), namely the twisted group rings. We have studied their structure in terms of generators and relations and their property as a left adjoint functor. Now we want to prove that all forms of group rings are twisted group rings under only minor restrictions. For this purpose we consider those forms H of group rings which are split by a free (F-)Galois extension L of K, i.e. $L \otimes_K H \cong L \otimes_K KG$ and we assume that L is connected, i.e. has only the idempotents 0 and 1.

THEOREM 7. *Let $L : K$ be a free Galois extension with finite Galois group F and let L be connected. Let G be a group and H be a K-Hopf algebra which is an L-form of the group ring KG. Then there is an F-structure on G such that H is isomorphic to a Γ-twisted group ring $K_\Gamma G$.*

PROOF: Since $L : K$ is faithfully flat we can identify H and KG with specific K-subalgebras of LG. In particular the H-module $LG(\cong L \otimes_K H)$ is freely generated over the basis $\{x_i\}$. Each element $g \in G$ has a representation $g = \sum_{i=1}^n x_i s_i(g)$ with uniquely determined coefficients $s_i(g) \in H$. This defines maps $s_i : G \longrightarrow H$. These maps correspond to the generators s_{ij} coming from the group generators g_j in the preceding paragraphs.

We define an F-action on G by $\pi(g) := \sum_i \pi(x_i)s_i(g)$. First we show that $\pi(g)$ is an element of G. Since L is connected it suffices to prove that $\pi(g)$ is a group-like element in LG. By Lemma 1 the diagonal of the elements $s_i(g)$ can be derived from $\sum_k x_k \Delta(s_k(g)) = \Delta(g) = g \otimes g = \sum_{ij} x_i x_j s_i(g) \otimes s_j(g) = \sum_k \sum_{ij} \alpha_{ij}^k x_k s_i(g) \otimes s_j(g)$ as $\Delta(s_k(g)) = \sum_{ij} \alpha_{ij}^k s_i(g) \otimes s_j(g)$. The augmentation is $\sum_k x_k \varepsilon(s_k(g)) = \varepsilon(g) = 1 = \sum x_k \varepsilon_k$ hence $\varepsilon(s_i(g) = \varepsilon_i$. So $(s_i(g)|i = 1, \ldots, n)$ is a Γ-group-like sequence in H. Then

$$\begin{aligned}\Delta(\pi(g)) &= \sum_k \pi(x_k)\Delta(s_k(g)) = \sum_k \sum_{ij} \alpha_{ij}^k \pi(x_k) s_i(g) \otimes s_j(g) \\ &= \sum_{ij} \pi(x_i x_j) s_i(g) \otimes s_j(g) = \pi(g) \otimes \pi(g)\end{aligned}$$

and $\varepsilon(\pi(g)) = \sum_i \pi(x_i)\varepsilon(s_i(g)) = \pi(\sum_i \varepsilon_i x_i) = \pi(1) = 1$ imply that $\pi(g)$ is group-like, hence $\pi(g) \in G$.

Now it is easy to verify that this is an action of F on G by automorphisms. From Lemma 1 we get

$$\tau(\sigma(g)) = \tau(\sum_i \sigma(x_i) s_i(g)) = \tau(\sum_i \sum_k \xi_{i,\sigma}^k x_k s_i(g))$$

$$= \sum_{i,k,\pi} \gamma_{i,\pi\sigma} \beta_{\pi,k} \gamma_{k,\tau} s_i(g) = \sum_i \gamma_{i,\tau\sigma} s_i(g) = (\tau\sigma)(g).$$

Furthermore we have

$$\begin{aligned}\pi(gh) &= \pi(\sum_{i,j} x_i x_j s_i(g) s_j(h)) = \pi(\sum_k x_k \sum_{i,j} \alpha_{ij}^k s_i(g) s_j(h)) \\ &= \sum_k \pi(x_k) \sum_{i,j} \alpha_{ij}^k s_i(g) s_j(h)) = \sum_{i,j} \pi(\sum_k \alpha_{ij}^k x_k) s_i(g) s_j(h)) \\ &= \sum_{i,j} \pi(x_i)\pi(x_j) s_i(g) s_j(h) = \pi(g)\pi(h).\end{aligned}$$

By definition we also have $e(g) = g$ so that G is an F-group.

The choice of F-monoid generators $\{g_j\}$ for G defines K-algebra generators $s_i(g_j)$ for H with the formulas $g_j = \sum_i x_i s_i(g_j)$ and $s_i(g_j) = \sum_\pi \beta_{\pi,i} \pi(g_j)$. The relations are transformed as in Theorem 3. $\square$

The connection with some known results should be mentioned here. In [6 Thm.6.4.] the category of finite etale group schemes over a field K was found to be equivalent to the category of finite $\mathcal{G}$-groups, where $\mathcal{G}$ is the profinite Galois group of the separable closure of K, acting continuously on the finite groups. These group schemes, represented by commutative Hopf algebras, become constant group schemes already after a finite Galois field extension L, i.e. $L \otimes H^* \cong LG$ for a finite group G. The action of $\mathcal{G}$ on G is given by the action of $F = \mathrm{Aut}(L/K)$ on G via the isomorphism

$$\begin{aligned} K\text{-Alg}(H, L) &\cong L\text{-Alg}(L \otimes_K H, L) \cong L\text{-Coalg}(L, L \otimes_K H^*) \\ &\cong \text{Group-Likes}(L \otimes H^*) \cong \text{Group-Likes}(LG) \cong G. \end{aligned}$$

Theorem 7 is a generalization of this theorem to infinite groups and away from the field requirements for L and K. On the other hand our theorem also generalizes part of the known antiequivalence between group schemes of multiplicative type and abelian $\mathcal{G}$-groups [6 Thm.7.3.] to noncommutative Hopf algebras. In fact our Theorem 3 gives a description of the representing algebras of group schemes of multiplicative type by generators and relations of the corresponding character group.

6. Two Examples

We want to illustrate our results with two examples. Let $C_2 = \{\pi_1, \pi_2\} = \{\epsilon, \sigma\}$ be the cyclic group with two elements. Assume that 2 is invertible in K. Consider the C_2-Galois extension $L = K(i)$ of K with $i^2 = -1$ and basis $x_1 = 1, x_2 = i$. We form the matrix

$$(\gamma_{i\pi}) = \begin{pmatrix} \pi_1(x_1) & \pi_2(x_1) \\ \pi_1(x_2) & \pi_2(x_2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

with the inverse matrix

$$(\beta_{ij}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix}$$

Then the multiplication coefficients are

$$(\alpha_{ij}^k) = \left(\sum_{l=1}^{2} \gamma_{kl}\beta_{li}\beta_{lj} \right) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array} \right)$$

where the two adjacent matices are (α_{ij}^1) and (α_{ij}^2). The coefficients of the action of F on L are

$$(\xi_{i,\sigma}^k) = \left(\sum_{\pi \in F} \gamma_{i,\pi\sigma}\beta_{\pi,k} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the coefficients of the unit are $(\varepsilon_1, \varepsilon_2) = (1, 0)$.

For a Γ-twisted group ring $K_\Gamma G$ we use the isomorphism $(LG)^n \cong (LG)^F$ as described at the end of paragraph 2 given by multiplication by B resp. C. We denote the multiplication in the algebra $(LG)^n$ by $\star$, the action of F with no special notation. We still consider $K_\Gamma G$ and KG embedded in LG. Each element $g \in G$ induces an element $(\pi(g)|\pi \in F) \in (LG)^F$ and the fact that F acts by automorphisms induces $(\pi(g)) \cdot (\pi(h)) = (\pi(gh))$. Each relation in G thus induces a relation in $(LG)^F$, hence n relations in LG.

We construct the twisted group ring $K_\Gamma \mathbf{Z}$. The group $G := \mathbf{Z}$ has one (F-monoid) generator g and C_2 acts on G by $\sigma(g) = g^{-1}$. So there is a relation $\sigma(g)g = 1$ (if we consider G as a multiplicative group). We define

$$(c, s) := (g, \sigma(g)) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix},$$

i.e. $c = \frac{1}{2}(g + g^{-1})$ and $s = \frac{1}{2i}(g - g^{-1})$ in LG. These elements generate a K-subalgebra $K[c, s] \subseteq LG$. There is a relation $c^2 + s^2 = 1$, in fact the subalgebra is isomorphic to $K_\Gamma G = K\langle C, S\rangle/(C^2 + S^2 - 1, CS - SC)$ where $K\langle C, S\rangle$ is the free algebra on C, S (the polynomial ring in non-commuting variables C, S). The relations arise from a translation of the relation $\sigma(g)g = 1$ to

$$\sigma(c, s) \star (c, s) = (\varepsilon_1, \varepsilon_2)$$

or using the definition of $\star$ and the operation of F on $(LG)^n$

$$(c, -s)(\alpha_{ij}^k)(c, s)^t = (1, 0)$$

hence $c^2 + s^2 = 1$ and $-sc + cs = 0$.

It is obvious that this algebra $K_\Gamma G$ represents the circle group functor which associates with each K-algebra S the "circle" $\{(c, s)|c, s \in S; c^2 + s^2 = 1\}$ with multiplication $(c, s) \cdot (c', s') = (cc' - ss', cs' + sc')$ and unit $(1, 0)$ which we studied in [2].

The second example uses again the Galois extension $L = K(i)$ over K with Galois group C_2, but this time we use the symmetric group $G = S_3$. The Γ-twisted group ring $K_\Gamma S_3$ is a specific form of KS_3. In the theory of [2] we were able to describe all forms of KS_3 by all $S_3 = \mathrm{Aut}(S_3)$-Galois extensions of K, which are difficult to describe. Here we need only a C_2-Galois extension of K, but we obtain only one specific form of KS_3. The

Galois group C_2 acts on S_3 in the following way. We represent S_3 by the generators g_1, g_2 and the relations $g_1^2 = 1$, $g_2^3 = 1$, and $g_1 g_2 = g_2^2 g_1$. The action of C_2 then is given by $\sigma(g_1) = g_1$ and $\sigma(g_2) = g_2^2$. Consider the equations

$$(c, s) := (g_1, \sigma(g_1)) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix},$$

and

$$(d, t) := (g_2, \sigma(g_2)) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix}.$$

They define elements $c = \frac{1}{2}(g_1 + g_1) = g_1$, $s = \frac{1}{2i}(g_1 - g_1) = 0$, $d = \frac{1}{2}(g_2 + g_2^2)$, and $t = \frac{1}{2i}(g_2 - g_2^2)$ in LG. The twisted group ring $K_\Gamma S_3$ is the subring $K[c, s, d, t] \subseteq LG$ which can be represented as $K_\Gamma S_3 = K\langle C, S, D, T\rangle / I$, where I is the ideal generated by the following expressions (simplified by $s = 0$):

$$\begin{array}{ll} C^2 - 1, & S, \\ D^3 - T^2 D - TDT - DT^2 - 1, & T^3 - D^2 T - DTD - TD^2, \\ D^2 C - T^2 C - CD, & DTC + TDC - CT, \\ D^2 - T^2 - D, & DT + TD + T. \end{array}$$

These expressions follow from the translation of the group relations

$$\begin{aligned} &(c, s) \star (c, s) = (1, 0), \\ &((d, t) \star (d, t)) \star (d, t) = (1, 0), \\ &(c, s) \star (d, t) = ((d, t) \star (d, t)) \star (c, s), \\ &\sigma(d, t) = (d, t) \star (d, t), \end{aligned}$$

or from

$$\begin{aligned} &(C^2 - S^2, SC + CS) = (1, 0), \\ &(D^3 - T^2 D - DT^2 - TDT, D^2 T - T^3 + DTD + TD^2) = (1, 0), \\ &(CD - ST, CT + SD) \\ &\qquad = (D^2 C - T^2 C - DTS - TDS, D^2 S - T^2 S + DTC + TDC), \\ &(D, -T) = (D^2 - T^2, DT + TD). \end{aligned}$$

The $\star$-multiplication is still defined by the matrix (α_{ij}^k) and the action of C_2 is as in Lemma 1.

As C. Greither pointed out, the particularly nice multiplication and operation matrices for the C_2-extension $K(i)/K$ as calculated at the beginning of this paragraph are due to the fact that $K(i)/K$ is a Kummer extension and hold for general Kummer extensions. Consider a Kummer extension $K(\alpha)/K$ with $\alpha^n = b \in K$, n invertible in K and ζ a primitive n-th root of unity in K. Choose $1, \alpha, \ldots, \alpha^{n-1}$ as a basis of $K(\alpha)$ over K. Then by a simple computation the matrices obtained for this basis are

$$C = (\pi(\alpha^i)) = ((\zeta^j \alpha)^i),$$

$$B = (\beta_{j,\pi}) = \frac{1}{n}((\zeta^i \alpha)^{-j}),$$

$$A = (\alpha_{ij}^k) \text{ with } \alpha_{ij}^k = \begin{cases} 1 & \text{for } i = j + k, \\ b^{-1} & \text{for } i + n = j + k, \\ 0 & \text{else,} \end{cases}$$

and

$$E = {}^t(1, 0, \ldots, 0).$$

References

[0] Anonymous: Referees Report (unpublished).

[1] S. Chase, D.K. Harrison, A. Rosenberg: Galois Theory and Galois Cohomology of Commutative Rings. Mem. Am. Math. Soc. 52 (1965).

[2] R. Haggenmüller, B. Pareigis: Hopf Algebra Forms of the Multiplicative Group and other Groups. manuscr. math. 55, 121-136 (1986).

[3] B. Pareigis: Descent Theory applied to Galois Theory. Technical Report Univ. California, San Diego, 1986.

[4] B. Pareigis: Categories and Functors. Academic Press 1970.

[5] M. Sweedler: Hopf Algebras. Benjamin – New York 1969.

[6] W.C. Waterhouse: Introduction to Affine Group Schemes. Springer Verlag – New York, Heidelberg, Berlin 1979