ON SYMBOLIC COMPUTATIONS IN BRAIDED MONOIDAL CATEGORIES

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Abstract. There are some powerful notations and tools to perform computations in with tensors, the Sweedler notation for coalgebras, the Einstein convention to reduce the number of summation signs in computations with tensors, the Penrose notation that has been further developed by Joyal and Street to a graphic calculus in braided monoidal categories. In 1977 I introduced a method of computation that looks very much like computation with ordinary elements or tensors, but can be performed in arbitrary monoidal categories, by using a Yoneda Lemma like technique. In the dual of the category of vector spaces this allows to work with ordinary coalgebras as if they were algebras. I will show how to expand this technique to braided monoidal categories, and develop some of the general rules of computation. As an application I will derive the well known result that the antipode of a Hopf algebra in a braided monoidal category is an algebra antihomomorphism which is expressed by the formulas $S(1) = 1$ and $S(ab) = \langle S(b)S(a), \tau\rangle$.

1. The Beginnings: The Sweedler-Heyneman notation

To describe the comultiplication of a $\mathbb{K}$-coalgebra in terms of elements we introduce a notation first introduced by Sweedler and Heyneman [?] similar to the notation $\nabla(a \otimes b) = ab$ used for algebras. Instead of $\Delta(c) = \sum c_i \otimes c'_i$ we write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}. \tag{1}$$

Observe that only the complete expression on the right hand side makes sense, not the components $c_{(1)}$ or $c_{(2)}$ which are *not* considered as families of elements of C. This notation alone does not help much in the calculations we have to perform later on. So we introduce a more general notation.

Definition 1.1. (Sweedler Notation) Let M be an arbitrary $\mathbb{K}$-module and C be a $\mathbb{K}$-coalgebra. Then there is a bijection between all multilinear maps

$$f : C \times \ldots \times C \longrightarrow M$$

and all linear maps

$$f^\sharp : C \otimes \ldots \otimes C \longrightarrow M.$$

These maps are associated to each other by the formula

$$f(c_1, \ldots, c_n) = f^\sharp(c_1 \otimes \ldots \otimes c_n) \tag{2}$$

or

$$f = f^\sharp \circ \otimes.$$

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This follows from the universal property of the tensor product. For $c \in C$ we define

$$\sum f(c_{(1)}, \ldots, c_{(n)}) := f^\sharp(\Delta^{n-1}(c)), \tag{3}$$

where Δ^{n-1} denotes the $(n-1)$-fold application of Δ, for example $\Delta^{n-1} = (\Delta \otimes 1 \otimes \ldots \otimes 1) \circ \ldots \circ (\Delta \otimes 1) \circ \Delta$.

In particular we obtain for the bilinear map $\otimes : C \times C \ni (c, d) \mapsto c \otimes d \in C \otimes C$ (with associated identity map)

$$\sum c_{(1)} \otimes c_{(2)} = \Delta(c), \tag{4}$$

and for the multilinear map $\otimes^2 : C \times C \times C \longrightarrow C \otimes C \otimes C$

$$\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = (\Delta \otimes 1)\Delta(c) = (1 \otimes \Delta)\Delta(c).$$

With this notation one verifies easily

$$\sum c_{(1)} \otimes \ldots \otimes \Delta(c_{(i)}) \otimes \ldots \otimes c_{(n)} = \sum c_{(1)} \otimes \ldots \otimes c_{(n+1)}$$

and

$$\begin{aligned}\sum c_{(1)} \otimes \ldots \otimes \epsilon(c_{(i)}) \otimes \ldots \otimes c_{(n)} &= \sum c_{(1)} \otimes \ldots \otimes 1 \otimes \ldots \otimes c_{(n-1)} \\ &= \sum c_{(1)} \otimes \ldots \otimes c_{(n-1)}\end{aligned}$$

This notation and its application to multilinear maps will also be used in more general contexts like comodules.

2. Symbolic Computations with tensors

Let $\mathcal{C}$ be a monoidal category. For objects $A, X \in \mathcal{C}$ define

$$A(X) := \mathrm{Mor}_{\mathcal{C}}(X, A).$$

We consider A as a "graded" or "variable" set with component $A(X)$ of "degree" X. Actually A is a (representable) functor from $\mathcal{C}$ into $\mathcal{S}et$.

Let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. Then we get "maps of variable sets" written by abuse of notation as $f : A(X) \longrightarrow B(X)$ with

$$f(a) := f \circ a. \tag{5}$$

This defines a natural transformation and by the Yoneda Lemma there is a bijection between the morphisms from A to B and the natural transformations from the functor A to the functor B.

In particular two morphisms $f, g : A \longrightarrow B$ are equal iff

$$\forall X \in \mathcal{C}, \forall a \in A(X) : f(a) = g(a).$$

Let $A, B, C \in \mathcal{C}$. Then $C(X \otimes Y)$ is a functor in two variables X and Y. Furthermore $A(X) \times B(Y)$ is also a functor in two variables denoted by $A \times B$. A natural transformation of functors in two variables $f : A \times B \longrightarrow C$ is called a *bimorphism.*

A special example of a bimorphism is

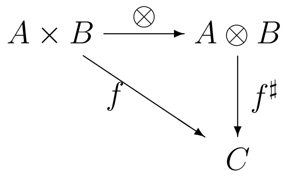
$$\otimes : A(X) \times B(Y) \longrightarrow A \otimes B(X \otimes Y) \text{ with } \otimes(a, b) := a \otimes b$$

where $a \otimes b : X \otimes Y \longrightarrow A \otimes B$. An element $a \otimes b \in A \otimes B(X \otimes Y)$ coming from two morphisms a, b is called a *decomposable tensor.*

If $f : A \times B \longrightarrow C$ is a bimorphism and $g : C \longrightarrow D$ is a morphism then $gf : A \times B \longrightarrow D$ is a bimorphism.

If $f : A \times B \longrightarrow C$ is a bimorphism and $g : U \longrightarrow A$ and $h : V \longrightarrow B$ are morphisms then $f(g \times h) : U \times V \longrightarrow C$ is a bimorphism.

Lemma 2.1. *For each bimorphism $f : A \times B \longrightarrow C$ there is exactly one morphism $f^\sharp : A \otimes B \longrightarrow C$ such that*



commutes.

Proof. This uses a Yoneda Lemma type argument. For details see [**?**, Lemma 1.1]. □

Occasionally if $h = f^\sharp$ is given then we write the associated bimorphism as $h^\flat := h \circ \otimes$, so that $(f^\sharp)^\flat = f$ and $(h^\flat)^\sharp = h$.

Given a bimorphism $f = f^\sharp \circ \otimes$ and $a \in A(X), b \in B(Y)$. Let $t = a \otimes b \in A \otimes B(X \otimes Y)$ be a decomposable tensor. Then $f(a, b) = f^\sharp(a \otimes b) = f^\sharp(t)$.

Similar remarks as above hold for *multimorphisms* $f : A_1 \times \ldots \times A_n \longrightarrow C$ and associated morphisms $f^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow C$. In particular we have for $a_i \in A_i(X_i), i = 1, \ldots, n$ and $t = a_1 \otimes \ldots \otimes a_n$

$$f(a_1, \ldots, a_n) = f^\sharp(t).$$

We introduce a first symbolic expression for all $t \in A_1 \otimes \ldots \otimes A_n(X)$ by

$$\boxed{f(t_1, \ldots, t_n) := f^\sharp(t).} \tag{6}$$

Observe that t is *not* a decomposable tensor in general. We have, however:

For the multimorphism $\otimes^{n-1} : A_1 \times \ldots \times A_n \longrightarrow A_1 \otimes \ldots \otimes A_n$ and the associated morphism $\otimes^\sharp = \mathrm{id} : A_1 \otimes \ldots \otimes A_n \longrightarrow A_1 \otimes \ldots \otimes A_n$ we get

$$t_1 \otimes \ldots \otimes t_n = t \tag{7}$$

for all "tensors" $t \in A_1 \otimes \ldots \otimes A_n(X)$. In particular we have then

$$f(t_1, \ldots, t_n) = f^\sharp(t_1 \otimes \ldots \otimes t_n). \tag{8}$$

Given $f^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow B_1 \otimes \ldots \otimes B_m$ and $t \in A_1 \otimes \ldots \otimes A_n(X)$. Then we may consider $f^\sharp(t)$ as an element of $B_1 \otimes \ldots \otimes B_m(X)$ hence

$$\begin{aligned} f^\sharp(t) = f^\sharp(t)_1 \otimes \ldots \otimes f^\sharp(t)_m = \\ = f(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)_1 \otimes \ldots \otimes f(t_1, \ldots, t_n)_m. \end{aligned} \tag{9}$$

Since $f^\sharp$ is also an element in $B_1 \otimes \ldots \otimes B_m(A_1 \otimes \ldots \otimes A_n)$ we can write $f^\sharp = f_1^\sharp \otimes \ldots \otimes f_m^\sharp$ and get

$$\begin{aligned} (f_1^\sharp \otimes \ldots \otimes f_m^\sharp)(t) = f^\sharp(t) = f^\sharp(t)_1 \otimes \ldots \otimes f^\sharp(t)_m \qquad \text{or} \\ (f_1^\sharp \otimes \ldots \otimes f_m^\sharp)(t_1 \otimes \ldots \otimes t_n) = f(t_1, \ldots, t_n)_1 \otimes \ldots \otimes f(t_1, \ldots, t_n)_m. \end{aligned} \tag{10}$$

If in addition $g^\sharp : B_1 \otimes \ldots \otimes B_m \longrightarrow C$ is given then we get

$$g(f^\sharp(t)_1, \ldots, f^\sharp(t)_m) = g^\sharp f(t_1, \ldots, t_n).$$

If $f_i : A_i \longrightarrow B_i, i = 1, \ldots, n$, $f^\sharp := f_1 \otimes \ldots \otimes f_n$, and $t \in A_1 \otimes \ldots \otimes A_n(X)$ are given then we have

$$f_1(t_1) \otimes \ldots \otimes f_n(t_n) = f^\sharp(t) = f^\sharp(t)_1 \otimes \ldots \otimes f^\sharp(t)_n.$$

Observe, we do not admit the same notation for an arbitrary morphism $f^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow B_1 \otimes \ldots \otimes B_m$. The problem is that certain natural transformations will commute with morphisms of the form $f_1 \otimes \ldots \otimes f_n : A_1 \otimes \ldots \otimes A_n \longrightarrow B_1 \otimes \ldots \otimes B_n$ but not with morphisms of the general form $f^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow B_1 \otimes \ldots \otimes B_m$ even if $m = n$.

Lemma 2.2. *Given multimorphisms $f, g : A_1 \times \ldots \times A_n \longrightarrow C$ with associated morphisms $f^\sharp, g^\sharp$. Then the following are equivalent:*

1. $f^\sharp = g^\sharp$,
2. $\forall X_i, \forall a_1, \ldots, a_n \in A_i(X_i) : f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$,
3. $\forall X, \forall t \in A_1 \otimes \ldots \otimes A_n(X) : f(t_1, \ldots, t_n) = g(t_1, \ldots, t_n)$.

Proof. (1) $\Longrightarrow f = g \Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow f = g \Longrightarrow$ (1). □

This notation will be used to express and compute certain identities of morphisms. We explain this by the following example. Let $(A, \mu : A \otimes A \longrightarrow A)$ be given. We want to express associativity by elements. Write $ab := \mu(a \otimes b) \in A(X \otimes Y)$. Then $(ab)c = \mu(\mu(a \otimes b) \otimes c) \in A((X \otimes Y) \otimes Z)$. Similarly $a(bc) = \mu(a \otimes \mu(b \otimes c)) \in A(X \otimes (Y \otimes Z))$. In order to compare these two products we apply $A(\alpha) : A(X \otimes (Y \otimes Z)) \longrightarrow A((X \otimes Y) \otimes Z)$ to get $(ab)c = a(bc) \circ \alpha = A(\alpha)(a(bc))$ iff (A, μ) is associative.

Since most such computations can be transferred to a strict monoidal category, we are going to assume from now on that $\mathcal{C}$ is a strict monoidal category.

Then $(A, \mu : A \otimes A \longrightarrow A)$ is associative iff $(ab)c = a(bc)$.

3. Braidings and tensors

Let $\mathcal{C}$ be a strict monoidal category that is braided. Let $\rho \in B_n$ be a braid in the braid group with canonical image $\overline{\rho} \in S_n$. Let $\sigma := \overline{\rho}^{-1}$. Let $\rho : A_1 \otimes \ldots \otimes A_n \longrightarrow A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)}$ also denote the associated braid action on the n-fold tensor product. So ρ is a natural transformation of functors in n variables.

Let $f^\sharp : A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)} \longrightarrow B$ be a morphism in $\mathcal{C}$ and $f := f^\sharp \circ \otimes^n : A_{\sigma(1)}(X_{\sigma(1)}) \times \ldots \times A_{\sigma(n)}(X_{\sigma(n)}) \longrightarrow B(X_{\sigma(1)} \otimes \ldots \otimes X_{\sigma(n)})$ be the associated multimorphism.

We want to study the application of $f^\sharp$ on a tensor $t \in A_1 \otimes \ldots \otimes A_n(X)$ if ρ is first applied to t. First assume that t is a decomposable tensor of the form $t = a_1 \otimes \ldots \otimes a_n \in A_1 \otimes \ldots \otimes A_n(X_1 \otimes \ldots \otimes X_n)$ with $a_i \in A_i(X_i)$. We get

$$f(\rho(t)_1, \ldots, \rho(t)_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})\rho \tag{11}$$

since $f(\rho(t)_1, \ldots, \rho(t)_n) = f^\sharp\rho(t) = f^\sharp\rho(a_1 \otimes \ldots \otimes a_n) = f^\sharp(a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)})\rho = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})\rho$ where we used that ρ is a natural transformation.

Observe that in the symbolic notation ρ is not really applied to $a_1 \otimes \ldots \otimes a_n$ as it is in ordinary computations in braided categories, it changes only the order of the components with $\sigma \in S_n$. We would only be interested in the expression $f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ in ordinary computations, but some information about ρ is lost, if we study this term in symbolic calculations. View ρ as an index for this expression and write

$$\langle f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \rho\rangle := f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})\rho. \tag{12}$$

In particular we have $\langle f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \rho\rangle = f^\sharp(\rho(a_1 \otimes \ldots \otimes a_n))$.

We extend this notation to arbitrary tensors $t = t_1 \otimes \ldots \otimes t_n \in A_1 \otimes \ldots \otimes A_n(X)$ (see equation (**??**)).

Definition 3.1. We define the map

$$\langle ., ., .\rangle : \mathrm{Nat}(A_{\sigma(1)} \times \ldots \times A_{\sigma(n)}, B) \times (A_1 \otimes \ldots \otimes A_n)(X) \times B_n \longrightarrow B(X)$$

by $\langle f, t, \rho\rangle := f^\sharp(\rho(t)) = f^\sharp \circ \rho \circ t$, where $f : A_{\sigma(1)} \times \ldots \times A_{\sigma(n)} \longrightarrow B$ is a multimorphism, $t = t_1 \otimes \ldots \otimes t_n \in A_1 \otimes \ldots \otimes A_n(X)$ is a variable or argument, and $\rho \in B_n$ is a braid. We write for $\langle f, t, \rho\rangle$ also $\langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle$ and define

$$\boxed{\langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle := f^\sharp(\rho(t)).} \tag{13}$$

The expression $f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$ taken separately is clearly not defined, except in the case where t is a decomposable tensor. Observe that $t : X \longrightarrow A_1 \otimes \ldots \otimes A_n$ and $f^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow B$ are morphisms so that ρ can operate on the range of t and on the domain of $f^\sharp$. We tacitly assume in writing down an expression $\langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle$ that the range and domain of $f^\sharp$ and t are given and fixed. (Otherwise an operation of ρ would not be well defined.) If this separation is not quite clear we also use the notation

$$\langle f[t_{\sigma(1)}, \ldots, t_{\sigma(n)}], \rho\rangle := f^\sharp(\rho(t)).$$

In some cases one has to name the arguments explicitly, which are used in a concrete computation.

Theorem 3.2. (Comparison theorem): *Given $f^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow B$ and $g^\sharp : A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)} \longrightarrow B$. Then the following are equivalent:*

(1) $g^\sharp \circ \rho = f^\sharp$,
(2) $\forall a_1, \ldots, a_n \in A_i(X_i) : f(a_1, \ldots, a_n) = \langle g(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \rho\rangle$,
(3) $\forall t \in A_1 \otimes \ldots \otimes A_n(X) : f(t_1, \ldots, t_n) = \langle g(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle$.

Proof. (1) $\Longrightarrow$ (3): $f(t_1, \ldots, t_n) = f^\sharp(t) = g^\sharp(\rho(t)) = \langle g(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle$.

(3) $\Longrightarrow$ (2): Take $t := a_1 \otimes \ldots \otimes a_n$. Then $f(a_1, \ldots, a_n) = f(t_1, \ldots, t_n) = \langle g(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle = g^\sharp(\rho(t)) = g^\sharp(\rho(a_1 \otimes \ldots \otimes a_n)) = \langle g(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \rho\rangle$ as in equation (**??**).

(2) $\Longrightarrow$ (1): Take $X_i = A_i$, $a_i = \mathrm{id}_i$. Then $f(a_1, \ldots, a_n) = \langle g(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \rho\rangle$ implies $f^\sharp = g^\sharp \circ \rho$. □

4. Rules of Computation

4.1. **Special cases:** With this symbolic notation we get the following *rules of computation.*

If $\rho = \mathrm{id}$ then

$$\langle f(t_1, \ldots, t_n), \mathrm{id}\rangle = f^\sharp(t) = f(t_1, \ldots, t_n). \tag{14}$$

So the identity braid in the pairing of our notation can be omitted.

If $f^\sharp = \mathrm{id}_{A_{\sigma(1)}\otimes\ldots\otimes A_{\sigma(n)}}$ then we get

$$\langle t_{\sigma(1)} \otimes \ldots \otimes t_{\sigma(n)}, \rho\rangle = \rho(t) = \rho \circ t. \tag{15}$$

4.2. **Equality and Substitution:** We begin with a warning. Usually certain terms in more complex expressions may be substituted by equal terms. However, separate components of the form $f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$ in our expression $\langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle$ may not be replaced, even if it looks as if they could be equal.

For an example let indecomposable tensors $a_1 \otimes a_2, b_1 \otimes b_2 \in A \otimes A(X)$ be given, and let $m^\sharp : A \otimes A \longrightarrow A$ be a multiplication. Assume $m^\sharp(a_1 \otimes a_2) = a_1a_2 = b_1b_2 = m^\sharp(b_1 \otimes b_2)$. Then in general

$$\langle a_1a_2, \tau^2\rangle \neq \langle b_1b_2, \tau^2\rangle.$$

We will find a certain replacement or substitution rule in (**??**). This expression, however, differs from (**??**) in that here we have "elements" (or a "function applied to specific elements") whereas we have "functions" in (**??**). In terms of morphisms we may have

$$(X \xrightarrow{a} A \otimes A \xrightarrow{m^\sharp} A) = (X \xrightarrow{b} A \otimes A \xrightarrow{m^\sharp} A)$$

and at the same time

$$(X \xrightarrow{a} A \otimes A \xrightarrow{\tau^2} A \otimes A \xrightarrow{m^\sharp} A) \neq (X \xrightarrow{b} A \otimes A \xrightarrow{\tau^2} A \otimes A \xrightarrow{m^\sharp} A).$$

If $a = a_1 \otimes a_2$, $b = b_1 \otimes b_2$ are decomposable tensors then we have indeed

$$(X \otimes X \xrightarrow{a} A \otimes A \xrightarrow{\tau^2} A \otimes A \xrightarrow{m^\sharp} A) = (X \otimes X \xrightarrow{b} A \otimes A \xrightarrow{\tau^2} A \otimes A \xrightarrow{m^\sharp} A)$$

since τ^2 is a natural transformation.

By the definition of $\langle ., ., .\rangle$ we may certainly substitute equal expressions for the separate components f, ρ, and t. The following gives a somewhat more general rule for substitutions in case we have decomposable tensors as arguments.

Proposition 4.1. *Given $f, g : A_{\sigma(1)} \times \ldots \times A_{\sigma(1)} \longrightarrow B$, $\rho \in B$ and $a_i, b_i \in A_i(X_i)$ defining decomposable tensors $a = a_1 \otimes \ldots \otimes a_n$ and $b = b_1 \otimes \ldots \otimes b_n$. If*

$$f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = g(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$$

then

$$\langle f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \rho\rangle = \langle g(b_{\sigma(1)}, \ldots, b_{\sigma(n)}), \rho\rangle.$$

Proof. This is a simple computation:

$$\begin{aligned}
\langle f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \rho\rangle &= f^\sharp \circ \rho \circ a \\
&= f^\sharp \circ (a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}) \circ \rho \\
&= f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \circ \rho \\
&= g(b_{\sigma(1)}, \ldots, b_{\sigma(n)}) \circ \rho \\
&= g^\sharp \circ (b_{\sigma(1)} \otimes \ldots \otimes b_{\sigma(n)}) \circ \rho \\
&= g^\sharp \circ \rho \circ b \\
&= \langle g(b_{\sigma(1)}, \ldots, b_{\sigma(n)}), \rho\rangle.
\end{aligned}$$

□

The proposition shows that $\langle f(a_{\sigma(1)},\ldots,a_{\sigma(n)}),\rho\rangle$ does indeed only depend on the value of $f(a_{\sigma(1)},\ldots,a_{\sigma(n)})$ whereas in general it depends separately on f and a. So we may replace the term $f(a_{\sigma(1)},\ldots,a_{\sigma(n)})$ by its value because the result does not depend on the particular representation.

If $t = \mathrm{id} : A_1 \otimes \ldots \otimes A_n \longrightarrow A_1 \otimes \ldots \otimes A_n$ then $t = \mathrm{id}_{A_1} \otimes \ldots \otimes \mathrm{id}_{A_n}$ is a decomposable tensor. In this case we may apply Proposition **??** and have $f(\mathrm{id}_{A_{\sigma(1)}},\ldots,\mathrm{id}_{A_{\sigma(n)}})$ alone in $\langle f(\mathrm{id}_{A_{\sigma(1)}},\ldots,\mathrm{id}_{A_{\sigma(n)}}),\rho\rangle$ is defined and we have $f(\mathrm{id}_{A_{\sigma(1)}},\ldots,\mathrm{id}_{A_{\sigma(n)}}) = f^\sharp(\mathrm{id}_{A_{\sigma(1)}} \otimes \ldots \otimes \mathrm{id}_{A_{\sigma(n)}}) = f^\sharp$ so that we we can write $\langle f^\sharp,\rho\rangle$ and get

$$\langle f^\sharp,\rho\rangle = f^\sharp \circ \rho. \tag{16}$$

Hence the expression $\langle f^\sharp,\rho\rangle$ makes sense without the argument t. The argument can safely be assumed to be $t = \mathrm{id}$.

If $f_1^\sharp, f_2^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow B$ then we have

$$\langle f_1^\sharp,\rho\rangle = \langle f_2^\sharp,\rho\rangle \iff f_1^\sharp = f_2^\sharp, \tag{17}$$

since ρ is an isomorphism, and

$$\langle f_1^\sharp,\rho_1\rangle = \langle f_2^\sharp,\rho_2\rangle \iff f_1^\sharp = \langle f_1^\sharp,\mathrm{id}\rangle = \langle f_2^\sharp,\rho_2\rho_1^{-1}\rangle. \tag{18}$$

4.3. **Compatibility with elements of the braid group:** If $t = a_1 \otimes \ldots \otimes a_n$ with $(a_1,\ldots,a_n) \in A_1(X_1) \times \ldots \times A_n(X_n)$ we get

$$a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)} = \rho(a_1 \otimes \ldots \otimes a_n)\rho^{-1} \tag{19}$$

since ρ is a natural transformation where the expression $a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}$ is the morphism $a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)} : X_{\sigma(1)} \otimes \ldots \otimes X_{\sigma(n)} \longrightarrow A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)}$.

If $t \in A_1 \otimes \ldots \otimes A_n(X_1 \otimes \ldots \otimes X_n)$ then we can use equation (**??**) to define $\langle \rho t \rho^{-1},\rho\rangle$ and get

$$\langle \rho t \rho^{-1},\rho\rangle = \rho t \rho^{-1} \circ \rho = \rho \circ t = \langle t_{\sigma(1)} \otimes \ldots \otimes t_{\sigma(n)},\rho\rangle$$

In view of equations (**??**) and (**??**) we define for $t \in A_1 \otimes \ldots \otimes A_n(X_1 \otimes \ldots \otimes X_n)$

$$t_{\sigma(1)} \otimes \ldots \otimes t_{\sigma(n)} := \rho t \rho^{-1}. \tag{20}$$

We will write $[\rho](t) := \rho t \rho^{-1} = t_{\sigma(1)} \otimes \ldots \otimes t_{\sigma(n)}$ if $t \in A_1 \otimes \ldots \otimes A_n(X_1 \otimes \ldots \otimes X_n)$. Then $[\rho](a_1 \otimes \ldots \otimes a_n) = a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}$.

4.4. **Composition:** *Remark:* The terms encountered in this section are complicated and usually have no simplification. They are given for completeness and will not be used in the sequal.

Certain of these expressions can be composed or applied to each other. In particular we get the following. Given $\rho_1,\rho_2 \in B_n$. If $f^\sharp : A_{\sigma_2\sigma_1(1)} \otimes \ldots \otimes A_{\sigma_2\sigma_1(n)} \longrightarrow B$ and $t \in A_1 \otimes \ldots \otimes A_n(X)$ then

$$\langle f^\sharp,\rho_1\rangle\left(\langle t_{\sigma_2(1)} \otimes \ldots \otimes t_{\sigma_2(n)},\rho_2\rangle\right) = \langle f(t_{\sigma_2\sigma_1(1)},\ldots,t_{\sigma_2\sigma_1(n)}),\rho_1\rho_2\rangle. \tag{21}$$

The following are immediately clear

$$\langle f^\sharp,\rho_1\rho_2\rangle = \langle\langle f^\sharp,\rho_1\rangle,\rho_2\rangle, \tag{22}$$

$$\langle f_1^\sharp,\rho_1\rangle \otimes \langle f_2^\sharp,\rho_2\rangle = \langle f_1^\sharp \otimes f_2^\sharp,\rho_1 \otimes \rho_2\rangle, \tag{23}$$

If $t \in A_1 \otimes \ldots \otimes A_n(X_1 \otimes \ldots \otimes X_n)$ then

$$\langle f^\sharp,\rho_1\rangle \circ \langle t,\rho_2\rangle = \langle f^\sharp \circ [\rho_1](t),\rho_1 \circ \rho_2\rangle. \tag{24}$$

If $f^\sharp : A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)} \longrightarrow B$ and $g : B \longrightarrow C$ are given then

$$g(\langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle) = \langle gf(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle. \tag{25}$$

If $t \in A_1 \otimes \ldots \otimes A_n(X)$ then we get from (**??**)

$$f^\sharp(\langle t_{\sigma(1)} \otimes \ldots \otimes t_{\sigma(n)}, \rho\rangle) = \langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle. \tag{26}$$

The naturality of braids leads to a very usefull rule for a *change of variables or of arguments*. Given $f_i : A_i \longrightarrow B_i, i = 1, \ldots, n$, $g^\sharp : B_1 \otimes \ldots \otimes B_n \longrightarrow C$, and $t \in A_1 \otimes \ldots \otimes A_n(X)$. Then we get

(27)
$$\langle g^\sharp(f_{\sigma(1)}(t_{\sigma(1)}) \otimes \ldots \otimes f_{\sigma(n)}(t_{\sigma(n)})), \rho\rangle = \langle g(f_{\sigma(1)}(t_{\sigma(1)}), \ldots, f_{\sigma(n)}(t_{\sigma(n)})), \rho\rangle$$
$$= \langle (g^\sharp(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}))^\flat(t_{\sigma(1)}, \ldots, t_{\sigma(n)}), \rho\rangle = \langle (g(f_{\sigma(1)}, \ldots, f_{\sigma(n)}))(t_{\sigma(n)}, \ldots, t_{\sigma(n)}), \rho\rangle.$$

or

$$\langle g[f_{\sigma(1)}(t_{\sigma(1)}), \ldots, f_{\sigma(n)}(t_{\sigma(n)})], \rho\rangle = \langle (g^\sharp(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)})\otimes^n)[t_{\sigma(1)}, \ldots, t_{\sigma(n)}], \rho\rangle.$$

where we change from the "arguments" $f_1(t_1), \ldots, f_n(t_n)$ to the "arguments" $t_1, \ldots, t_n$.

More general terms for composition are obtained from $f^\sharp : A_1 \otimes \ldots \otimes A_n \longrightarrow B_1 \otimes \ldots \otimes B_m$ as

$$\langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})_1 \otimes \ldots \otimes f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})_m, \rho\rangle.$$

We get compositions of such terms which in general cannot be simplified. For $g^\sharp : B_1 \otimes \ldots \otimes B_m \longrightarrow C$ we get

(28)
$$\big\langle g\big(\langle f(t_{\sigma_2(1)}, \ldots, t_{\sigma_2(n)})_{\sigma_1(1)}, \ldots, f(t_{\sigma_2(1)}, \ldots, t_{\sigma_2(n)})_{\sigma_1(m)}, \rho_2\rangle\big), \rho_1\big\rangle =$$
$$\langle g(f_{\sigma_1(1)}, \ldots, f_{\sigma_1(m)}), \rho_1\rangle \circ \langle f^\sharp(t_{\sigma_2(1)}, \ldots, t_{\sigma_2(n)})_1 \otimes \ldots \otimes f^\sharp(t_{\sigma_2(1)}, \ldots, t_{\sigma_2(n)})_m, \rho_2\rangle.$$

Furthermore we may take tensor products of terms as follows:

$$\begin{aligned} &\langle f(t_{\sigma_1(1)}, \ldots, t_{\sigma_1(n)})_1 \otimes \ldots \otimes f(t_{\sigma_1(1)}, \ldots, t_{\sigma_1(n)})_m, \rho_1\rangle \otimes \\ &\qquad \langle g(s_{\sigma_2(1)}, \ldots, s_{\sigma_2(r)})_1 \otimes \ldots \otimes g(u_{\sigma_2(1)}, \ldots, u_{\sigma_2(r)})_s, \rho_2\rangle = \\ &\langle f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}, s_{\sigma(n+1)}, \ldots, s_{\sigma 2(n+r)})_1 \otimes \ldots \\ &\qquad \ldots \otimes g(t_{\sigma(1)}, \ldots, t_{\sigma(n)}, s_{\sigma(n+1)}, \ldots, s_{\sigma 2(n+r)})_{m+s}, \rho_1 \otimes \rho_2\rangle. \end{aligned} \tag{29}$$

5. Coalgebras, Hopf Algebras, and beyond

5.1. **Linear algebra:** First we observe some rules from Linear Algebra. Let $\kappa \in I(X)$, $a_i \in A_i(Y_i)$, and $f : A_1 \times \ldots \times A_n \longrightarrow B$ be a multimorphism. Let κa_i resp $a_i\kappa$ denote the multiplication given by $\lambda : I \otimes A_i = A_i$ and $\rho : A_i \otimes I = A_i$. Then we have

$$f(a_1, \ldots, a_i\kappa, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_i, \kappa a_{i+1}, \ldots, a_n) \tag{30}$$

and for any morphism $g : A \longrightarrow B$

$$\kappa f(a) = f(\kappa a) \text{ and } f(a)\kappa = f(a\kappa). \tag{31}$$

A more interesting formula for a braiding is obtained as

$$\kappa a = \langle a\kappa, \tau\rangle \text{ and } \kappa a = \langle a\kappa, \tau^{-1}\rangle. \tag{32}$$

Let (A, ∇, η) be an algebra then

$$\eta(\kappa) \cdot a = \kappa a \text{ and } a \cdot \eta(\kappa) = a\kappa. \tag{33}$$

5.2. **The Sweedler-Heyneman Notation:** Let H be a Hopf algebra in $\mathcal{C}$. For $a \in H(X)$ we want to have $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Let $f : H \times \ldots \times H \longrightarrow M$ with associated morphism $f^\sharp : H \otimes \ldots \otimes H \longrightarrow M$ be given. Let $a \in H(X)$, then $\Delta^{n-1}(a) \in H \otimes \ldots \otimes H(X)$. Using the definition in equation (**??**) we define

$$\boxed{f(a_{(1)}, \ldots, a_{(n)}) := f^\sharp(\Delta^{n-1}(a)).} \tag{34}$$

As in equation (**??**) (and also as in (**??**)) this gives the formula $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Then by equation (**??**)

$$\langle f(a_{(\sigma(1))}, \ldots, a_{(\sigma(n))}), \rho\rangle = f^\sharp(\rho(a_{(1)} \otimes \ldots \otimes a_{(n)})).$$

Using the coassociativity of Δ we get the following rule for a change of the number of arguments

$$\langle f(a_{(\sigma(1))}, \ldots, \Delta(a_{(\sigma(i))}), \ldots, a_{(\sigma(n))}), \rho\rangle = \langle f(a_{(\sigma(1))}, \ldots, a_{(\sigma(n+1))}), \rho_i\rangle \tag{35}$$

where ρ_i acts like ρ but switches the braids i and $i+1$ in parallel.

Observe, however, that the braid does not change in

$$\begin{aligned} &\langle f(a_{(\sigma(1))}, \ldots, a_{(\sigma(i))}, \ldots, a_{(\sigma(n+1))}), \rho\rangle \\ = &\langle f(a_{(\sigma(1))}, \ldots, a_{(\sigma(i))(1)}, \ldots, a_{(\sigma(i))(2)}, \ldots, a_{(\sigma(n))}), \rho\rangle. \end{aligned} \tag{36}$$

5.3. **Hopf Algebras:** As usual one gets

$$\eta\varepsilon(a_{(1)})a_{(2)} = a = a_{(1)}\eta\varepsilon(a_{(2)}) \tag{37}$$

and

$$a_{(1)}S(a_{(2)}) = \eta\varepsilon(a) = S(a_{(1)})a_{(2)}. \tag{38}$$

These last equations are to be considered as functions in one argument a, so they allow substitution at any position where a occurs.

The compatibility of multiplication and comultiplication is expressed by

$$(ab)_{(1)} \otimes (ab)_{(2)} = \langle a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \tau_{23}\rangle \tag{39}$$

where $a \otimes b \in H \otimes H(X)$ and τ is the basic braid map interchanging two factors. Furthermore we have from (**??**)

$$\eta\varepsilon(a_{(1)})a_{(2)} = a = \langle a_{(2)}\eta\varepsilon(a_{(1)}), \tau\rangle. \tag{40}$$

Theorem 5.1. *If H is a braided Hopf algebra then the antipode S of the Hopf algebra H is an algebra τ-antihomomorphism, i.e.*

$$S(ab) = \langle S(b)S(a), \tau\rangle. \tag{41}$$

Proof. We compute

$$
\begin{aligned}
S(ab) &= S((ab)_{(1)}\eta\varepsilon((ab)_{(2)})) \\
&\quad \text{(by (\textbf{??}),} \quad \text{the arguments are } a,\, b) \\
&= S((ab)_{(1)})\ \eta\varepsilon((ab)_{(2)}) \\
&\quad \text{(by (\textbf{??}) and (\textbf{??}))} \\
&= \langle S(a_{(1)}b_{(1)})\ \eta\varepsilon(a_{(2)}b_{(2)}), \tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}),} \quad \text{change to 4 arguments } a_{(1)}, a_{(2)}, b_{(1)}, b_{(2)}) \\
&= \langle S(a_{(1)}b_{(1)})\ \varepsilon(a_{(2)}b_{(2)}), \tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}))} \\
&= \langle S(a_{(1)}b_{(1)})\ \varepsilon(a_{(2)})\ \varepsilon(b_{(2)}), \tau_{23}\rangle \\
&\quad (\varepsilon \text{ is multiplicative for } \textit{all} \text{ elements}) \\
&= \langle S(a_{(1)}b_{(1)})\ \eta\varepsilon(a_{(2)})\ \varepsilon(b_{(2)}), \tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}))} \\
&= \langle S(a_{(1)}b_{(1)})\ a_{(2)(1)}S(a_{(2)(2)})\ \varepsilon(b_{(2)}), \tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}),} \quad \text{the arguments are still } a_{(1)}, a_{(2)}, b_{(1)}, b_{(2)}) \\
&= \langle S(a_{(1)}b_{(1)})\ a_{(2)}S(a_{(3)})\ \varepsilon(b_{(2)}), \tau_{23}\tau_{34}\rangle \\
&\quad \text{(by (\textbf{??}), change of arguments to } a_{(1)}, a_{(2)}, a_{(3)}, b_{(1)}, b_{(2)}) \\
&\quad \text{(change of arguments by (\textbf{??}) to } a_{(1)}, a_{(2)}, S(a_{(3)}), b_{(1)}, \varepsilon(b_{(2)})) \\
&= \langle S(a_{(1)}b_{(1)})\ a_{(2)}\ \varepsilon(b_{(2)})\ S(a_{(3)}), \tau_{45}\tau_{23}\tau_{34}\rangle \\
&\quad \text{(by (\textbf{??}), change arguments back to } a_{(1)}, a_{(2)}, a_{(3)}, b_{(1)}, b_{(2)} \text{ by (\textbf{??}))} \\
&= \langle S(a_{(1)}b_{(1)})\ a_{(2)}\ \eta\varepsilon(b_{(2)})\ S(a_{(3)}), \tau_{45}\tau_{23}\tau_{34}\rangle \\
&\quad \text{(by (\textbf{??}))} \\
&= \langle S(a_{(1)}b_{(1)})\ a_{(2)}\ b_{(2)}\ S(b_{(3)})\ S(a_{(3)}), \tau_{56}\tau_{45}\tau_{23}\tau_{34}\rangle \\
&\quad \text{(as above by (\textbf{??}), (\textbf{??}))} \\
&= \langle S(a_{(1)}b_{(1)})\ a_{(2)}\ b_{(2)}\ S(b_{(3)})\ S(a_{(3)}), \tau_{23}\tau_{56}\tau_{45}\tau_{34}\rangle \\
&\quad \text{(change of braid map)} \\
&= \langle S(a_{(1)(1)}b_{(1)(1)})\ a_{(1)(2)}\ b_{(1)(2)}\ S(b_{(2)})\ S(a_{(2)}), \tau_{23}\tau_{56}\tau_{45}\tau_{34}\rangle \\
&\quad \text{(by (\textbf{??}), the arguments are } a_{(1)(1)}, a_{(1)(2)}, a_{(2)}, b_{(1)(1)}, b_{(1)(2)}, b_{(2)}) \\
&= \langle S((a_{(1)}b_{(1)})_{(1)})\ (a_{(1)}b_{(1)})_{(2)}\ S(b_{(2)})\ S(a_{(2)}), \tau_{34}\tau_{23}\rangle \quad (\textbf{??}) \\
&\quad \text{(change to 4 arguments } a_{(1)}, a_{(2)}, b_{(1)}, b_{(2)}, \text{ apply (\textbf{??}) twice)} \\
&\quad \text{(read from lower line to upper line)} \\
&= \langle \eta\varepsilon(a_{(1)}b_{(1)})\ S(b_{(2)})\ S(a_{(2)}), \tau_{34}\tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}))} \\
&= \langle \varepsilon(a_{(1)})\ \varepsilon(b_{(1)})\ S(b_{(2)})\ S(a_{(2)}), \tau_{34}\tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}) and multiplicativity of } \varepsilon) \\
&= \langle \varepsilon(a_{(1)})\ S(b)\ S(a_{(2)}), \tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}) together with change of arguments to } a_{(1)}, a_{(2)}, b) \\
&= \langle S(b)\ \varepsilon(a_{(1)})\ S(a_{(2)}), \tau_{12}\tau_{23}\rangle \\
&\quad \text{(by (\textbf{??}))}
\end{aligned}
$$

$= \langle S(b)\ S(a), \tau\rangle$
(by (**??**) together with change of arguments to a, b).

$\square$

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