SKEW-PRIMITIVE ELEMENTS OF QUANTUM GROUPS AND BRAIDED LIE ALGEBRAS

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Contents

In the study of Lie groups, of algebraic groups or of formal groups, the concept of Lie algebras plays a central role. These Lie algebras consist of the primitive elements. It is difficult to introduce a similar concept for quantum groups. Many important quantum groups have braided Hopf algebras as building blocks. As we will see most primitive elements live in these braided Hopf algebras. In [P1] and [P2] we introduced the concept of braided Lie algebras for this type of Hopf algebras. In this paper we will give a survey of and a motivation for this concept together with some interesting examples.

By the work of Yetter [Y] we know that the category of Yetter-Drinfel'd modules is in a sense the most general category of modules carrying a natural braiding on the tensor power of each module (instead of a symmetric structure). The study of algebraic structures in such a category is a generalization of the study of group graded algebraic structures. We will describe the braid structure in the category of Yetter-Drinfel'd modules, the concept of a Hopf algebra in this category, and explain the reason why we want to study such braided Hopf algebras.

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One of the big obstacles in this theory is the fact, that the set of primitive elements $P(H)$ of a braided Hopf algebra H does not form a Lie algebra in the ordinary or slightly generalized sense. We will show, however, that there is still an algebraic structure on $P(H)$ consisting of partially defined n-ary bracket operations, satisfying certain generalizations of the anti-symmetry and Jacobi relations. We call this structure a braided Lie algebra. This will generalize ordinary Lie algebras, Lie super algebras, and Lie color algebras. Furthermore we will show that the universal enveloping algebra of a braided Lie algebra is again a braided Hopf algebra leading us back to quantum groups.

Primitive elements of an ordinary Hopf algebra L are elements $x \in L$ satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$. The set of primitive elements $P(L)$ of L forms a Lie algebra induced by the Lie algebra structure $[x, y] := xy - yx$ on L. In fact one verifies that $\Delta([x, y]) = [x, y] \otimes 1 + 1 \otimes [x, y]$ if $x, y \in P(L)$.

Since primitive elements are cocommutative they can only generate a cocommutative Hopf subalgebra of L. More general elements, skew-primitive elements with $\Delta(x) = x \otimes g + g' \otimes x$, are needed to generate quantum groups or general (noncommutative noncocommutative) Hopf algebras. But the skew-primitive elements do not form a Lie algebra anymore.

Many quantum groups are Hopf algebras of the special form $L = kG \star H = kG \otimes H$ where H is a braided graded Hopf algebra over a commutative group G [L, Ma, R, S]. In this situation the primitive elements of H are skew-primitive elements of L. So the structure of a braided Lie algebra on the set of primitive elements in H induces a similar structure on a subset of the skew-primitive elements of L.

The central idea leading to the structure of braided Lie algebras is the concept of symmetrization. For any module P in the category of Yetter-Drinfel'd modules the n-th tensor power P^n of P has a natural braid structure. We construct submodules $P^n(\zeta) \subseteq P^n$ for any nonzero ζ in the base field k, that carry a (symmetric) S_n-structure. This is essentially an eigenspace construction for a family of operators. The Lie algebra multiplications will be defined on these S_n-modules $P^n(\zeta)$ for primitive n-th roots of unity ζ.

In the group graded case, there is a fairly explicit construction of these symmetrizations. In the last section of this paper we describe them in the C_{p^n}-graded case for a cyclic group of prime power order.

Apart from the explicit examples of braided Lie algebras we gave in [P1] we showed in [P2] that the set of derivations $Der(A)$ of an algebra A in $\mathcal{YD}_K^K$ forms a braided Lie algebra. This is based on the existence of inner hom-objects in $\mathcal{YD}_K^K$. In Theorem 6.3 we will show that the category $\mathcal{YD}_K^K$ is a closed monoidal category. We also construct another large family of braided Lie algebras consisting of skew-symmetric endomorphisms of a Yetter-Drinfel'd module with a bilinear form. This generalizes the construction of Lie algebras of classical groups.

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1. Quantum groups, Yetter-Drinfel'd algebras, and G-graded algebras

Quantum groups arise from deformations of universal enveloping algebras of Lie algebras. They often have the following form.

Let $K = kG$ be the group algebra of a commutative group. Let H be a Hopf algebra in the category of Yetter-Drinfel'd modules over K. Then the biproduct $K \star H$ is a Hopf algebra [R, Ma, FM], which in general is neither commutative nor cocommutative. More generally quantum groups are Hopf algebras of the form $H_- \star K \star H_+$ where H_+ and H_- are dual to each other [L, S].

We investigate the question what the primitive (Lie) elements of these quantum groups are and whether they carry a specific structure (of a Lie algebra).

Let us first introduce the concept of Yetter-Drinfel'd modules over a Hopf algebra K with bijective antipode (see [M] 10.6.10). A *Yetter-Drinfel'd module* or *crossed module* over a Hopf algebra K is a vector space M which is a right K-module and a right K-comodule such that

$$\sum (x \cdot c)_{[0]} \otimes (x \cdot c)_{[1]} = \sum x_{[0]} \cdot c_{(2)} \otimes S(c_{(1)}) x_{[1]} c_{(3)} \tag{1}$$

for all $x \in M$ and all $c \in K$. Here we use the Sweedler notation $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ with $\Delta : K \longrightarrow K \otimes K$ and $\delta(x) = \sum x_{[0]} \otimes x_{[1]}$ with $\delta : M \longrightarrow M \otimes K$. The Yetter-Drinfel'd modules form a category $\mathcal{YD}_K^K$ in the obvious way (morphisms are the K-module homomorphisms which are also K-comodule homomorphisms).

The most interesting structure on $\mathcal{YD}_K^K$ is given by its tensor products. It is well known that the tensor product $M \otimes N$ of two vector spaces which are K-modules is again a K-module (via the comultiplication or diagonal of K). If M and N are K-comodules then their tensor product is also a K-comodule (via the multiplication of K). If M and N are Yetter-Drinfel'd modules over K, then their tensor product is a Yetter-Drinfel'd module, too. So with this tensor product $\mathcal{YD}_K^K$ is a monoidal category.

A monoidal or tensor product structure on an arbitrary category $\mathcal{C}$ allows to define the notion of an algebra A with a multiplication $\nabla : A \otimes A \longrightarrow A$ which is associative and unitary (by $u : k \longrightarrow A$). Similarly one can define coalgebras in $\mathcal{C}$. There is, however, a problem with defining a bialgebra or Hopf algebra H in $\mathcal{C}$. In the compatibility condition between multiplication and comultiplication of H

$$\sum (h \cdot h')_{(1)} \otimes (h \cdot h')_{(2)} = \sum h_{(1)} \cdot h'_{(1)} \otimes h_{(2)} \cdot h'_{(2)}$$

one uses in the formation of the right hand side $\sum h_{(1)} \cdot h'_{(1)} \otimes h_{(2)} \cdot h'_{(2)} = (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\sum h_{(1)} \otimes h_{(2)} \otimes h'_{(1)} \otimes h'_{(2)}) = (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)(h \otimes h')$ a switch or exchange function $\tau : H \otimes H \longrightarrow H \otimes H$ in the category $\mathcal{C}$.

There exists such a nontrivial morphism $\tau : M \otimes N \longrightarrow N \otimes M$ in the category $\mathcal{YD}^K_K$ of Yetter-Drinfel'd modules. It is given by

$$\tau : M \otimes N \ni m \otimes n \mapsto \sum n_{[0]} \otimes m \cdot n_{[1]} \in N \otimes M. \tag{2}$$

This is a natural transformation with the additional property of a braiding which we will discuss later. So we know now how to define a Hopf algebra H in $\mathcal{YD}^K_K$. Observe that these Hopf algebras are not ordinary Hopf algebras since the condition for the compatibility between multiplication and comultiplication involves the new switch morphism.

Given a Hopf algebra H in $\mathcal{YD}^K_K$ we can define the biproduct $K \star H$ [R] between K and H. The underlying vector space is the tensor product $K \otimes H$. We denote the elements by $\sum c_i \otimes h_i =: \sum c_i \# h_i$. The (smash product) multiplication is given by

$$(c\#h)(c'\#h') := \sum cc'_{(1)}\#(h \cdot c'_{(2)})h' \tag{3}$$

and the (smash coproduct) comultiplication is given by

$$\Delta(c\#h) = \sum (c_{(1)}\#(h_{(1)})_{[0]}) \otimes (c_{(2)}(h_{(1)})_{[1]}\#h_{(2)}). \tag{4}$$

If K is a Hopf algebra and H is a Hopf algebra in $\mathcal{YD}^K_K$ then $K \otimes H$ becomes a Hopf algebra with this multiplication and comultiplication, called the *biproduct* $K \star H$ (see [R, M, Ma]).

We will be mainly interested in the case where $K = kG$ is the group ring of a commutative group. It is well known that the kG-comodules are precisely the G-graded vector spaces ([M] Example 1.6.7). We denote this category by $\mathcal{M}^{kG}$. From the comodule structure on a G-graded vector space $M = \oplus_{h \in G} M_h$ we can construct a kG-module structure such that M becomes a Yetter-Drinfel'd module. This construction depends on a bicharacter $\chi : G \times G \longrightarrow k^\times$ given by a group homomorphism $\chi : G \otimes_{\mathbf{Z}} G \longrightarrow k^\times$. Then it is easy to verify that M is in $\mathcal{YD}^{kG}_{kG}$ with the kG-module structure

$$m_h \cdot g := \chi(h, g) m_h$$

for homogeneous elements $m_h \in M_h$, $h \in G$ and $g \in G$. So any bicharacter χ defines a functor $\mathcal{M}^{kG} \longrightarrow \mathcal{YD}^{kG}_{kG}$. This functor preserves tensor products. In particular any algebra or coalgebra in $\mathcal{M}^{kG}$ is also an algebra resp. coalgebra in $\mathcal{YD}^{kG}_{kG}$. Since $\mathcal{M}^{kG}$ can be considered as a monoidal subcategory of $\mathcal{YD}^{kG}_{kG}$ via χ and thus has a switch map $\tau : M \otimes N \longrightarrow N \otimes M$ we can also define Hopf algebras in $\mathcal{M}^{kG}$ and they are also preserved by the functor induced by χ. In this situation τ turns out to be simply

$$\tau(m_h \otimes n_g) = \chi(h, g) n_g \otimes m_h.$$

Although one may define Yetter-Drinfel'd categories $\mathcal{YD}^K_K$ for arbitrary Hopf algebras K with bijective antipode hence in particular for arbitrary group rings kG (where G is not commutative) the above functor that induces Yetter-Drinfel'd modules from kG-comodules with a bicharacter can only be constructed for commutative groups G.

If A is an algebra in $\mathcal{M}^{kG}$ then $kG\#A$ carries the induced algebra structure

$$(g\#a_h)(g'\#a_{h'}) = gg'\#\chi(h,g')a_h a_{h'}. \tag{5}$$

If H is a Hopf algebra in $\mathcal{M}^{kG}$ then the Hopf algebra $L = kG \star H$ has the comultiplication

$$\Delta_L(g\#a_h) = \sum_{k\in G}(g\#b_k)\otimes(gk\#b_{hk^{-1}}) \tag{6}$$

if $\Delta_H(a_h) = \sum_{k\in G} b_k \otimes b_{hk^{-1}}$ where lower indices stand for the degree of homogeneous elements.

Example 1.1. Let $G = C_3 = \langle t\rangle$ be the cyclic group with three elements, generator t, and let $\chi(t,t) = \xi \in k$ be a primitive 3-rd root of unity. Then $H = k[x]/(x^3)$ with $\Delta(x) = x\otimes 1 + 1\otimes x$ is a Hopf algebra in $\mathcal{YD}_{kG}^{kG}$ where x has degree t, the generator of C_3.

To see that $k[x]/(x^3)$ is a Hopf algebra in $\mathcal{YD}_{kG}^{kG}$ we check that $\Delta(x^3) = \Delta(x)^3$. We observe that $(x\otimes 1)(1\otimes x) = x\otimes x$ but $(1\otimes x)(x\otimes 1) = \xi x\otimes x$. So we have $\Delta(x)^3 = (x\otimes 1+1\otimes x)^3 = x^3\otimes 1 + (1+\xi+\xi^2)(x\otimes x^2) + (1+\xi+\xi^2)(x^2\otimes x) + (1\otimes x^3) = 0 = \Delta(0) = \Delta(x^3)$.

This example shows, that products and powers of primitive elements behave totally different in $\mathcal{YD}_{kG}^{kG}$ from how they behave in Vec, the category of vector spaces. This new behavior is central to the following observations on Lie algebras.

The biproduct $kC_3 \star k[x]/(x^3)$ is isomorphic to the Hopf algebra $k\langle t,x\rangle/(t^3 - 1, x^3, xt - \xi tx)$ if we associate $t\#1$ and t resp. $1\#x$ and x.

2. Skew primitive elements

The last example shows the importance of elements $x \in H$ (a Hopf algebra in $\mathcal{YD}_K^K$) with $\Delta(x) = x\otimes 1 + 1\otimes x$. These elements are called *primitive elements*. They act like derivations. The primitive elements, homogeneous of degree $g \in G$, form a vector space $P_g(H)$. In [P1] (after the proof of 3.2) and [P2] Lemma 5.1 we proved

Lemma 2.1. *The set of primitive elements of a Hopf algebra H in $\mathcal{YD}_K^K$ is a Yetter-Drinfel'd module $P(H)$.*

If $K = kG$ and $H \in \mathcal{M}^{kG}$ then $P(H) = \bigoplus_{g\in G} P_g(H)$ is also in $\mathcal{M}^{kG}$ (via the same bicharacter χ).

In general and especially in Hopf algebras of the form $K \star H$ we have to consider more general conditions for primitive elements. An element $g \neq 0$ in a Hopf algebra L (in Vec) is called a *group-like element* if

$$\Delta(g) = g\otimes g.$$

This implies $\varepsilon(g) = 1$. By (4) a group-like element $g \in K$ defines a group-like element $g\#1 \in L = K \star H$.

Let $g, g' \in L$ be group-like elements. An element $x \in L$ is called a (g', g)-*primitive element* or a *skew primitive element* if

$$\Delta(x) = x \otimes g + g' \otimes x.$$

This implies $\varepsilon(x) = 0$. Observe that a primitive element is $(1,1)$-primitive, since 1 is a group-like element.

The (g', g)-primitive elements form a vector space $P_{(g',g)}(L)$. For $x \in P_{(g',g)}(L)$ we have

$$\Delta(g'^{-1}x) = (g'^{-1} \otimes g'^{-1})(x \otimes g + g' \otimes x) = g'^{-1}x \otimes g'^{-1}g + 1 \otimes g'^{-1}x,$$

so we get an isomorphism $P_{(g',g)}(L) \ni x \mapsto g'^{-1}x \in P_{(1,g'^{-1}g)}(L)$. Thus it suffices to study the spaces $P_{(1,g)}(L)$.

If $f : L \longrightarrow L'$ is a Hopf algebra homomorphism then f obviously preserves group-like elements $g \in L$ and (g', g)-primitive elements $x \in L$ are mapped into $(f(g'), f(g))$-primitive elements $f(x) \in L'$. So we get a homomorphism $f : P_{(g',g)}(L) \longrightarrow P_{(f(g'),f(g))}(L')$ and in particular a homomorphism $f : P_{(1,g)}(L) \longrightarrow P_{(1,f(g))}(L')$.

Now let $L = K \star H$ and let $h \in H$ be primitive and homogeneous of degree $g \in K$, a group-like element in K, i.e. $\Delta_H(h) = h \otimes 1 + 1 \otimes h$ and $\delta(h) = \sum h_{[0]} \otimes h_{[1]} = h \otimes g$ with respect to the K-comodule structure $\delta : H \longrightarrow H \otimes K$ of H. Then by (6)

$$\Delta_L(1\#h) = 1\#h \otimes g\#1 + 1\#1 \otimes 1\#h$$

hence $1\#h \in P_{(1,g)}(L)$. So the following is a monomorphism

$$P_g(H) \ni h \mapsto 1\#h \in P_{(1,g)}(L).$$

Theorem 2.2. *Let K be a Hopf algebra with bijective antipode and H be a Hopf algebra in $\mathcal{YD}_K^K$. Let $L = K \star H$. For every group-like element $g \in K$ we have*

$$P_{(1,g)}(L) = P_{(1,g)}(K)\#1 \oplus 1\#P_g(H).$$

Proof. Let $\pi : K \star H \longrightarrow K$ be defined by $\pi(c\#h) = c\varepsilon_H(h)$ and $\iota : K \longrightarrow K \star H$ by $\iota(c) = c\#1$. Then one checks easily that π and ι are Hopf algebra homomorphisms and that $\pi\iota = \mathrm{id}_K$. Thus π and ι preserve group-like elements and skew-primitive elements. In particular we have for any group-like element $g \in K$ that $\iota(g) = g\#1 \in L$ is group-like. We identify the group-like elements g in K with the group-like elements $\iota(g)$ in L. For any $(1, g)$-primitive element $x \in L$ the element $\pi(x) \in K$ is also $(1, g)$-primitive. Furthermore π and ι define a direct sum decomposition $K \star H = \mathrm{Im}(\iota) \oplus \mathrm{Ker}(\pi)$.

We have already seen $1\#P_g(H) \subseteq P_{(1,g)}(L)$. Furthermore if $c \in P_{(1,g)}(K)$ then $\iota(c) = c\#1 \in P_{(1,g)}(L)$ so that $P_{(1,g)}(L) \supseteq P_{(1,g)}(K)\#1 + 1\#P_g(H)$.

Given a $(1, g)$-primitive element $x \in L = K \star H$ for $g \in K$. We study how x decomposes with respect to the direct sum decomposition $x = \iota\pi(x) + (x - \iota\pi(x))$. The element $\pi(x)$ is $(1, g)$-primitive since x is $(1, g)$-primitive. So $\iota\pi(x) \in P_{(1,g)}(L)\#1$.

Furthermore $\iota\pi(x) \in P_{(1,g)}(L)$ implies $y := x - \iota\pi(x) \in P_{(1,g)}(L) \cap \mathrm{Ker}(\pi)$. We have $\Delta_L(y) = y \otimes g + 1 \otimes y$ and $(\pi \otimes 1)\Delta_L(y) = 1 \otimes y$, since $y \in \mathrm{Ker}(\pi)$. Let $y = \sum c\# h \in L = K \star H$ then

$$\begin{aligned} 1 \otimes y = (\pi \otimes 1)\Delta_L(y) &= (\pi \otimes 1)(\sum (c_{(1)} \# (h_{(1)})_{[0]}) \otimes (c_{(2)}(h_{(1)})_{[1]} \# h_{(2)})) \\ &= \sum c_{(1)}\varepsilon_H((h_{(1)})_{[0]}) \otimes c_{(2)}(h_{(1)})_{[1]} \# h_{(2)}. \end{aligned}$$

We apply $1 \otimes \varepsilon_K \# 1$ and get

$$\begin{aligned} \sum 1 \# \varepsilon_K(c) h &= (1 \otimes \varepsilon_K \# 1)(\sum 1 \otimes c \# h) \\ &= \sum c_{(1)}\varepsilon_K(c_{(2)}) \# \varepsilon_H((h_{(1)})_{[0]})\varepsilon_K((h_{(1)})_{[1]}) h_{(2)} = \sum c \# h, \end{aligned}$$

so we know

$$y = x - \iota\pi(x) = 1 \# h$$

for some $h \in H$.

Since y is skew-primitive we get $0 = \varepsilon_L(y) = \varepsilon_H(h)$. Furthermore $\Delta_L(y) = y \otimes g + 1 \otimes y = 1\#h \otimes g\#1 + 1\#1 \otimes 1\#h = \sum (1\#(h_{(1)})_{[0]}) \otimes ((h_{(1)})_{[1]} \# h_{(2)})$ implies

$$h \otimes g\#1 + 1_H \otimes 1_K \# h = \sum (h_{(1)})_{[0]} \otimes (h_{(1)})_{[1]} \# h_{(2)}$$

so that by applying $1 \otimes \varepsilon \# 1$ we get $h \otimes 1 + 1 \otimes h = \sum h_{(1)} \otimes h_{(2)} = \Delta_H(h)$. i.e. h is primitive. Furthermore with $1 \otimes 1 \# \varepsilon$ we get $h \otimes g = \sum h_{[0]} \otimes h_{[1]} = \delta(h)$ where $\delta : H \longrightarrow H \otimes K$ is the given comodule struktur of H. Thus h is homogeneous of degree g so that

$$x - \iota\pi(x) \in 1 \# P_g(H).$$

So we have shown $P_{(1,g)}(L) \subseteq P_{(1,g)}(K)\#1 \oplus 1\#P_g(H)$. □

Corollary 2.3. *Let G be a commutative group, χ be a bicharacter of G. Let H be a Hopf algebra in $\mathcal{M}^{kG}$. Let $L = kG \star H$. For every $g \in G$ we have*

$$P_{(1,g)}(L) = k(g-1)\#1 \oplus 1\#P_g(H).$$

Proof. The only thing to check is $P_{(1,g)}(kG) = k(g-1)$. We have $\Delta(g-1) = g \otimes g - 1 \otimes 1 = (g-1) \otimes g + 1 \otimes (g-1)$. Conversely if $x = \sum \alpha_i g_i$ is in $P_{(1,g_0)}(kG)$ then by comparing coefficients one obtains $x = \alpha_0(g_0 - 1)$. □

In particular we have $P_{(1,1)}(kG \star H) = 1\#P_1(H)$.

In order to study the (g', g)-primitive elements in $L = kG \otimes H$ it suffices now to study $P(H)$. We are interested in obtaining an algebraic structure on $P(H)$ similar to the Lie algebra structure on the primitive elements of an ordinary Hopf algebra in Vec. The usual Lie multiplication on $P(H)$ induced by the multiplication of the Hopf algebra H in $\mathcal{YD}^{kG}_{kG}$ cannot be used as the following example shows.

Example 2.4. Let $x, y \in P(H)$ and H a Hopf algebra in $\mathcal{YD}^K_K$. If we define $[x, y] := xy - \nabla\tau(x \otimes y)$ then

$$\Delta_H[x, y] = [x, y] \otimes 1 + 1 \otimes [x, y] + x \otimes y - \tau^2(x \otimes y).$$

So in general the element $[x, y] \in H$ will not be a primitive element unless $\tau^2(x \otimes y) = x \otimes y$.

However, in Example 1.1 we found the fact that x^3 may be primitive if x is primitive.

3. Symmetrization of B_n-modules

We want to find a reasonable algebraic structure (of a generalized Lie algebra) on the set of primitive elements of a Hopf algebra in the category $\mathcal{YD}_K^K$ of Yetter-Drinfel'd modules. We also want to get a generalized Lie algebra from every (noncommutative) algebra A in $\mathcal{YD}_K^K$ by suitable definition of Lie multiplications with the help of the algebra multiplication. We expect that the Lie products are (skew-)commutative and satisfy some kind of Jacobi identity. The (skew-)commutativity of an ordinary Lie algebra P results from the action of S_2 (the symmetric group) on $P \otimes P$. In the case of an algebra A made into a Lie algebra this skew-commutativity results from the following composition of maps

$$[.,.] = \nabla \circ \mathrm{SkSymm} : A \otimes A \longrightarrow \mathrm{SkSymm}(A \otimes A) \longrightarrow A$$

where SkSymm denotes the set of anti-symmetric tensors in $A \otimes A$ and the anti-symmetrization process itself. In general the Lie multiplication must only be defined on $\mathrm{SkSymm}(P \otimes P)$ since

$$[x, y] = [x \otimes y] = \frac{1}{2}[x \otimes y - y \otimes x]$$

where $\frac{1}{2}(x \otimes y - y \otimes x) \in \mathrm{SkSymm}(P \otimes P)$.

This is a special case of the following more general observation. If a finite group G acts on a module M then the map $M \ni m \mapsto \sum_{g \in G} gm \in G\text{-Inv}(M)$ sends any $m \in M$ into the set of G-invariant elements $G\text{-Inv}(M) = \{m \in M | \forall g \in G : gm = m\}$. This process is only possible for finite groups G. In the above case S_2 acts on $P \otimes P$ by $\sigma(x \otimes y) = -y \otimes x$. We want to use this process to define a generalized Lie algebra. We will not restrict ourselves to binary multiplications, since the Jacobi identity indicates that higher order multiplications might be of interest, too. Furthermore generalized Lie multiplications will only be partially defined, on subspaces of $P \otimes \ldots \otimes P$.

The reason for the fact that the Lie bracket $[x, y]$ of two primitive elements $x, y \in P(H)$ is not primitive in Example 2.4 results from the following observation. The operation of the switch map $\tau : P \otimes P \longrightarrow P \otimes P$ induces only an operation of the group $\mathbb{Z}$ rather than $\mathbb{Z}/(2)$ on $P \otimes P$. Observe that $\mathbb{Z} = B_2$ is the 2-nd braid group, whereas $\mathbb{Z}/(2) = S_2$ is the 2-nd symmetric group.

The switch morphism $\tau : P \otimes P \longrightarrow P \otimes P$ satisfies the (quantum-) Yang-Baxter equation

$$(\tau \otimes 1)(1 \otimes \tau)(\tau \otimes 1) = (1 \otimes \tau)(\tau \otimes 1)(1 \otimes \tau)$$

hence it induces the action of the n-th braid group B_n on P^n. The braid group B_n is generated by elements $\tau_1, \ldots, \tau_{n-1}$ (τ_i acting on P^n by switching the i-th and $(i+1)$-st component) and has relations

$$\tau_i\tau_j = \tau_j\tau_i \quad \text{for } |i-j| \geq 2,$$
$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}.$$

So the n-th symmetric group S_n with generators $\sigma_1, \ldots, \sigma_{n-1}$ is a canonical quotient of B_n (by $\tau_i \mapsto \sigma_i$) by $\tau_i^2 = \mathrm{id}$. This observation is the reason that τ is called a *braiding* for the category $\mathcal{YD}_K^K$.

Any S_n-module is a B_n-module by the residue homomorphism $B_n \longrightarrow S_n$. Conversely, is there a way to construct an S_n-module from a given B_n-module in a canonical way? Why do we want to consider S_n-modules rather than B_n-modules? The main reason is, as we saw above, that B_n is an infinite group and S_n is a finite group.

The first question leads to the process of *symmetrization* of a B_n-module or braid module as follows. Any element $\zeta \in k^\times$ induces an algebra automorphism $\zeta : kB_n \longrightarrow kB_n$ by $\zeta(\tau_i) := \zeta \cdot \tau_i$, due to the fact that the relations for the braid group are homogeneous. The algebra homomorphism $\nu\zeta : kB_n \longrightarrow kB_n \longrightarrow kS_n$ induces a forgetful functor ${}_{kS_n}\mathcal{M} \longrightarrow {}_{kB_n}\mathcal{M}$ which has the right adjoint $\mathrm{Hom}_{kB_n}(kS_n, -) : {}_{kB_n}\mathcal{M} \longrightarrow {}_{kS_n}\mathcal{M}$. So any braid module M and any $\zeta \in k^\times$ induces a module $\mathrm{Hom}_{kB_n}(kS_n, M)$ over the symmetric group. Since the algebra homomorphism $\nu\zeta$ is surjective we get a submodule

$$M(\zeta) := \mathrm{Hom}_{kB_n}(kS_n, M) \subseteq \mathrm{Hom}_{kB_n}(kB_n, M) = M.$$

In [P2] following Definition 2.3 we proved

$$M(\zeta) = \{m \in M | \phi^{-1}\tau_i^2\phi(m) = \zeta^2 m \quad \forall \phi \in B_n,\ i = 1, \ldots, n-1\}$$

and computed the action of S_n on $M(\zeta)$ as

$$\sigma_i(m) = \zeta^{-1}\tau_i(m). \tag{7}$$

So we have kS_n-submodules $M(\zeta) \subseteq M$ for every $\zeta \in k^\times$. Since they are constructed similar to eigenspaces for the eigenvalues ζ^2 they form direct sums in M.

If $P \in \mathcal{YD}_K^K$ then $P^n = P \otimes \ldots \otimes P$ is in $\mathcal{YD}_K^K$ and B_n acts on P^n. The symmetrization with respect to $\zeta \in k^\times$ gives a module $P^n(\zeta) \in \mathcal{YD}_K^K$ ([P2] Theorem 2.5).

Now we consider the special case $K = kG$ and a G-graded vector space $M \in \mathcal{M}^{kG}$ for a commutative group G with a bicharacter $\chi : G \otimes_{\mathbf{Z}} G \longrightarrow k^\times$. As we saw in the first section M can be considered as a Yetter-Drinfel'd module in $\mathcal{YD}_{kG}^{kG}$. The space M decomposes into homogeneous components $M = \oplus_{g \in G} M_g$. The components themselves are again Yetter-Drinfel'd modules, since M_g is G-graded.

Assume that M is in $\mathcal{YD}_{kG}^{kG}$ and that B_n operates on M by morphisms in the category $\mathcal{YD}_{kG}^{kG}$. Then B_n operates also on the homogeneous components M_g. Since $M(\zeta)$ is a Yetter-Drinfel'd module it decomposes into homogeneous components $M(\zeta)_g = M_g(\zeta)$.

Let $P \in \mathcal{M}^{kG}$ and $\sum x_{g_1} \otimes \ldots \otimes x_{g_n} \in P^n$ be an element with the x_{g_i} homogeneous of degree $g_i \in G$. Then the action of $\tau_i^2 \in B_n$ looks particularly simple

$$\tau_i^2(\sum x_{g_1} \otimes \ldots \otimes x_{g_n}) = \chi(g_{i+1}, g_i)\chi(g_i, g_{i+1})(\sum x_{g_1} \otimes \ldots \otimes x_{g_n}).$$

One shows that the element $\sum x_{g_1} \otimes \ldots \otimes x_{g_n}$ is in $P^n(\zeta)$ iff $\chi(g_j, g_i)\chi(g_i, g_j)(\sum x_{g_1} \otimes \ldots \otimes x_{g_n}) = \zeta^2(\sum x_{g_1} \otimes \ldots \otimes x_{g_n})$ for all $i \neq j$. This implies $\chi(g_j, g_i)\chi(g_i, g_j) = \zeta^2$ for all $i \neq j$, if the given element is not zero. In [P2] Proposition 7.2 we proved

Proposition 3.1. *Let $\zeta \in k^*$ be given. Then*

$$P^n(\zeta) = \bigoplus_{\{(g_1,\ldots,g_n)\ \zeta\text{-family}\}} P_{g_1} \otimes \ldots \otimes P_{g_n}.$$

Here $(g_1, \ldots, g_n)$ is called a *ζ-family* if $\chi(g_i, g_j)\chi(g_j, g_i) = \zeta^2$ for all $i \neq j$.

Example 3.2. We close this section with some important examples.

1. If $G = \{0\}$ then $\chi(0,0) = 1$ and $\mathcal{YD}^{kG}_{kG} \cong \mathrm{Vec}$. The braiding in Vec is the usual switch map $\tau(x \otimes y) = y \otimes x$ hence $\tau^2 = \mathrm{id}$. If M is a B_n-module then the equation $\phi^{-1}\tau_i^2\phi(m) = \zeta^2 m$ has a non-trivial solution only if $\zeta = \pm 1$. Hence $P^n(\zeta) = 0$ for all $\zeta \neq \pm 1$. So the only nontrivial "symmetrization" is $P^n(1) = P^n(-1) = P^n$. Since $\tau^2 = \mathrm{id}$ the braid group B_n acts on P^n by ordinary permutations generated by the action of the canonical switch map. If $\phi \in B_n$ and σ its canonical image in S_n then the action (7) given by the symmetrization on $z \in P^n$ is $\sigma(z) = \mathrm{sgn}(\sigma)\phi(z)$, where ϕ is acting as ordinary switch permutation.
2. Let $G = \mathbb{Z}/2\mathbb{Z} = \{0,1\}$ the cyclic group of order two with the (only) nontrivial bicharacter $\chi(i,j) = (-1)^{ij}$. Then $\mathcal{M}^{kG}$ is the category of (2-graded) super vector spaces with the braiding $\tau(x_i \otimes y_j) = (-1)^{ij} y_j \otimes x_i$. Again $\tau^2 = \mathrm{id}$ so that $P^n(\zeta) = 0$ for all $\zeta \neq \pm 1$. The only symmetrization is $P^n(1) = P^n(-1) = P^n$.
3. Let G be an arbitrary finite abelian group with bicharacter χ such that $\chi(h,g) = \chi(g,h)^{-1}$. Then $\mathcal{M}^{kG}$ is the category of (G-graded) color vector spaces with the braiding $\tau(x_h \otimes y_g) = \chi(h,g) y_g \otimes x_h$. Again $\tau^2 = \mathrm{id}$ so that $P^n(\zeta) = 0$ for all $\zeta \neq \pm 1$. The only symmetrization is $P^n(1) = P^n(-1) = P^n$.
4. The first interesting example is $G = \mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ with the bicharacter $\chi(i,j) = \xi^{ij}$ where ξ is a primitive 3-rd root of unity. Then $\mathcal{M}^{kG}$ is the category of 3-graded vector spaces with the braiding $\tau(x_i \otimes y_j) = \xi^{ij} y_j \otimes x_i$. The homogeneous elements $m \in P^n(\zeta) = \bigoplus_{\{(g_1,\ldots,g_n)\ \zeta\ \text{-family}\}} P_{g_1} \otimes \ldots \otimes P_{g_n}$ for $P \in \mathcal{M}^{kG}$ have to satisfy $\phi^{-1}\tau^2\phi(m) = \xi^{2ij} m = \zeta^2 m$. The possibilities for ζ are $1, -1, \xi, -\xi, \xi^2, -\xi^2$. By computing all possible ζ-families we get for example

$$(P \otimes P)(-1) = (P \otimes P_0) + (P_0 \otimes P),$$
$$(P \otimes P \otimes P)(\xi) = (P_1 \otimes P_1 \otimes P_1) \oplus (P_2 \otimes P_2 \otimes P_2),$$
$$P^6(-\xi) = P_1^6 \oplus P_2^6,$$
$$(P \otimes P \otimes P)(\xi^2) = 0,$$
$$P^6(-\xi^2) = 0.$$

The particular choice of the number of tensor factors in this example will become clear in the next section. The action of the symmetric group on these symmetrizations is

$$\sigma(x_i \otimes y_j) = -y_j \otimes x_i,$$

$$\sigma_1(x_1 \otimes y_1 \otimes z_1) = y_1 \otimes x_1 \otimes z_1,$$

i.e. the ordinary permutation – and this holds for all elements in S_3 and also for elements in $P_2 \otimes P_2 \otimes P_2$,

$$\sigma_1(x_1 \otimes y_1 \otimes z_1 \otimes u_1 \otimes v_1 \otimes w_1) = -y_1 \otimes x_1 \otimes z_1 \otimes u_1 \otimes v_1 \otimes w_1.$$

4. Lie algebras

Let P be a Yetter-Drinfel'd module in $\mathcal{YD}^K_K$. Then $P^n(\zeta)$ is an S_n-module. We will have to consider morphisms $[\ ] : P^n(\zeta) \longrightarrow P$ in $\mathcal{YD}^K_K$. If we suppress the summation index and the summation sign then we may write the bracket operation on elements $z = x_1 \otimes \ldots \otimes x_n \in P^n(\zeta)$ as $[x_1, \ldots, x_n] := [z]$. Furthermore we define

$$x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)} := \sigma^{-1}(z) \tag{8}$$

Observe that the components $x_1, \ldots, x_n$ in these expressions are interchanged and *changed* according to the action of the braid group resp. the symmetric group on $P^n(\zeta)$, so $x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$ is only a symbolic expression, not the usual permutation of the tensor factors given by the permutation of the indices.

We need another submodule of P^n whose special properties will not be investigated. Define

$$P^{n+1}(-1, \zeta) := P \otimes P^n(\zeta) \cap \{z \in P^{n+1} | \forall \phi \in S_n : (1 \otimes \phi)^{-1} \tau_1^2 (1 \otimes \phi)(z) = z\}.$$

Since this is a kernel (limit) construction in $\mathcal{YD}^K_K$, $P^{n+1}(-1, \zeta)$ is again an object in $\mathcal{YD}^K_K$.

For $z = x \otimes y_1 \otimes \ldots \otimes y_n \in P^{n+1}(-1, \zeta)$ we write $y_1 \otimes \ldots \otimes y_{i-1} \otimes x \otimes y_i \otimes \ldots \otimes y_n := \tau_{i-1} \ldots \tau_1(z)$. If the morphisms $[\ ]_n : P^n \longrightarrow P$ and $[\ ]_2 : P^2 \longrightarrow P$ are suitably defined then we write

$$[y_1, \ldots, [x, y_i], \ldots, y_n] := [., [., .]_2, .]_n \tau_{i-1} \ldots \tau_1(z). \tag{9}$$

Now we have the tools to give the definition of a braided Lie algebra.

Definition 4.1. A Yetter-Drinfel'd module P together with operations in $\mathcal{YD}^K_K$

$$[., .] : (P \otimes \ldots \otimes P)(\zeta) = P^n(\zeta) \longrightarrow P$$

for all $n \in \mathbb{N}$ and all primitive n-th roots of unity $\zeta \neq 1$ is called a *braided Lie algebra* or a *Lie algebra in* $\mathcal{YD}^K_K$ if the following identities hold:

(1) (*"anti"-symmetry*) for all $n \in \mathbb{N}$, for all primitive n-th roots of unity $\zeta \neq 1$, for all $\sigma \in S_n$, and for all $z \in P^n(\zeta)$

$$[z] = [\sigma(z)],$$

(2) (*1. Jacobi identity*) for all $n \in \mathbb{N}$, for all primitive n-th roots of unity $\zeta \neq 1$, and for all $z = x_1 \otimes \ldots \otimes x_{n+1} \in P^{n+1}(\zeta)$

$$\sum_{i=1}^{n+1}[x_i,[x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}]] = \sum_{i=1}^{n+1}[.,[.,.]](1\ldots i)(z) = 0,$$

where we use notation (8) (and where $(1\ldots i)$ is a cycle in S_n),

(3) (*2. Jacobi identity*) for all $n \in \mathbb{N}$, for all primitive n-th roots of unity $\zeta \neq 1$, and for all $z = x \otimes y_1 \otimes \ldots \otimes y_n \in P^{n+1}(-1,\zeta)$ we have

$$[x,[y_1,\ldots,y_n]] = \sum_{i=1}^{n}[y_1,\ldots,[x,y_i],\ldots,y_n]$$

where we use notation (9).

Observe that the bracket operations are only partially defined and should *not* be considered as multilinear operations, since $P^n(\zeta) \subseteq P^n$ is just a submodule in $\mathcal{YD}_K^K$ and does not necessarily decompose into an n-fold tensor product. The elements in $P^n(\zeta)$ are, however, of the form $z = \sum_k x_{k,1} \otimes \ldots \otimes x_{k,n}$.

Clearly the braided Lie algebras in $\mathcal{YD}_K^K$ form a category $\mathcal{LYD}_K^K$. Before we investigate its properties we discuss some examples.

Example 4.2. 1. If $G = \{0\}$ as in Example 3.2.1. then the only required morphism for a braided Lie algebra P is $[\,] : P^2(-1) \longrightarrow P$ since -1 is a primitive 2-nd root of unity. This is the usual bracket operation of Lie algebras. The action of B_2 on $P \otimes P$ is given by the canonical switch map $\tau(x \otimes y) = y \otimes x$. The induced action of S_2 with respect to $\zeta = -1$ is then

$$\sigma(x \otimes y) = -y \otimes x$$

by (7). Thus axiom 1. gives $[x,y] = [\sigma(x \otimes y)] = -[y,x]$, the usual anti-symmetry relation. With this action of S_2 on $P^2(-1) = P \otimes P$ one gets the usual Jacobi identity from both braided Jacobi identities.

2. Let $G = \mathbb{Z}/2\mathbb{Z} = \{0,1\}$ with the nontrivial bicharacter $\chi(i,j) = (-1)^{ij}$. In Example 3.2.2. we saw that the only non-trivial symmetrization occurs with respect to $\zeta = -1$, a primitive 2-nd root of unity. So the only bracket is defined on $(P \otimes P)(-1) = P \otimes P$. The operation of B_2 on $P \otimes P$ is the braid action and $\sigma(x_i \otimes y_j) = (-1)(-1)^{ij} y_j \otimes x_i$. So we get $[x_i,y_j] = [\sigma(x_i \otimes y_j)] = -[y_j,x_i]$ if at least one of the degrees i or j is zero and we get $[x_1,y_1] = [\sigma(x_1 \otimes y_1)] = [y_1,x_1]$. In this case we get the notion of Lie super algebras since the braided Jacobi identities translate to the Jacobi identity for Lie super algebras.

3. Let G be an arbitrary finite abelian group with bicharacter χ such that $\chi(h,g) = \chi(g,h)^{-1}$. Again we get only one bracket operation $[\,] : P \otimes P \longrightarrow P$ and anti-symmetry and Jacobi identities translate to those for Lie color algebras.

4. The example $G = \mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ with bicharacter $\chi(i,j) = \xi^{ij}$ where ξ is a primitive 3-rd root of unity has three bracket operations

$$\begin{gathered} [\,] : P^2(-1) = (P \otimes P_0) + (P_0 \otimes P) \longrightarrow P, \\ [\,] : P^3(\xi) = (P_1 \otimes P_1 \otimes P_1) \oplus (P_2 \otimes P_2 \otimes P_2) \longrightarrow P, \\ [\,] : P^6(-\xi) = P_1^6 \oplus P_2^6 \longrightarrow P. \end{gathered}$$

Here the 1. Jacobi identity means for example

$$[x_1,[x_2,x_3,x_4]] + [x_2,[x_1,x_3,x_4]] + [x_3,[x_1,x_2,x_4]] + [x_4,[x_1,x_2,x_3]] = 0,$$

and the 2. Jacobi identity

$$[x,[y_1,y_2,y_3]] = [[x,y_1],y_2,y_3] + [y_1,[x,y_2],y_3] + [y_1,y_2,[x,y_3]].$$

Further explicit examples of braided Lie algebras can be found in [P1].

5. Properties of Lie algebras

The definition of braided Lie algebras, although it generalizes the notion of the known Lie algebras, Lie super algebras, and Lie color algebras, gains its interest from the properties that these Lie algebras have. We cite some of these properties in brief.

Theorem 5.1. ([P2] Corollary 4.2) *Let A be an algebra in $\mathcal{YD}_K^K$. Then A carries the structure of a Lie algebra A^L with the symmetric multiplications*

$$[\text{-}] : A^n(\zeta) \longrightarrow A \qquad \textit{defined by} \qquad [z] := \sum_{\sigma \in S_n} \nabla^n \sigma(z).$$

for all $n \in \mathbb{N}$ and all roots of unity $\zeta \neq 1$ in $k^\times$.

This defines a functor $-^L : \mathcal{AYD}_K^K \longrightarrow \mathcal{LYD}_K^K$ from the category $\mathcal{AYD}_K^K$ of algebras in $\mathcal{YD}_K^K$ to $\mathcal{LYD}_K^K$.

In [P2] Theorem 5.3 we proved

Theorem 5.2. *For any algebra A the morphism $p : A \ni a \mapsto a \otimes 1 + 1 \otimes a \in A \otimes A$ is a Lie algebra homomorphism in $\mathcal{YD}_K^K$.*

An easy consequence of this theorem is

Theorem 5.3. ([P2] Corollary 5.4) *Let H be a Hopf algebra in $\mathcal{YD}_K^K$. Then the set of primitive elements $P(H)$ forms a Lie algebra in $\mathcal{YD}_K^K$.*

This defines a functor $P : \mathcal{HYD}_K^K \longrightarrow \mathcal{LYD}_K^K$ from the category $\mathcal{HYD}_K^K$ of Hopf algebras in $\mathcal{YD}_K^K$ to $\mathcal{LYD}_K^K$. This is the most interesting result which solves the question for the algebraic structure of the primitive elements of a Hopf algebra in $\mathcal{YD}_K^K$. In particular the braided Lie brackets live also on the set of skew primitive elements $K \star H$ as partially defined operations.

Theorem 5.4. *The functor $-^L : \mathcal{AYD}_K^K \longrightarrow \mathcal{LYD}_K^K$ has a left adjoint $U : \mathcal{LYD}_K^K \longrightarrow \mathcal{AYD}_K^K$, called the* universal enveloping algebra.

Theorem 5.5. ([P2] Theorem 6.1) *The universal enveloping algebra $U(P)$ of a braided Lie algebra P is a Hopf algebra in $\mathcal{YD}_K^K$.*

This defines a left adjoint functor $U : \mathcal{LYD}_K^K \longrightarrow \mathcal{HYD}_K^K$ to $P : \mathcal{HYD}_K^K \longrightarrow \mathcal{LYD}_K^K$.

Example 5.6. The one dimensional vector space kx considered as a $\mathbb{Z}/3\mathbb{Z}$-graded space with x of degree $1 \in G = \mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ is a braided Lie algebra in $\mathcal{YD}_{kG}^{kG}$ with $[x \otimes x \otimes x] = 0$ and $[x \otimes x \otimes x \otimes x \otimes x \otimes x] = 0$. The universal enveloping algebra of kx is $k[x]/(x^3)$, the Hopf algebra discussed in Example 1.1.

6. Derivations and skew-symmetric endomorphisms

We shall give two examples which show how to construct large families of Lie algebras from Yetter-Drinfel'd algebras and from Yetter-Drinfel'd modules with a bilinear form in a similar way as one does for classical Lie algebras.

For this purpose we need inner hom-objects in $\mathcal{YD}_K^K$. Let V, W be Yetter-Drinfel'd modules in $\mathcal{YD}_K^K$. Then $\mathrm{Hom}(V,W)$ is a right K-module by

$$(fh)(v) = f(vS^{-1}(h_{(2)}))h_{(1)}. \tag{10}$$

This is equivalent to

$$(fh_{(1)})(vh_{(2)}) = f(vh_{(3)}S^{-1}(h_{(2)}))h_{(1)} = f(v)h, \tag{11}$$

i.e. the evaluation $\mathrm{Hom}(V,W) \otimes V \longrightarrow W$ is a K-module homomorphism.

We define a map $\delta_0 : \mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(V, W \otimes K)$ by

$$\delta_0(f)(v) := f(v_{[0]})_{[0]} \otimes f(v_{[0]})_{[1]}S(v_{[1]}) \tag{12}$$

that "dualizes" the right module structure on $\mathrm{Hom}(V,W)$.

Let $hom(V,W)$ be the pullback (in Vec) in the diagram

$$\begin{tikzcd} hom(V,W) \arrow[r, "\delta_1"] \arrow[d, "\iota"'] & \mathrm{Hom}(V,W) \otimes K \arrow[d, "j"] \\ \mathrm{Hom}(V,W) \arrow[r, "\delta_0"] & \mathrm{Hom}(V, W \otimes K) \end{tikzcd}$$

$hom(V,W)$ thus can be written as

$$hom(V,W) = \{f \in \mathrm{Hom}(V,W) | \exists \sum f_0 \otimes f_1 \in \mathrm{Hom}(V,W) \otimes K \forall v \in V : \\ \sum f_0(v) \otimes f_1 = \sum f(v_{[0]})_{[0]} \otimes f(v_{[0]})_{[1]}S(v_{[1]}) = \delta_0(f)(v)\} \tag{13}$$

Lemma 6.1. *$hom(V,W)$ is a K-comodule.*

Proof. Since K is faithfully flat we get that

$$\begin{tikzcd} hom(V,W) \otimes K \arrow[r, "\delta_1 \otimes K"] \arrow[d, "\iota \otimes K"'] & \mathrm{Hom}(V,W) \otimes K \otimes K \arrow[d, "j \otimes K"] \\ \mathrm{Hom}(V,W) \otimes K \arrow[r, "\delta_0 \otimes K"] & \mathrm{Hom}(V, W \otimes K) \otimes K \end{tikzcd}$$

again is a pullback. So we get a uniquely determined homomorphism $\delta : hom(V,W) \longrightarrow hom(V,W)\otimes K$ such that

$$\begin{tikzcd}
hom(V,W) \arrow[dr, "\delta"] \arrow[drr, "(1\otimes\Delta)\delta_1"] \arrow[ddr, "\delta_1"'] & & \\
& hom(V,W)\otimes K \arrow[r, "\delta_1\otimes K"] \arrow[d, "\iota\otimes K"'] & \mathrm{Hom}(V,W)\otimes K\otimes K \arrow[d, "j\otimes K"] \\
& \mathrm{Hom}(V,W)\otimes K \arrow[r, "\delta_0\otimes K"] & \mathrm{Hom}(V,W\otimes K)\otimes K
\end{tikzcd}$$

commutes. To show that the outer diagram commutes we compute

$$\begin{aligned}
((\delta_0\otimes 1)\delta_1(f))(v) &= (\delta_0\otimes 1)(f_0\otimes f_1)(v)\\
&= (\delta_0(f_0))(v)\otimes f_1\\
&= f_0(v_{[0]})_{[0]}\otimes f_0(v_{[0]})_{[1]}S(v_{[1]})\otimes f_1\\
&= f(v_{[0]})_{[0]}\otimes f(v_{[0]})_{[1]}S(v_{[2]})\otimes f(v_{[0]})_{[2]}S(v_{[1]})\\
&= f(v_{[0]})_{[0]}\otimes \Delta(f(v_{[0]})_{[1]}S(v_{[1]}))\\
&= ((1\otimes\Delta)\delta_1(f))(v)
\end{aligned}$$

hence $(\delta_0\otimes 1)\delta_1(f) = (1\otimes\Delta)\delta_1(f)$. Since $\delta_1 = (\iota\otimes K)\delta$ and $(\delta_1\otimes K)\delta = (1\otimes\Delta)\delta_1$ we get for the induced map δ the equality $(\iota\otimes K\otimes K)(\delta\otimes K)\delta = (1\otimes\Delta)(\iota\otimes K)\delta = (\iota\otimes K\otimes K)(1\otimes\Delta)\delta$ and thus $(\delta\otimes K)\delta = (1\otimes\Delta)\delta$. Consequently $hom(V,W)$ is a K-comodule. □

Lemma 6.2. *$hom(V,W)$ is a Yetter-Drinfel'd module.*

Proof. We first show that $hom(V,W)$ is a K-module. Let $f\in hom(V,W)$ and $h\in K$. Then

$$\delta_0(fh) = j(f_0h_{(2)}\otimes S(h_{(1)})f_1h_{(3)}) \tag{14}$$

since

$$\begin{aligned}
&\delta_0(fh)(v) = (fh)(v_{[0]})_{[0]}\otimes (fh)(v_{[0]})_{[1]}S(v_{[1]})\\
&= (f(v_{[0]}S^{-1}(h_{(2)}))h_{(1)})_{[0]}\otimes (f(v_{[0]}S^{-1}(h_{(2)}))h_{(1)})_{(1)}S(v_{[1]})\\
&= f(v_{[0]}S^{-1}(h_{(4)}))_{[0]}h_{(2)}\otimes S(h_{(1)})f(v_{[0]}S^{-1}(h_{(4)}))_{[1]}h_{(3)}S(v_{[1]})\\
&= f(v_{[0]}S^{-1}(h_{(4)}))_{[0]}h_{(2)}\otimes S(h_{(1)})f(v_{[0]}S^{-1}(h_{(4)}))_{[1]}h_{(3)}S(v_{[1]})S(h_{(5)})h_{(6)}\\
&= f(v_{[0]}S^{-1}(h_{(4)}))_{[0]}h_{(2)}\otimes S(h_{(1)})f(v_{[0]}S^{-1}(h_{(4)}))_{[1]}S(S(S^{-1}(h_{(5)}))v_{[1]}S^{-1}(h_{(3)}))h_{(6)}\\
&= f((vS^{-1}(h_{(3)}))_{[0]})_{[0]}h_{(2)}\otimes S(h_{(1)})f((vS^{-1}(h_{(3)}))_{[0]})_{[1]}S((vS^{-1}(h_{(3)}))_{[1]})h_{(4)}\\
&= f_0(vS^{-1}(h_{(3)}))h_{(2)}\otimes S(h_{(1)})f_1h_{(4)}\\
&= (f_0h_{(2)})(v)\otimes S(h_{(1)})f_1h_{(3)}
\end{aligned}$$

Thus $fh\in hom(V,W)$ by definition of $hom(V,W)$. So $hom(V,W)$ is a K-submodule of $\mathrm{Hom}(V,W)$. Furthermore (14) shows also that $hom(V,W)$ is a Yetter-Drinfel'd module. □

Theorem 6.3. *The category of Yetter-Drinfel'd modules $\mathcal{YD}_K^K$ is a closed monoidal category.*

Proof. It suffices to show for a homomorphism $g : X \otimes V \longrightarrow W$ in $\mathcal{YD}_K^K$ that the induced map $\tilde{g} : X \longrightarrow \mathrm{Hom}(V, W)$ factors through a Yetter-Drinfel'd homomorphism $\bar{g} : X \longrightarrow hom(V, W)$. We have

$$\begin{aligned}\delta_0(\tilde{g}(x))(v) &= \tilde{g}(x)(v_{[0]})_{[0]} \otimes \tilde{g}(x)(v_{[0]})_{[1]} S(v_{[1]}) \\ &= g(x \otimes v_{[0]})_{[0]} \otimes g(x \otimes v_{[0]})_{[1]} S(v_{[1]}) \\ &= g(x_{[0]} \otimes v_{[0]}) \otimes x_{[1]} v_{[1]} S(v_{[2]}) \\ &= \tilde{g}(x_{[0]})(v) \otimes x_{[1]} \\ &= j(\tilde{g}(x_{[0]}) \otimes x_{[1]})(v)\end{aligned}$$

or $\delta_0(\tilde{g}(x)) = j(\tilde{g}(x_{[0]}) \otimes x_{[1]})$ so that $\tilde{g}(x) \in hom(V, W)$ which defines a homomorphism $\bar{g} : X \longrightarrow hom(V, W)$. Furthermore this proves $\delta(\bar{g}(x)) = \bar{g}(x_{[0]}) \otimes x_{[1]} = (\bar{g} \otimes 1)\delta(x)$ which shows that $\bar{g} : X \longrightarrow hom(V, W)$ is a comodule homomorphism. Finally we have

$$\begin{aligned}\bar{g}(xh)(v) &= g(xh \otimes v) \\ &= g(xh_{(1)} \otimes vS^{-1}(h_{(3)})h_{(2)}) \\ &= g((x \otimes vS^{-1}(h_{(2)}))h_{(1)}) \\ &= g(x \otimes vS^{-1}(h_{(2)}))h_{(1)} \\ &= \bar{g}(x)(vS^{-1}(h_{(2)}))h_{(1)} \\ &= (\bar{g}(x)h)(v)\end{aligned}$$

so that $\bar{g}(xh) = \bar{g}(x)h$, i.e. $\bar{g}$ is a Yetter-Drinfel'd homomorphism. □

We denote the evaluation map corresponding to $\mathrm{id} \in \mathrm{Hom}(hom(V, W), hom(V, W))$ by $\vartheta : hom(V, W) \otimes V \longrightarrow W$.

Now we consider derivations on algebras A in $\mathcal{YD}_K^K$. A *derivation* from A to A is a linear map $(d : A \longrightarrow A) \in hom(A, A)$ such that

$$d(ab) = d(a)b + \nabla(1 \otimes \vartheta)(\tau \otimes 1)(d \otimes a \otimes b)$$

for all $a, b \in A$. Observe that in the symmetric situation this means $d(ab) = d(a)b + ad(b)$.

Lemma 6.4. *Let A be an algebra in $\mathcal{YD}_K^K$. Then the set*

$$Der(A) := \{d \in hom(A, A) | d(ab) = d(a)b + \nabla(1 \otimes \vartheta)(\tau \otimes 1)(d \otimes a \otimes b) \forall a, b \in A\}$$

is a Yetter-Drinfel'd module in $\mathcal{YD}_K^K$.

In fact $Der(A)$ is the kernel in $\mathcal{YD}_K^K$ of the morphism in $\mathrm{Hom}(end(A), hom(A \otimes A, A)) \cong \mathrm{Hom}(end(A) \otimes A \otimes A, A)$ given by $\vartheta(1 \otimes \nabla) - \nabla(\vartheta \otimes 1) - \nabla(1 \otimes \vartheta)(\tau \otimes 1))$. It is easily checked that $Der(A)$ together with its operation on A is the universal derivation module on A, i.e. module M in $\mathcal{YD}_K^K$ together with an operation $\vartheta : M \otimes A \longrightarrow A$ such that $(1 \otimes \nabla)\vartheta = \nabla(\vartheta \otimes 1) + \nabla(1 \otimes \vartheta)(\tau \otimes 1)$.

Theorem 6.5. ([P2] Corollary 5.6) *The Yetter-Drinfel'd module of derivations $Der(A)$ of an algebra A is a Lie algebra.*

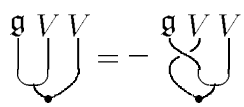
Now let V be in $\mathcal{YD}_K^K$ with inner endomorphism object $A := end(V)$. Given a bilinear form $\langle .,. \rangle : V \otimes V \longrightarrow k$ in $\mathcal{YD}_K^K$. We collect the set $\mathfrak{g}$ of *skew-symmetric* endomorphisms $f \in end(V)$ for which in principle the following hold: $\langle f(v), w\rangle = -\langle v, f(w)\rangle$ for all $v, w \in V$. Since there is a switch between f and v in these expressions the correct condition is

$$\langle f(v), w\rangle = -\sum \langle v_i, f_i(w)\rangle$$

where $\tau(f \otimes v) = \sum v_i \otimes f_i$ resp.

$$\mu(\vartheta \otimes 1)(f \otimes v \otimes w) = -\mu(1 \otimes \vartheta)(\tau \otimes 1)(f \otimes v \otimes w.$$

Thus $\mathfrak{g}$ is the difference kernel of two morphisms in $\mathcal{YD}_K^K$. Hence $\mathfrak{g}$ is an (universal) object satisfying the diagrammatic condition

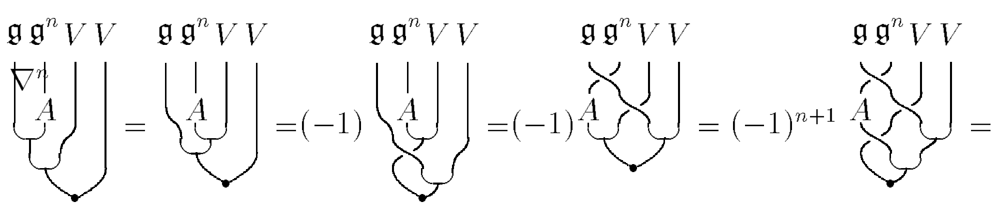


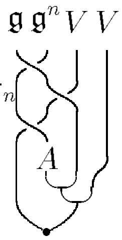
Theorem 6.6. *For a Yetter-Drinfel'd module $V \in \mathcal{YD}_K^K$ with bilinear form $\mu = \langle .,. \rangle$: $V \otimes V \longrightarrow k$ the Yetter-Drinfel'd module $\mathfrak{g}$ of skew-symmetric endomorphisms is a Lie subalgebra of $\mathrm{end}(V)^L$ in $\mathcal{YD}_K^K$.*

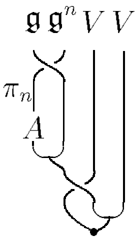
Proof. We give a diagrammatic proof where we use the multiplication $\nabla : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow A = end(V)$ and the fact that the evaluation $\vartheta : \mathfrak{g} \otimes V \longrightarrow V$ of $\mathfrak{g}$ on V is associative with respect to this multiplication. For the n-fold multiplication we write $\nabla^n : \mathfrak{g}^n \longrightarrow A$. Let $\pi_n : \mathfrak{g}^n \longrightarrow \mathfrak{g}^n$ be given by $\pi_2 := \tau_{\mathfrak{g}\otimes\mathfrak{g}}$, $\pi_{n+1} = (\pi_n \otimes 1)\tau_{\mathfrak{g}\otimes\mathfrak{g}^n}$. Then

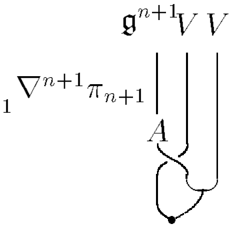
$$\mu(\vartheta \otimes 1)(\nabla^n \otimes 1 \otimes 1) = (-1)^n \mu(1 \otimes \vartheta)(\tau \otimes 1)(\nabla^n \pi_n \otimes 1 \otimes 1).$$

We prove this by induction. For $n = 1$ this is the defining condition for $\mathfrak{g}$. The induction step is



$= (-1)^{n+1}$ 

$= (-1)^{n+1}$ 

$= (-1)^{n+1}$ 

In the first and the last term we indicated the multiplication $\nabla^n : \mathfrak{g}^n \longrightarrow A$ resp. $\nabla^{n+1} : \mathfrak{g}^{n+1} \longrightarrow A$. Furthermore we indicated the use of π_n where appropriate.

Now let ζ be a primitive n-th root of unity and $z \in \mathfrak{g}^n(\zeta)$. Then $\zeta^{\frac{n(n-1)}{2}} = (-1)^{n-1}$ and $\pi_n(z) = \zeta^{\frac{n(n-1)}{2}}\rho_n(z)$ by (7) and the definition of π_n, where ρ_n is the image of π_n under the canonical map $B_n \longrightarrow S_n$. So we get

$$\begin{aligned}
\mu(\vartheta \otimes 1)([z] \otimes v \otimes w) = &\\
&= \textstyle\sum_\sigma \mu(\vartheta \otimes 1)(\nabla^n \sigma(z) \otimes v \otimes w)\\
&= \textstyle\sum_\sigma \mu(\vartheta \otimes 1)(\nabla^n \otimes 1 \otimes 1)(\sigma(z) \otimes v \otimes w)\\
&= (-1)^n \textstyle\sum_\sigma \mu(1 \otimes \vartheta)(\tau \otimes 1)(\nabla^n \pi_n \otimes 1 \otimes 1)(\sigma(z) \otimes v \otimes w)\\
&= (-1) \textstyle\sum_\sigma \mu(1 \otimes \vartheta)(\tau \otimes 1)(\nabla^n \otimes 1 \otimes 1)(\rho_n \sigma(z) \otimes v \otimes w)\\
&= (-1)\mu(1 \otimes \vartheta)(\tau \otimes 1)([z] \otimes v \otimes w),
\end{aligned}$$

hence $[z] \in \mathfrak{g}$. Thus $\mathfrak{g}$ is a Lie subalgebra of $end(V)$. □

7. Lie structures on C_{p^n}-graded modules

In this section we assume that $G = C_{p^t} = \mathbb{Z}/(p^t)$ is the cyclic group with p^t elements where $p \neq 2$ is prime and that the field k has characteristic $\neq 2$ and contains a p^t-th primitive root of unity ξ. We want to get information on the nontrivial symmetrizations of G-comodules.

A bicharacter $\chi : G \otimes_{\mathbb{Z}} G \longrightarrow k^\times$ is uniquely defined by the value of $\chi(1,1)$ as $\chi(i,j) = \chi(1,1)^{ij} \in k$. Since $\mathbb{Z}/(p^t) \otimes_{\mathbb{Z}} \mathbb{Z}/(p^t) \cong \mathbb{Z}/(p^t)$ this amounts to a homomorphism $\varphi : \mathbb{Z}/(p^t) \ni g \mapsto \xi^g \in k^\times$ for some element $\xi \in k^\times$. Such a homomorphism has a unique image factorization $\mathbb{Z}/(p^t) \longrightarrow \mathbb{Z}/(p^s) \longrightarrow k^\times$ with $g \mapsto g \mapsto \xi^g$ with the second homomorphism injective. Then ξ has order p^s so it is a primitive p^s-th root of unity. Without loss of generality we may assume $s = t$ and ξ a primitive p^t-th root of unity.

We wish to compute the ζ-symmetrization $(P \otimes \ldots \otimes P)(\zeta) = P^n(\zeta)$ where P is a $G = \mathbb{Z}/(p^t)$-graded vector space and ζ is an n-th primitive root of unity (with $n > 1$) so that we can determine the domain of the possible Lie multiplications.

To determine $P^n(\zeta) = \bigoplus_{\{(g_1,\ldots,g_n)\zeta\text{-family}\}} P_{g_1} \otimes \ldots \otimes P_{g_n}$ (Proposition 3.1) we have to find the ζ-families $(g_1, \ldots, g_n)$ in G i.e. families with $\chi(g_i, g_j)^2 = \zeta^2$ for all $i \neq j$. Since $\chi(g_i, g_j)$ has order p^r for some r, there is only a restricted choice for the primitive root of unity ζ and for the arity of the possible Lie multiplication on P.

We have $n > 1$ and $\zeta \in k^\times$ a primitive n-th root of unity. We wish to determine all ζ-families $(g_1, \ldots, g_n)$ of elements in $G = \mathbb{Z}/(p^t)$ satisfying $\chi(g_i, g_j)^2 = \zeta^2$ or $\xi^{2g_ig_j} = \zeta^2$ for all $i \neq j$. This amounts to $\xi^{g_ig_j} = \varepsilon\zeta$ with $\varepsilon = \pm 1$. In the case $\varepsilon = +1$ we get $\zeta = \xi^{g_ig_j} \in \mathrm{Im}(\varphi)$. Hence the order of ζ is $n = p^{t'}$ and $\zeta = \xi^b$ for $b \in \mathbb{Z}/(p^t)$. In the case $\varepsilon = -1$ we get $-\zeta = \xi^{g_ig_j} \in \mathrm{Im}(\varphi)$ (and $\zeta \notin \mathrm{Im}(\varphi)$ since p is odd). Hence the order of ζ is $n = 2p^{t'}$ and $\zeta = -\xi^b$ for $b \in \mathbb{Z}/(p^t)$. Together this says $\zeta = \varepsilon\xi^b$ has order $n = (\frac{3}{2} - \frac{1}{2}\varepsilon)p^{t'}$ for $\varepsilon = \pm 1$.

If $t' = 0$ then $\zeta = \pm 1$. If $\zeta = 1$ then we get the trivial case $P^1(1) = P$. If $\zeta = -1$ the primitive 2-nd root of unity, then (g_1, g_2) is a ζ-family iff it satisfies $\chi(g_1, g_2) = 1$ (the value -1 is not possible) iff $g_1g_2 \equiv 0(p^t)$.

Assume now that $t' > 0$ hence $b \neq 0$. Then there are $n > 2$ components g_i in a ζ-family $(g_1, \ldots, g_n)$. Choose representatives $g_i \in \mathbb{N}$ with $0 < g_i < p^t$. Observe that $g_i \neq 0$ since $b \neq 0$ and $b \equiv g_ig_j(p^t)$. So we can write $g_i = n_ip^{r_i}$ with $(n_i, p) = 1$ and $0 \leq r_i < t$ and $b = qp^s$ with $(q, p) = 1$ and $0 \leq s < t$.

We have $g_ig_j \equiv b(p^t)$ for all $i \neq j$ hence $n_in_jp^{r_i}p^{r_j} \equiv qp^s(p^t)$. Since $g_ig_j \neq 0$ in $\mathbb{Z}/(p^t)$ we get $r_i + r_j < t$ hence $r_i + r_j = s$ for all $i \neq j$. Thus $r_i = r$ for all $i = 1, \ldots, n$ and $s = 2r < t$. Since $b = qp^s$ has order p^{t-s} in $\mathbb{Z}/(p^t)$ we get $t' = t - 2r$ and $n = (\frac{3}{2} - \frac{1}{2}\varepsilon)p^{t-2r}$.

Finally we have $g_i(g_j - g_k) = n_i(n_j - n_k)p^{2r} \equiv 0(p^t)$ hence $n_j - n_k \equiv 0(p^{t-2r})$ or

$$n_i \equiv n_j(p^{t-2r}), \forall i \neq j$$

and $b \equiv n_in_ip^{2r}(p^t)$. It is easy to check that any family $(g_1, \ldots, g_n)$ satisfying these equations is a ζ-family. This proves the following

Theorem 7.1. *Let $p \neq 2$ be a prime and $t \geq 1$. Then the (-1)-families (g_1, g_2) are those with $g_1g_2 \equiv 0(p^n)$.*

For any choice of

- *r such that $0 \leq 2r < t$,*
- *$\varepsilon \in \{+1, -1\}$,*
- *$m \in \{1, \ldots, p^n - 1\}$ with $(m, p) = 1$,*

there are ζ-families $(g_1, \ldots, g_n)$ with $\zeta = \varepsilon\xi^b$, $b \equiv m^2p^{2r}(p^n)$, and $n = (\frac{3}{2} - \frac{1}{2}\varepsilon)p^{t-2r}$. The g_i can be chosen as $g_i \equiv mp^r + a_ip^{t-r}(p^n)$ with $a_i \in \{0, \ldots, p^r - 1\}$.

These are all families on which a Lie multiplication can be defined.

Example 7.2. 1. Let $p = 3$ and $t = 1$. Then there is the (-1)-symmetrization

$$P^2(-1) = P_0 \otimes P + P \otimes P_0,$$

since $g_1g_2 \equiv 0(3)$ iff one of the factors g_i is zero. To get all other symmetrizations observe that $r = 0$ hence $b \equiv g_1g_1 \equiv 1(3)$. So there are 2 cases $\zeta = \xi$ and $n = 3$ or $\zeta = -\xi$ and $n = 6$. The corresponding possible ζ-families are $(1,1,1)$ and $(2,2,2)$ resp. $(1,1,1,1,1,1)$ and $(2,2,2,2,2,2)$ hence

$$P^3(\xi) = P_1 \otimes P_1 \otimes P_1 \oplus P_2 \otimes P_2 \otimes P_2$$

and

$$\begin{aligned} P^6(-\xi) = \quad & P_1 \otimes P_1 \otimes P_1 \otimes P_1 \otimes P_1 \otimes P_1 \\ & \oplus P_2 \otimes P_2 \otimes P_2 \otimes P_2 \otimes P_2 \otimes P_2. \end{aligned}$$

2. Let $p = 3$ and $t = 2$. Then $P^2(-1) = P_0 \otimes P + P \otimes P_0 + (P_3 + P_6) \otimes (P_3 + P_6)$. For larger ζ-families the only choice for r is $r = 0$. Since $m \in \{1, 2, 4, 5, 7, 8\}$ we get $b \in \{1, 4, 7\}$. We get families with 9 or 18 elements g_i and these elements must all be equal, $g_i \in \{1, 2, 4, 5, 7, 8\}$.

3. Let $p = 3$ and $t = 3$. We consider the families of length > 2. There are two choices for $r \in \{0, 1\}$. So we get families with 3, 6, 27, and 54 elements. For the choice $r = 1$, $m = 4$, $\varepsilon = +1$ the ξ^9-families (g_1, g_2, g_3) are composed of $g_i = 4 \cdot 3 + a_i \cdot 9$ with $a_i \in \{0, 1, 2\}$. There is for example the ξ^9-family $(3, 12, 21)$. So any braided Lie algebra P in $\mathcal{M}^{kG}$ with $G = \mathbb{Z}/(27)$ has a Lie operation $[\,] : P_3 \otimes P_{12} \otimes P_{21} \longrightarrow P_9$.

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