RECONSTRUCTION OF HIDDEN SYMMETRIES

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Contents

1. Introduction

Groups G are often obtained as groups of symmetries (or automorphisms) of mathematical structures like a vector space (over a fixed field $\mathbb{K}$) or two vector spaces together with a linear map between them or a whole diagram of vector spaces, where a symmetry of such a diagram is a family of automorphisms one for each vector space which are compatible with the linear maps of the diagram (a natural automorphism). This process of constructing the group of symmetries is a special case of the notion of (Tannaka-Krein) reconstruction.

Conversely given a group G one considers its representations $G \longrightarrow GL(V)$ in vector spaces V over the field $\mathbb{K}$. All representations of G form the category ${}_{\mathbb{K}G}\mathcal{M}$ of modules, which we may consider as a huge diagram of vector spaces. The category ${}_{\mathbb{K}G}\mathcal{M}$ has an additional interesting structure – the tensor product $V \otimes W$ of two representations is again a representation in a canonical way, ${}_{\mathbb{K}G}\mathcal{M}$ is a monoidal category. A special consequence of reconstruction theory is the fact that G may be recovered as the full group of those symmetries of this huge diagram which are compatible with the tensor product. This process seems to be the inverse of the first one. In a more general setting there are, however, subtle deviations. One may reconstruct much larger groups of symmetries than what one started out with.

More generally we know that algebras A, Lie algebras $\mathfrak{g}$ and Hopf algebras H can be reconstructed from their categories of modules. For the reconstruction of an algebra A one actually needs not only the category of A-modules A-Mod but also the underlying functor $\omega : A\text{-Mod} \longrightarrow \text{Vec}$. Then A (as an algebra) can be reconstructed (up to isomorphism) as $\text{end}(\omega)$, the categorical end of the underlying functor. For the reconstruction of a Hopf algebra H one needs in addition the monoidal structure of H-Mod. Then the full Hopf algebra structure can be reconstructed [DM82, Pa81, Ul89].

This stands in pointed contrast to another similar result, the Morita theorems [Ba68], which show that the knowledge of the category of modules A-Mod of an algebra A does not determine A up to isomorphism.

As we remarked before, the forgetful functor $\omega : A\text{-Mod} \longrightarrow \text{Vec}$ is essential in the process of reconstruction. In particular one has to consider representations of the given objects (algebras, groups, Lie algebras, Hopf algebras) in vector spaces. Representations in categories of objects with a richer structure like super vector spaces, $\star$-spaces, graded vector spaces, comodules over Hopf algebras have a different behavior.

Instead of the base category Vec we wish to use an arbitrary braided monoidal category $\mathcal{A}$. There are many examples for such a base category $\mathcal{A}$ such as L-Mod, the category of modules over a quasitriangular Hopf algebra L, L-Comod, the category of comodules over a coquasitriangular Hopf algebra L, ${}_{L}^{L}\mathcal{YD}$, the category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode, or dys $\mathcal{C}_A$, the category of dyslectic modules over a commutative algebra in a braided monoidal category $\mathcal{C}$ [Pa95]. We study the following problem: given an algebra, a bialgebra, or a Hopf algebra H in $\mathcal{A}$, can it be reconstructed from ${}_H\mathcal{A}$, the underlying functor $\omega : {}_H\mathcal{A} \longrightarrow \mathcal{A}$ and the monoidal structure? A special case is the reconstruction of a super algebra from its super representations.

The surprising solution of this problem shows that one usually reconstructs an object $\text{end}(\omega)$ from the underlying functor $\omega : {}_H\mathcal{A} \longrightarrow \mathcal{A}$ in $\mathcal{A}$ that is much bigger than H. In the group case (for $H = kG$) this amounts to additional symmetries which we call hidden symmetries. A concrete case of such a hidden symmetry is given in 5.1.8. In the (Hopf) algebra case the situation is more complex but we also talk about hidden symmetries. In certain cases we describe precisely the additional hidden symmetries by a smash product decomposition of the reconstructed object.

We control the process of reconstruction by a control category $\mathcal{C}$ which operates on $\omega : {}_H\mathcal{A} \longrightarrow \mathcal{A}$. With different choices of the control category $\mathcal{C}$ we obtain different reconstructed objects $\text{end}_{\mathcal{C}}(\omega)$ from one and the same underlying functor $\omega : {}_H\mathcal{A} \longrightarrow \mathcal{A}$. We study the properties of $\text{end}_{\mathcal{C}}(\omega)$.

The second section of this paper is devoted to some basic notions from the theory of braided monoidal categories $\mathcal{C}$ and the notion of $\mathcal{C}$-categories. The most interesting examples for $\mathcal{C}$ are the categories of modules resp. comodules over Hopf algebras with an additional structure known as a quasitriangular structure resp. braiding, one example is the category of super vector spaces, furthermore the category of Yetter-Drinfeld modules.

In the third section we study the general algebraic structure of reconstructed objects in a braided monoidal category. We separate the discussion of the properties of reconstructed objects strictly from the existence theorems that are treated in the fourth section. We have decided to base our investigations on coalgebras and (right) comodules instead of algebras and (left) modules, because the fundamental structure theorem for comodules makes certain constructions in vector spaces in this case much easier. So we study the coend of a functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ as the universal natural transformation $\omega \longrightarrow \omega \otimes U$ and show that such a universal $\mathrm{coend}(\omega) = U \in \mathcal{A}$ carries the structure of a coalgebra, a bialgebra, or even a Hopf algebra depending on the properties of ω. These structures on the coend of a functor have already been studied in various papers [Mj93b, Mj94a, Mj94b, Pa93, Sch92a]. Our techniques allow us to restrict the class of natural transformations (by the notion of $\mathcal{C}$-morphisms and by choosing different control categories $\mathcal{C}$), which gives us a family of different universal transformations $\omega \longrightarrow \omega \otimes U_{\mathcal{C}}$ parametrized by the choice of control category $\mathcal{C}$ and thus different coalgebras, bialgebras, or Hopf algebras $U_{\mathcal{C}}$. This technique was first used in [Pa78, Pa81]. Now we expand this technique to the case of braided monoidal categories to study the structure of hidden symmetries. It turns out that some of the structure is connected with coadjoint coactions, cosmash products and transmutation.

In the fourth section we show under which conditions a coalgebra C in $\mathcal{A}$ can be reconstructed from the category of C-comodules $\mathcal{A}^C$ in $\mathcal{A}$ and the functor $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$. If C is a coalgebra in the $\mathcal{C}$-monoidal category $\mathcal{A} = \mathcal{C}$ and $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ is the underlying functor then $C = \mathrm{coend}_{\mathcal{C}}(\omega)$ ([Pa81] and Cor. 4.3 with $\mathcal{C}_0 = \mathcal{C}$). One would like to have a more general theorem of the form: if C is a coalgebra in an *arbitrary* $\mathcal{C}$-monoidal category $\mathcal{A}$ and $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ is the underlying functor then $C = \mathrm{coend}_{\mathcal{C}}(\omega)$. This, however, is not true because of hidden symmetries, which live in part in $\mathcal{A}$ (or more precisely in $\mathrm{coend}_{\mathcal{C}}(\mathrm{id} : \mathcal{A} \longrightarrow \mathcal{A})$, see section 5). Our use of control categories allows us to reconstruct a coalgebra from its representations (comodules) in an arbitrary monoidal category $\mathcal{A}$, namely $C = \mathrm{coend}_{\mathcal{A}}(\omega : \mathcal{A}^C \longrightarrow \mathcal{A})$ (no braiding of $\mathcal{A}$ is needed). This fact allows us now to wonder about and study the additional hidden symmetries appearing in $\mathrm{coend}_{\mathcal{C}}(\omega)$. Various refinements of this reconstruction can be found in Theorems 4.1 and 4.2.

The second existence theorem deals with the (re-)construction of C in the case of an arbitrary $\mathcal{C}$-functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$. Conditions for such a reconstruction have been developed in many papers, such as [Ul89, Ul90] in the case $\mathcal{A} = \mathrm{Vec}$. In our situation we can prove the following: if $\omega : \mathcal{B} \longrightarrow \mathcal{C}$ is a $\mathcal{C}_0$-functor which factors through the category $\mathcal{C}_0$ of rigid objects in $\mathcal{C}$ and if $\mathcal{C}$ is cocomplete then $U_{\mathcal{C}_0} = \mathrm{coend}_{\mathcal{C}_0}(\omega)$ exists. A weak variant of this theorem was proved in [Mj93b] (Theorem 2.2) where $\mathcal{B}$ is essentially a category with only a finite number of objects and morphisms, the control category is trivial, and only rigid objects are reconstructed (for more details see the remarks in 4.1.2).

In section five we show as one of the main results of this paper that the universal object $\mathrm{coend}_{\mathcal{C}}(\omega) = U_{\mathcal{C}}$ for a functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ tends to decompose into a cosmash product of a Hopf algebra with a coalgebra. In particular we show the following. If H is a braided Hopf algebra in a braided monoidal category $\mathcal{D}$, C is a coalgebra in $\mathcal{D}^H$, and $\omega : (\mathcal{D}^H)^C \longrightarrow \mathcal{D}^H$ is the underlying functor. Then there is a canonical isomorphism

$$f : \mathrm{coend}_{\mathcal{D}}(\omega) \longrightarrow \mathrm{coend}_{\mathcal{D}}(\mathrm{id}_{\mathcal{D}^H})\widetilde{\#}^c\mathrm{coend}_{\mathcal{D}^H}(\omega),$$

where we use a cosmash product with respect to the transmutation multiplication in H. As a special example one obtains for a braided Hopf algebra H over a field $\mathbb{K}$, an H-comodule coalgebra C, and the functor $\omega : (\mathrm{Vec}^H)^C \longrightarrow \mathrm{Vec}^H$, that the coend of ω is the cosmash product $H\widetilde{\#}^cC \cong H\#^cC$.

In the Appendix we study certain connections between $\mathbb{K}$-additive categories and our notion of $\mathcal{C}$-categories and show in particular (Theorem 6.4) why there are no hidden symmetries in the case of representations in ordinary vector spaces.

2. Braided categories and $\mathcal{C}$-categories

Throughout this paper let $\mathcal{A}$ be a *monoidal category*, i.e. a category together with a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, a neutral object $I \in \mathcal{A}$, and natural isomorphisms $\alpha : (P \otimes Q) \otimes R \longrightarrow P \otimes (Q \otimes R)$, $\lambda : I \otimes P \longrightarrow P$, and $\rho : P \otimes I \longrightarrow P$, satisfying the well-known coherence (constraint) conditions. Without loss of generality (by Mac Lane's coherence theorem [ML71] Theorem 15.1) we shall assume that $\mathcal{A}$ is a strict monoidal category, so that all associativity and unit isomorphisms are identities. Similarly $\mathcal{C}$ will be a monoidal category throughout.

We are mainly interested in the case where $\mathcal{A}$ is the category of (right) modules $\mathcal{M}_B$ or comodules $\mathcal{M}^B$ over a bialgebra B (over a field $\mathbb{K}$) with the canonical monoidal structure. Further examples are Vec the category of vector spaces over a field $\mathbb{K}$, vec the category of finite-dimensional vector spaces, mod-B, the category of finite-dimensional (over $\mathbb{K}$) B-modules, and comod-B, the category of finite-dimensional B-comodules. Some interesting topological examples may be found in [Ye90].

2.1. $\mathcal{C}$-categories. We will need the notion of categories, functors, and natural transformations "over" a monoidal category $\mathcal{C}$, which we call $\mathcal{C}$-categories, $\mathcal{C}$-functors, and $\mathcal{C}$-morphisms. Many of their properties have been investigated in [Pa77, Pa81]. They are built in analogy to G-sets and their morphisms or R-modules and their morphisms.

Definition 2.1. A category $\mathcal{B}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{B} \longrightarrow \mathcal{B}$ and coherent[1] natural isomorphisms $\beta : (X \otimes Y) \otimes P \longrightarrow X \otimes (Y \otimes P)$ (for $X, Y \in \mathcal{C}, P \in \mathcal{B}$) and $\pi : I \otimes P \longrightarrow P$ will be called a (left) $\mathcal{C}$-*category.* (For the coherence conditions see [Se79].) In such a context we will call $\mathcal{C}$ a *control category.*

Some of our main examples are:

2.1.1. A monoidal category $\mathcal{A}$ is an $\mathcal{A}$-category.

[1]Whenever we use the term "coherent" we mean that the given natural transformation is coherent also with respect to the already existing coherent natural transformations, in this case with α, λ, and ρ. The minimal requirements for coherence are obvious in most cases. We do not further investigate them.

2.1.2. Let A (with $m_A : A \otimes A \longrightarrow A$ and $u_A : I \longrightarrow A$) be an *algebra* (a monoid) in $\mathcal{A}$, i.e. the multiplication m_A is associative and unital (with unit morphism u_A). In the situation $\mathcal{A} = \mathcal{M}^B$, such an algebra A is called a *B-comodule algebra.* In the case $\mathcal{A} = \mathcal{M}_B$, such an algebra A is called a *B-module algebra* [Sw69].

2.1.3. The category $\mathcal{B} = \mathcal{A}_A$ of (right) *A-modules* $(P, \kappa : P \otimes A \longrightarrow P)$ in $\mathcal{A}$ is a (left) $\mathcal{A}$-category, since $X \otimes P$ carries the structure of a right A-module by $(X \otimes P) \otimes A \cong X \otimes (P \otimes A) \longrightarrow X \otimes P$.

2.1.4. A vector space P is in $(\mathcal{M}^B)_A$ if and only if P is a right B-comodule and a right A-module such that $\delta(pa) = \sum p_{(0)}a_{(0)} \otimes p_{(1)}a_{(1)}$, a B-A-Hopf module. A vector space P is in $(\mathcal{M}_B)_A$ iff P is a right B-module and a right A-module such that $(pa)b = \sum(pb_{(1)})(ab_{(2)})$, i.e. a $B\#A$-module.

2.1.5. Furthermore let C (with $\Delta_C : C \longrightarrow C \otimes C$ and $\varepsilon_C : C \longrightarrow I$) be a coalgebra in $\mathcal{A}$. In the situation $\mathcal{A} = \mathcal{M}^B$, such a coalgebra C is called a *B-comodule coalgebra.* In the case $\mathcal{A} = \mathcal{M}_B$, such a coalgebra C is called a *B-module coalgebra.*

2.1.6. The category $\mathcal{B} = \mathcal{A}^C$ of (right) *C-comodules* $(P, \delta : P \longrightarrow P \otimes C)$ in $\mathcal{A}$ is a (left) $\mathcal{A}$-category, since $X \otimes P$ carries the structure of a right C-comodule by $X \otimes P \longrightarrow X \otimes (P \otimes C) \cong (X \otimes P) \otimes C$.

2.1.7. A vector space P is in $(\mathcal{M}^B)^C$ if and only if P is a right B-comodule and a right C-comodule such that

$$\begin{tikzcd} P \arrow[r, "\delta_B"] \arrow[d, "\delta_C"] & P \otimes B \arrow[d, "\delta_C \otimes \mathrm{id}_B"] \\ P \otimes C \arrow[r, "\delta_B"] & (P \otimes C) \otimes B \end{tikzcd}$$

commutes, i.e. P is a $B\#^c C$-comodule, a comodule over the cosmash product. A vector space P is in $(\mathcal{M}_B)^C$ iff P is a right B-module and a right C-comodule such that $\delta_C(pb) = \sum p_{(0)}b_{(1)} \otimes p_{(C,1)}b_{(2)}$.

Definition 2.2. Let $\mathcal{B}$ and $\mathcal{B}'$ be $\mathcal{C}$-categories. A functor $\omega : \mathcal{B} \longrightarrow \mathcal{B}'$ together with a coherent natural isomorphism $\xi : \omega(X \otimes P) \longrightarrow X \otimes \omega(P)$ is called a *$\mathcal{C}$-functor.*

Observe that our assumption on coherence implies in particular that $\pi'\xi(I, P) = \omega(\pi)$.

2.1.8. The identity functor $\mathrm{id} : \mathcal{A} \longrightarrow \mathcal{A}$ is an $\mathcal{A}$-functor. Furthermore the forgetful (underlying) functors $\omega : \mathcal{A}_A \longrightarrow \mathcal{A}$ resp. $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ are easily seen to be $\mathcal{A}$-functors. If $f : A \longrightarrow A'$ is an algebra morphism in $\mathcal{A}$, then the induced functor $\omega : \mathcal{A}_{A'} \longrightarrow \mathcal{A}_A$ is an $\mathcal{A}$-functor. Similarly if $f : C \longrightarrow C'$ is a coalgebra morphism in $\mathcal{A}$, then the induced functor $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}^{C'}$ is an $\mathcal{A}$-functor.

2.1.9. We will use additional $\mathcal{C}$-functors. Let $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ be an $\mathcal{C}$-functor and let $M \in \mathcal{A}$. Then $\omega \otimes M : \mathcal{B} \ni P \mapsto \omega(P) \otimes M \in \mathcal{A}$ is again a $\mathcal{C}$-functor.

Definition 2.3. Let $\mathcal{B}$ and $\mathcal{B}'$ be $\mathcal{C}$-categories and $\omega : \mathcal{B} \longrightarrow \mathcal{B}'$ and $\omega' : \mathcal{B} \longrightarrow \mathcal{B}'$ be $\mathcal{C}$-functors. A natural transformation $\varphi : \omega \longrightarrow \omega'$ is a *$\mathcal{C}$-morphism* if the following diagram commutes

$$\begin{array}{ccc} \omega(X\otimes P) & \xrightarrow{\varphi(X\otimes P)} & \omega'(X\otimes P) \\ \downarrow{\scriptstyle \xi(X,P)} & & \downarrow{\scriptstyle \xi'(X,P)} \\ X\otimes\omega(P) & \xrightarrow{X\otimes\varphi(P)} & X\otimes\omega'(P). \end{array}$$

We will denote the set[2] of natural transformation from ω to ω' by $\mathrm{Nat}(\omega,\omega')$ and the subset of $\mathcal{C}$-morphisms by $\mathrm{Nat}_{\mathcal{C}}(\omega,\omega')$.

2.1.10. These $\mathcal{C}$-morphisms will be of central importance for reconstruction, so we will give an example. Let B be a bialgebra in Vec and $\mathcal{C} = \mathcal{A} := \mathcal{M}_B$. Let A be an algebra in Vec. It can be considered as a B-module algebra by the trivial action $ab := a\varepsilon(b)$. Let $\mathcal{B} := (\mathcal{M}_B)_A$. Consider the $\mathcal{C}$-functor $\omega : (\mathcal{M}_B)_A \longrightarrow \mathcal{M}_B$ with $\omega(P) = P$, the forgetful functor. Then for any $a \in A$ the morphism $\varphi_a : \omega \longrightarrow \omega$, $\varphi_a(P) : \omega(P) \longrightarrow \omega(P)$, $\varphi_a(p) = pa$ is a natural transformation and in fact a $\mathcal{C}$-morphism. For any $b \in \mathrm{center}(B)$ the morphism $\varphi_b : \omega \longrightarrow \omega$, $\varphi_b(P) : \omega(P) \longrightarrow \omega(P)$, $\varphi_b(p) = pb$ is a natural transformation, but in general it is not a $\mathcal{C}$-morphism. If φ_b is a $\mathcal{C}$-morphism then for the special choice $X = B$, $P = B\otimes A$, $x = 1_B$, and $p = 1_B \otimes 1_A$ we have $\sum b_{(1)} \otimes b_{(2)} \otimes 1_A = \sum b_{(1)} \otimes b_{(2)} \otimes 1_A b_{(3)} = \sum x b_{(1)} \otimes p b_{(2)} = \varphi_b(x\otimes p) = x \otimes \varphi_b(p) = x\otimes pb = 1_B \otimes b \otimes 1_A$, and hence $\Delta(b) = 1\otimes b$ which implies $b = \alpha \cdot 1_B$ ($\alpha \in \mathbb{K}$). Conversely for $b = \alpha\cdot 1_B$ it is easy to see that φ_b is a $\mathcal{C}$-morphism.

2.2. Braided categories. For the definition and study of more complicated objects, like bialgebras and Hopf algebras in $\mathcal{C}$, we assume that the monoidal category $\mathcal{C}$ is *braided* (or a *quasitensor* category) with a natural isomorphism of bifunctors $\sigma_{X,Y} : X\otimes Y \cong Y \otimes X$, the *braiding*, such that $(1_Y \otimes \sigma_{X,Z})(\sigma_{X,Y}\otimes 1_Z) = \sigma_{X,Y\otimes Z}$ and $(\sigma_{X,Z}\otimes 1_Y)(1_X\otimes\sigma_{Y,Z}) = \sigma_{X\otimes Y,Z}$.

2.2.1. A *quasitriangular structure* or *universal R-matrix* [Dr86] for a bialgebra $B = (B, m, u, \Delta, \varepsilon)$ in Vec is an invertible element $R = \sum R_1 \otimes R_2 \in B\otimes B$ such that

(1) $\forall b \in B : \tau\Delta(b) = R\Delta(b)R^{-1}$,
(2) $(\Delta\otimes 1)(R) = R_{13}R_{23}$,
(3) $(1\otimes\Delta)(R) = R_{13}R_{12}$

where $R_{12} = R\otimes 1_B$, $R_{13} = \sum R_1 \otimes 1_B \otimes R_2$, and $R_{23} = 1_B \otimes R$.

2.2.2. A *coquasitriangular structure* ([Sch92b] Definition 2.4.4 and [LT91]) or *braiding* is a convolution-invertible homomorphism $r : B\otimes B \longrightarrow \mathbb{K}$ such that

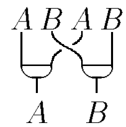
(1) $m\tau = r * m * r^{-1}$,
(2) $r(m\otimes 1) = r^{13}r^{23}$,
(3) $r(1\otimes m) = r^{13}r^{12}$.

2.2.3. If B is quasitriangular then $\mathcal{M}_B$ is a braided monoidal category with $\sigma_{X,Y}(x\otimes y) = \sum(yR_2 \otimes xR_1)$ [Dr86].

2.2.4. If B is coquasitriangular then $\mathcal{M}^B$ is a braided monoidal category with $\sigma_{X,Y}(x\otimes y) = \sum(y_{(0)}\otimes x_{(0)})r(x_1\otimes y_1)$ ([Sch92b] Remark 2.4.6; see also the last paragraph in [Pa81]).

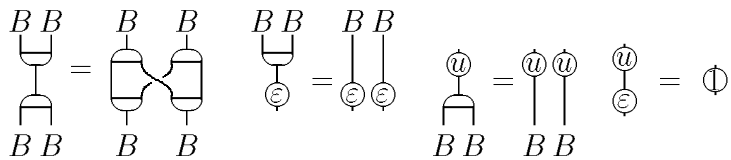
[2]There are well known standard methods to handle the set theoretic difficulties of this construction.

2.2.5. Here are some observations from [Mj94a] about algebras, bialgebras and Hopf algebras in braided monoidal categories $\mathcal{C}$. If A and B are algebras in $\mathcal{C}$ then so is $A \otimes B$. We use the graphical calculus [Ye90] to describe the algebra multiplication as



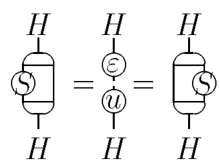
which represents the morphism $(m_A \otimes m_B)(1_A \otimes \sigma_{B,A} \otimes 1_B)$. One checks that $A \otimes B$ becomes an algebra with this multiplication.

This allows us to define a *bialgebra* in $\mathcal{C}$ which is an algebra (B, m, u) and a coalgebra (B, Δ, ε) such that

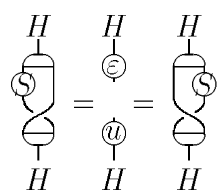


i.e. $\Delta m = (m \otimes m)(1 \otimes \sigma \otimes 1)(\Delta \otimes \Delta)$, $\varepsilon m = \varepsilon \otimes \varepsilon$, $\Delta u = u \otimes u$, and $\varepsilon u = 1_{\mathbb{K}}$.

A bialgebra H in $\mathcal{C}$ is a *Hopf algebra* with *antipode* $S : H \longrightarrow H$ in $\mathcal{C}$ if it also satisfies



A bialgebra H in $\mathcal{C}$ has a *twisted antipode* S in $\mathcal{C}$ if it satisfies



The notion of a braided bialgebra in $\mathcal{C}$ is somewhat more subtle and has been studied in [Mj93a].

2.3. $\mathcal{C}$-monoidal categories.

Now let $\mathcal{C}$ be a braided monoidal category.

Definition 2.4. Let $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$ be $\mathcal{C}$-categories. A bifunctor $\omega : \mathcal{B} \times \mathcal{B}' \longrightarrow \mathcal{B}''$ together with natural isomorphisms coherent with the $\mathcal{C}$-structures on $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$,

(1) $\xi_{X,P,Q} : \omega(X \otimes P, Q) \longrightarrow X \otimes \omega(P, Q)$,
(2) $\tau_{P,X,Q} : \omega(P, X \otimes Q) \longrightarrow X \otimes \omega(P, Q)$, and
(3) $\tilde{\tau}_{X,P,Q} : X \otimes \omega(P, Q) \longrightarrow \omega(P, X \otimes Q)$

is called a $\mathcal{C}$-*bifunctor*, if the following diagrams commute

$$\begin{tikzcd}
\omega(X \otimes P, Y \otimes Q) \arrow[r, "\tau_{X\otimes P,Y,Q}"] \arrow[d, "\xi"'] & Y \otimes \omega(X \otimes P, Q) \arrow[r, "1\otimes\xi"] & Y \otimes X \otimes \omega(P, Q) \\
X \otimes \omega(P, Y \otimes Q) \arrow[r, "1\otimes\tau_{P,Y,Q}"] & X \otimes Y \otimes \omega(P, Q) \arrow[ur, "\sigma_{X,Y}\otimes 1_{\omega(P,Q)}"'] &
\end{tikzcd}$$

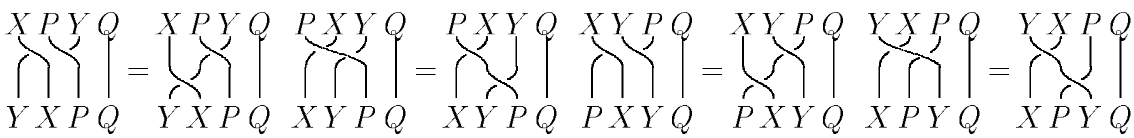
$$\begin{tikzcd}
\omega(P, X \otimes Y \otimes Q) \arrow[rr, "\tau_{P,X\otimes Y,Q}"] \arrow[dr, "\tau_{P,X,Y\otimes Q}"'] & & X \otimes Y \otimes \omega(P,Q) \\
& X \otimes \omega(P, Y \otimes Q). \arrow[ur, "1_X \otimes \tau_{P,Y,Q}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \otimes \omega(P,Q) \arrow[rr, "\widetilde{\tau}_{X\otimes Y,P,Q}"] \arrow[dr, "1_X \otimes \widetilde{\tau}_{Y,P,Q}"'] & & \omega(P, X \otimes Y \otimes Q) \\
& X \otimes \omega(P, Y \otimes Q). \arrow[ur, "\widetilde{\tau}_{X,P,Y\otimes Q}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y \otimes X \otimes \omega(P,Q) \arrow[r, "1\otimes\xi^{-1}"] \arrow[dr, "\sigma_{Y,X}\otimes 1_{\omega(P,Q)}"'] & Y \otimes \omega(X \otimes P, Q) \arrow[r, "\widetilde{\tau}_{Y,X\otimes P,Q}"] & \omega(X \otimes P, Y \otimes Q) \\
& X \otimes Y \otimes \omega(P,Q) \arrow[r, "1\otimes\widetilde{\tau}_{Y,P,Q}"] & X \otimes \omega(P, Y \otimes Q) \arrow[u, "\xi^{-1}"']
\end{tikzcd}$$

(suppressing the coherence isomorphisms α and β from Definition 2.1, i.e. going to the strict case.) The corresponding braid diagrams are



Observe that $\widetilde{\tau}_{X,P,Q}$ is not the inverse of $\tau_{P,X,Q}$. Both morphisms are associated with σ in the control category, so in braid diagrams they will be represented by a braid with the same orientation as σ.

2.3.1. If $\mathcal{A} = \mathcal{C}$ is a braided monoidal category, $\mathcal{B}$ is an $\mathcal{A}$-category and $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ is an $\mathcal{A}$-functor, then the bifunctor $\omega \otimes \omega : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{A}$ given by $(\omega \otimes \omega)(P,Q) = \omega(P) \otimes \omega(Q)$ is an $\mathcal{A}$-bifunctor. The bifunctor $(\omega \otimes \omega \otimes M)(P,Q) = \omega(P) \otimes \omega(Q) \otimes M$ for $M \in \mathcal{A}$ is also an $\mathcal{A}$-bifunctor.

2.3.2. In a similar way define a $\mathcal{C}$-multifunctor property for multifunctors $\omega : \mathcal{B}_1 \times \ldots \times \mathcal{B}_n \longrightarrow \mathcal{B}$. In particular functors of the form $\omega \otimes \ldots \otimes \omega = \omega^n : \mathcal{B} \times \ldots \times \mathcal{B} \longrightarrow \mathcal{A}$ and $\omega \otimes \ldots \otimes \omega \otimes M = \omega^n \otimes M : \mathcal{B} \times \ldots \times \mathcal{B} \longrightarrow \mathcal{A}$ are $\mathcal{A}$-multifunctors, if $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ is an $\mathcal{A}$-functor and $M \in \mathcal{A}$.

Definition 2.5. Let $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$ be $\mathcal{C}$-categories and $\omega, \omega' : \mathcal{B} \times \mathcal{B}' \longrightarrow \mathcal{B}''$ be $\mathcal{C}$-bifunctors. A natural transformation $\varphi : \omega \longrightarrow \omega'$ is a $\mathcal{C}$-*bimorphism*, if the following diagrams commute

$$\begin{tikzcd}
\omega(X \otimes P, Q) \arrow[r, "\varphi(X\otimes P,Q)"] \arrow[d, "\xi"] & \omega'(X \otimes P, Q) \arrow[d, "\xi'"] \\
X \otimes \omega(P,Q) \arrow[r, "X\otimes\varphi(P,Q)"] & X \otimes \omega'(P,Q)
\end{tikzcd}$$

$$\begin{tikzcd}
\omega(P, Y \otimes Q) \arrow[r, "\varphi(P,Y\otimes Q)"] \arrow[d, "\tau_{P,Y,Q}"'] & \omega'(P, Y \otimes Q) \arrow[d, "\tau'_{P,Y,Q}"] \\
Y \otimes \omega(P,Q) \arrow[r, "Y\otimes\varphi(P,Q)"] & Y \otimes \omega'(P,Q)
\end{tikzcd}$$

$$\begin{tikzcd}
Y \otimes \omega(P,Q) \arrow[r, "Y\otimes\varphi(P,Q)"] \arrow[d, "\widetilde{\tau}_{Y,P,Q}"'] & Y \otimes \omega'(P,Q) \arrow[d, "\widetilde{\tau}'_{Y,P,Q}"] \\
\omega(P, Y \otimes Q) \arrow[r, "\varphi(P,Y\otimes Q)"] & \omega'(P, Y \otimes Q)
\end{tikzcd}$$

Let $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega')$ denote the set of $\mathcal{C}$-bimorphisms.

For multifunctors $\omega, \omega' : \mathcal{B}_1 \times \ldots \times \mathcal{B}_n \longrightarrow \mathcal{B}$ we proceed in a similar way. A natural transformation of multifunctors $\varphi : \omega \longrightarrow \omega'$ is called a $\mathcal{C}$-multimorphism, if commutative diagrams as above hold for all variables.

Definition 2.6. Let $\mathcal{A}$ be a $\mathcal{C}$-category with a coherent structure of a monoidal category with tensor product $P \mathbin{\widehat{\otimes}} Q$. If $\widehat{\otimes} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is a coherent $\mathcal{C}$-bifunctor, then $\mathcal{A}$ is called a *$\mathcal{C}$-monoidal category*. In particular the structural morphisms α, λ, and ρ for $\mathcal{A}$ are $\mathcal{C}$-morphisms in each variable from $\mathcal{A}$.

If we go to the strict case we assume $\alpha_{\mathcal{C}}$, $\lambda_{\mathcal{C}}$, $\rho_{\mathcal{C}}$, β, π, $\alpha_{\mathcal{A}}$, $\lambda_{\mathcal{A}}$, $\rho_{\mathcal{A}}$, and ξ to be identities. Then the necessary equalities for the strict case are

$$\begin{aligned}
&\tau(I, X, P) = \mathrm{id},\\
&\widetilde{\tau}((X, I, P) = \mathrm{id},\\
&\tau(P, X, Q) \mathbin{\widehat{\otimes}} 1_R = \tau(P, X, Q \mathbin{\widehat{\otimes}} R),\\
&\widetilde{\tau}(X, P, Q \mathbin{\widehat{\otimes}} R) = \widetilde{\tau}(X, P, Q) \mathbin{\widehat{\otimes}} 1_R,\\
&(\tau(P, X, Q) \mathbin{\widehat{\otimes}} 1_R)(1_P \mathbin{\widehat{\otimes}} \tau(Q, X, R)) = \tau(P \mathbin{\widehat{\otimes}} Q, X, R),\\
&(1_P \mathbin{\widehat{\otimes}} \widetilde{\tau}(X, Q, R))(\widetilde{\tau}(X, P, Q) \mathbin{\widehat{\otimes}} 1_R) = \widetilde{\tau}(X, P \mathbin{\widehat{\otimes}} Q, R).
\end{aligned}$$

Observe that $\mathcal{C}$ is a $\mathcal{C}$-monoidal category (since $\mathcal{C}$ is braided).

Definition 2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories. A *monoidal functor* is a functor $\omega : \mathcal{A} \longrightarrow \mathcal{B}$ together with coherent natural isomorphisms $\upsilon : \omega(P \mathbin{\widehat{\otimes}} Q) \cong \omega(P) \mathbin{\widehat{\otimes}} \omega(Q)$ and $\varsigma : \omega(I_{\mathcal{A}}) \cong I_{\mathcal{B}}$.

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{C}$-monoidal categories, $\omega : \mathcal{A} \longrightarrow \mathcal{B}$ is a monoidal functor and a $\mathcal{C}$-functor, and $\upsilon : \omega(P \mathbin{\widehat{\otimes}} Q) \cong \omega(P) \mathbin{\widehat{\otimes}} \omega(Q)$ is a $\mathcal{C}$-bimorphism, then ω is called a *$\mathcal{C}$-monoidal functor.*

Let $\omega, \omega' : \mathcal{A} \longrightarrow \mathcal{B}$ be $\mathcal{C}$-monoidal functors. A natural transformation $\varphi : \omega \longrightarrow \omega'$ is a *$\mathcal{C}$-monoidal morphism*, if φ is a $\mathcal{C}$-morphism and monoidal.

Definition 2.8. Let $\mathcal{A}$ be a $\mathcal{C}$-monoidal category together with a braiding $\sigma = \sigma_{\mathcal{A}} : P \mathbin{\widehat{\otimes}} Q \longrightarrow Q \mathbin{\widehat{\otimes}} P$. We call $\mathcal{A}$ a *$\mathcal{C}$-braided* $\mathcal{C}$-monoidal category, if the braid morphisms in both categories are coherent w.r.t. the braid group, in particular if

$$\begin{tikzcd}
P \mathbin{\widehat{\otimes}} (X \otimes Q) \arrow[r, "\sigma"] \arrow[d, "\tau"'] & (X \otimes Q) \mathbin{\widehat{\otimes}} P \arrow[d, "\xi"] \\
X \otimes (P \mathbin{\widehat{\otimes}} Q) \arrow[r, "1\otimes\sigma"] & X \otimes (Q \mathbin{\widehat{\otimes}} P)
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \otimes (P \mathbin{\widehat{\otimes}} Q) \arrow[r, "1\otimes\sigma"] \arrow[d, "\xi^{-1}"'] & X \otimes (Q \mathbin{\widehat{\otimes}} P) \arrow[d, "\widetilde{\tau}"] \\
(X \otimes P) \mathbin{\widehat{\otimes}} Q \arrow[r, "\sigma"] & Q \mathbin{\widehat{\otimes}} (X \otimes P)
\end{tikzcd}$$

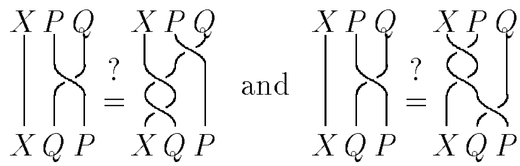
commute. Observe, however, that the diagrams

$$\begin{tikzcd}
(X \otimes P) \hat{\otimes} Q \arrow[r, "\sigma"] \arrow[d, "\xi"'] & Q \hat{\otimes} (X \otimes P) \arrow[d, "\tau"] \\
X \otimes (P \hat{\otimes} Q) \arrow[r, "1\otimes\sigma"] & X \otimes (Q \hat{\otimes} P)
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \otimes (P \hat{\otimes} Q) \arrow[r, "1\otimes\sigma"] \arrow[d, "\widetilde{\tau}"'] & X \otimes (Q \hat{\otimes} P) \arrow[d, "\xi^{-1}"] \\
P \hat{\otimes} (X \otimes Q) \arrow[r, "\sigma"] & (X \otimes Q) \hat{\otimes} P
\end{tikzcd}$$

do not necessarily commute since their braid diagrams are



In principle arbitrary tensor products of objects from $\mathcal{C}$ and from $\mathcal{A}$ can be formed and twisted by elements of the braid group with the exception of tensor factors from $\mathcal{C}$ appearing on the far right of a tensor product containing tensor factors from $\mathcal{A}$.

Definition 2.9. An object P in a $\mathcal{C}$-monoidal category $\mathcal{A}$ is called *$\mathcal{C}$-central*, if

$$(X \otimes P \otimes Q \overset{\widetilde{\tau}(X,P,Q)}{\longrightarrow} P \otimes X \otimes Q \overset{\tau(P,X,Q)}{\longrightarrow} X \otimes P \otimes Q) = \mathrm{id}$$

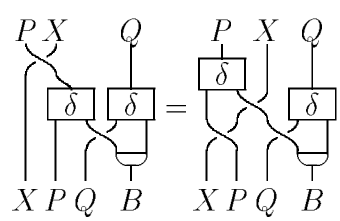
holds for all $X \in \mathcal{C}$ and $Q \in \mathcal{A}$.

Theorem 2.10. *Let $\mathcal{A}$ be a $\mathcal{C}$-braided $\mathcal{C}$-monoidal category. Let B be a $\mathcal{C}$-central bialgebra in $\mathcal{A}$, C be a coalgebra in $\mathcal{A}$ and $z : C \longrightarrow B$ be a coalgebra morphism. Then*

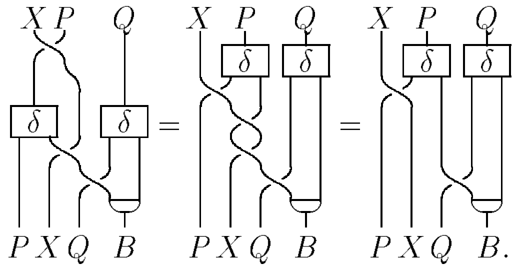
(1) *$\mathcal{A}^C$ is a $\mathcal{C}$-category;*
(2) *$\mathcal{A}^B$ is a $\mathcal{C}$-monoidal category;*
(3) *$\omega := \mathcal{A}^z : \mathcal{A}^C \longrightarrow \mathcal{A}^B$ is a $\mathcal{C}$-functor;*
(4) *the forgetful functor $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ is a $\mathcal{C}$-functor;*
(5) *if C is a $\mathcal{C}$-central bialgebra and $z : C \longrightarrow B$ is a bialgebra morphism then $\omega := \mathcal{A}^z : \mathcal{A}^C \longrightarrow \mathcal{A}^B$ is a $\mathcal{C}$-monoidal functor;*
(6) *the forgetful functor $\omega : \mathcal{A}^B \longrightarrow \mathcal{A}$ is a $\mathcal{C}$-monoidal functor.*

Proof. (1) similar to 2.1.6.

(2) A little calculation shows that $\mathcal{A}^B$ is a monoidal category ([Mj94a] Prop. 2.5) with the comultiplication on the tensor product given by $(1_P \otimes 1_Q \otimes m_B)(1_P \otimes \sigma_{B,Q} \otimes 1_B)(\delta_P \otimes \delta_Q) : P \otimes Q \longrightarrow P \otimes Q \otimes B$. $\mathcal{A}^B$ is also a $\mathcal{C}$-category by (1). The natural transformation $\xi : (X \otimes P) \hat{\otimes} Q \longrightarrow X \otimes (P \hat{\otimes} Q)$ is compatible with the comultiplication with B from the right. So it is in $\mathcal{A}^B$. The natural transformation $\tau : P \hat{\otimes} (X \otimes Q) \cong X \otimes (P \hat{\otimes} Q)$ satisfies



hence it is in $\mathcal{A}^B$, too. Finally the natural transformation $\tilde{\tau} : X \otimes (P \mathbin{\hat{\otimes}} Q) \cong P \mathbin{\hat{\otimes}} (X \otimes Q)$ satisfies



Since the diagrams defining the structure of a $\mathcal{C}$-monoidal category on $\mathcal{A}^B$ commute in $\mathcal{A}$ and consist of morphisms of B-comodules they also commute as diagrams in $\mathcal{A}^B$. Thus $\mathcal{A}^B$ is a $\mathcal{C}$-monoidal category.

(3) and (4) similar to 2.1.8

(5) Since the tensor products in $\mathcal{A}^C$ and $\mathcal{A}^B$ are induced by the tensor product in $\mathcal{A}$ the natural transformation $\upsilon : \omega(P \mathbin{\hat{\otimes}} Q) \longrightarrow \omega(P) \otimes \omega(Q)$ is the identity which makes ω a $\mathcal{C}$-monoidal functor.

(6) is a special case of (5). $\square$

A corresponding result holds by duality for the category $\mathcal{A}_B$ of modules over a bialgebra B in $\mathcal{A}$.

2.4. Rigid categories.

2.4.1. Another important categorical notion is that of a (right) dual object. This is a generalization of finite-dimensional vector spaces. An object $X \in \mathcal{C}$ is *rigid* or has a *dual* (X^*, ev) where $X^* \in \mathcal{C}$ and $\mathrm{ev} : X^* \otimes X \longrightarrow I$ is called the *evaluation*, if there is a morphism $\mathrm{db} : I \longrightarrow X \otimes X^*$, the *dual basis*, such that

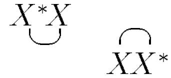
$$(X \xrightarrow{\mathrm{db}\otimes 1} X \otimes X^* \otimes X \xrightarrow{1\otimes\mathrm{ev}} X) = 1_X \ ,$$
$$(X^* \xrightarrow{1\otimes\mathrm{db}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}\otimes 1} X^*) = 1_{X^*} \ .$$

The monoidal category $\mathcal{C}$ is *rigid* or a *tensor category* if every object of $\mathcal{C}$ has a dual. The full subcategory of objects in $\mathcal{C}$ having duals is denoted by $\mathcal{C}_0$. An adjoint functor argument shows that the dual of an object is unique up to isomorphism if it exists.

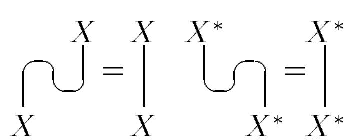
2.4.2. If $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ is a monoidal functor and $P \in \mathcal{B}$ is rigid then $\omega(P) \in \mathcal{A}$ is rigid with dual object $\omega(P^*)$, evaluation $\omega(P^*)\otimes\omega(P) \cong \omega(P^*\otimes P) \longrightarrow \omega(I_{\mathcal{B}}) \cong I_{\mathcal{A}}$, and dual basis $I_{\mathcal{A}} \cong \omega(I_{\mathcal{B}}) \longrightarrow \omega(P \otimes P^*) \cong \omega(P) \otimes \omega(P^*)$.

Proposition 2.11. *Let $\mathcal{C}$ be a braided monoidal category. Then the full subcategory $\mathcal{C}_0$ of rigid objects in $\mathcal{C}$ is a rigid braided monoidal category.*

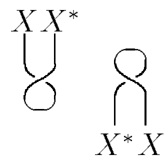
Proof. If the evaluation resp. the dual basis are morphisms represented by



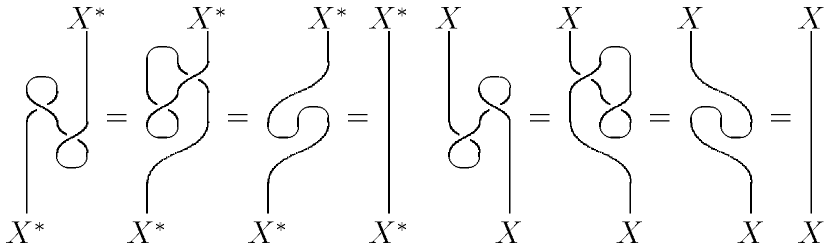
then the conditions are



If $X \in \mathcal{C}$ has a dual (X^*, ev) then X^* has the dual $(X, \mathrm{ev} \circ \sigma_{X,X^*})$ with the dual basis $\sigma^{-1}_{X,X^*} \circ \mathrm{db}$. The corresponding morphisms for X^* are



and the relations are



If X and Y are in $\mathcal{C}_0$ then $X \otimes Y$ has the dual $(Y^* \otimes X^*, \mathrm{ev}_Y(1_{Y^*} \otimes \mathrm{ev}_X \otimes 1_Y))$. The reader may try the easy graphic and the diagrammatic proofs. Thus $\mathcal{C}_0$ is a full monoidal subcategory of $\mathcal{C}$ which inherits the braiding and contains the duals for every object. $\square$

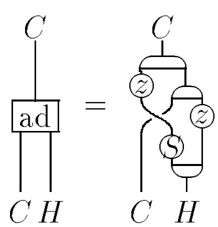
2.5. Coadjoint coactions.

2.5.1. Let $\mathcal{A}$ be a braided monoidal category. Let C be a coalgebra in $\mathcal{A}$, H be a Hopf algebra in $\mathcal{A}$ and $z : C \longrightarrow H$ be a coalgebra homomorphism. We define a *right coadjoint coaction* of H on C by

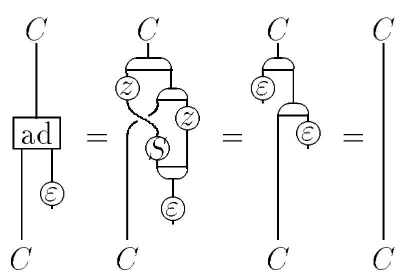
$$\begin{aligned}\mathrm{ad} := \ & (1_C \otimes m_H)(1_C \otimes S \otimes 1_H)(\sigma_{H,C} \otimes 1_H) \\ & (z \otimes 1_C \otimes z)(1_C \otimes \Delta_C)\Delta_C : C \longrightarrow C \otimes H.\end{aligned}$$

Proposition 2.12. *C with the right coadjoint coaction is an H-comodule coalgebra.*

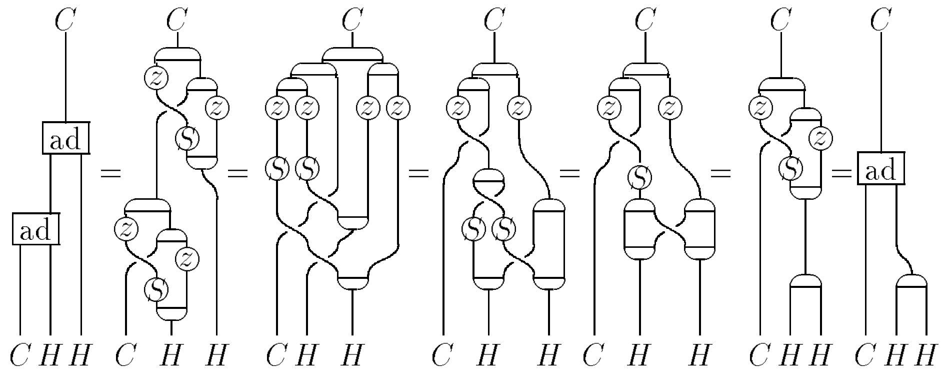
Proof. In graphical notation the right coadjoint coaction is



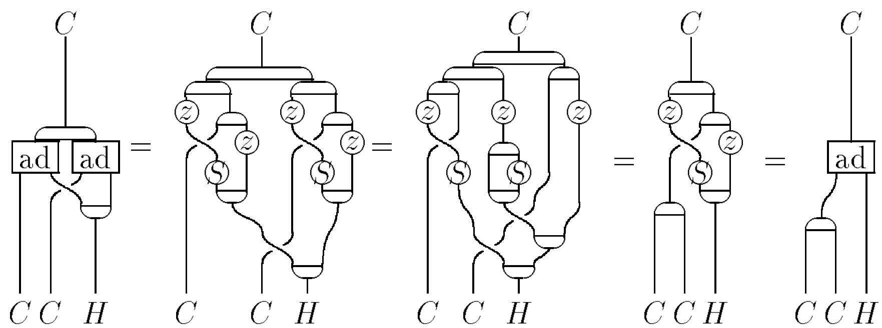
The right coadjoint action is a counary action by



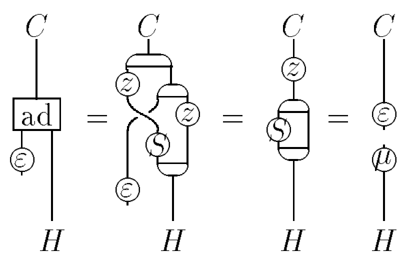
The coaction is coassociative:



The comultiplication is an H-comodule morphism by



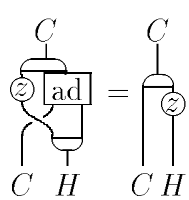
and preserves the counit of C



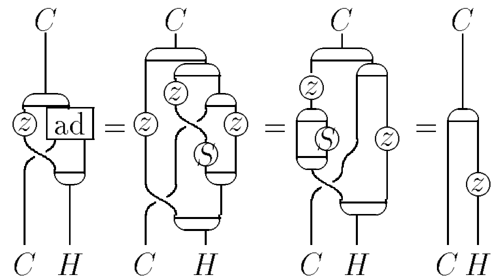
□

Now we want to slightly generalize the notion of a right coadjoint coaction to the case where H is only a bialgebra.

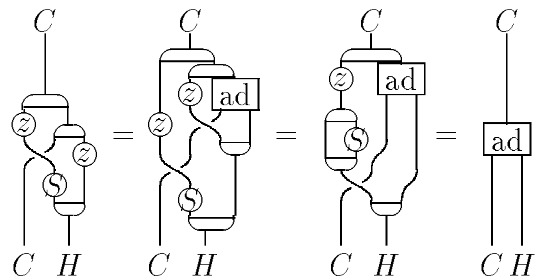
Lemma 2.13. *Let H be a Hopf algebra, C be a coalgebra, and $z : C \longrightarrow H$ be a coalgebra morphism. A right coaction* $\mathrm{ad} : C \longrightarrow C \otimes H$ *is the right coadjoint coaction iff*



Proof. If ad is the right coadjoint coaction then the equation of the lemma holds by



Conversely if this equality holds then the right coaction $\mathrm{ad} : C \longrightarrow C \otimes H$ is the right coadjoint coaction since



□

2.5.2. We say that a coaction $\mathrm{ad} : C \longrightarrow C \otimes B$ for a given $z : C \longrightarrow B$, B a bialgebra, is a *right coadjoint coaction* if the equation in Lemma 2.13 holds.

More generally if $z : C \longrightarrow B$ is $\star$-invertible then a coadjoint coaction can be constructed in the same way as above. We don't know if there are more general conditions for $z : C \longrightarrow B$ such that a right coadjoint coaction exists nor whether it is unique then.

2.5.3. In the dual situation let $f : H \longrightarrow A$ be an algebra homomorphism with a Hopf algebra H. The right adjoint action $ah = \sum f(S(h_1)) \cdot a \cdot f(h_2)$ is characterized by the equation $\sum f(h_1) \cdot (ah_2) = a \cdot f(h)$.

2.6. $\mathcal{C}_0$-generated coalgebras. We still need another somewhat more general setup. Let $\mathcal{C}$ be a monoidal category, $\mathcal{C}_0$ be a full monoidal subcategory of $\mathcal{C}$. In this situation we consider a special type of coalgebra in $\mathcal{C}$.

Definition 2.14. Let $C \in \mathcal{C}$ be a coalgebra satisfying the following conditions:

(1) C is a colimit in $\mathcal{C}$ of a diagram of objects C_i in $\mathcal{C}_0$.
(2) All morphisms $X \otimes \iota_i \otimes M : X \otimes C_i \otimes M \longrightarrow X \otimes C \otimes M$ are monomorphisms in $\mathcal{C}$ where $X \in C_0$, $M \in \mathcal{C}$ and the $\iota_i : C_i \longrightarrow C$ are the injections of the colimit diagram.
(3) Every C_i is a subcoalgebra of C via $\iota_i : C_i \longrightarrow C$.
(4) If $(P, \delta_P : P \longrightarrow P \otimes C)$ is a comodule over C and $P \in \mathcal{C}_0$, then there exists a C_i in the diagram for C and a morphism $\delta_{P,i} : P \longrightarrow P \otimes C_i$ such that

$$\begin{tikzcd} P \arrow[r, "\delta_{P,i}"] \arrow[dr, "\delta_P"'] & P \otimes C_i \arrow[d, "1 \otimes \iota_i"] \\ & P \otimes C \end{tikzcd}$$

commutes.

Then C is called a $\mathcal{C}_0$-*generated coalgebra.*

2.6.1. If $\mathcal{C}_0 = \mathcal{C}$ then the conditions in the previous definition are trivially satisfied. If $\mathcal{C} = \mathrm{Vec}$ and $C_0 = \mathrm{vec}$, the category of finite-dimensional vector spaces, then every coalgebra in $\mathcal{C}$ is a $\mathcal{C}_0$-generated coalgebra by the fundamental theorem for coalgebras ([Sw69] Thm. 2.2.1) and its generalization to the fundamental theorem for comodules.

2.6.2. We denote by $\mathcal{C}_0^C$ the category of C-comodules in $\mathcal{C}_0$. Then $\mathcal{C}_0^C$ is a $\mathcal{C}_0$-category and the forgetful functor $\omega : \mathcal{C}_0^C \longrightarrow \mathcal{C}_0$ is a $\mathcal{C}_0$-functor.

2.6.3. It is an easy exercise to show for a $\mathcal{C}_0$-generated coalgebra, that the $(P, \delta_{P,i} : P \longrightarrow P \otimes C_i)$ are comodules.

3. Reconstruction properties

For the rest of this paper let the control category $\mathcal{C}$ be a braided monoidal category and the base category $\mathcal{A}$ be a $\mathcal{C}$-monoidal category.

3.1. Reconstruction of coalgebras.

Definition 3.1. We define the category $\overline{\mathfrak{A}}(\mathcal{C})$ of all $\mathcal{C}$-categories "over" $\mathcal{A}$ as follows. The objects are pairs $(\mathcal{B}, \omega)$ consisting of a $\mathcal{C}$-category $\mathcal{B}$ and of a $\mathcal{C}$-functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$. A morphism $[\chi, \zeta] : (\mathcal{B}, \omega) \longrightarrow (\mathcal{B}', \omega')$ is an equivalence class of pairs (χ, ζ) with $\chi : \mathcal{B} \longrightarrow \mathcal{B}'$ a $\mathcal{C}$-functor and $\zeta : \omega \longrightarrow \omega'\chi$ a $\mathcal{C}$-isomorphism. Two such pairs (χ, ζ) and (χ', ζ') are equivalent if there is a $\mathcal{C}$-isomorphism $\varphi : \chi \longrightarrow \chi'$ with $\zeta' = \omega'\varphi \circ \zeta$. Composition is given by $[\chi', \zeta'] \circ [\chi, \zeta] = [\chi'\chi, \zeta'\chi \circ \zeta]$.

Let $\mathfrak{A}(\mathcal{C})$ be a full subcategory of $\overline{\mathfrak{A}}(\mathcal{C})$.

3.1.1. Theorem 2.10 defines a functor $\mathcal{A}^{\text{-}} : \mathcal{A}\text{-coalg} \longrightarrow \overline{\mathfrak{A}}(\mathcal{C})$ by $\mathcal{A}^{\text{-}}(C) := (\mathcal{A}^C, \omega)$ where ω is the forgetful functor. Furthermore $\mathcal{A}^{\text{-}}(z) := [\mathcal{A}^z, \mathrm{id}]$.

3.1.2. If $\mathcal{A}_0$ is a full $\mathcal{C}$-monoidal subcategory of $\mathcal{A}$, then we define the full subcategory $\mathfrak{A}_0(\mathcal{C})$ of $\overline{\mathfrak{A}}(\mathcal{C})$ to consist of those $\mathcal{C}$-categories $\mathcal{B}$ over $\mathcal{A}$ whose forgetful functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ factors through $\mathcal{A}_0$.

3.1.3. In this case we obtain a functor $\mathcal{A}_0^{\text{-}} : \mathcal{A}\text{-coalg} \longrightarrow \mathfrak{A}_0(\mathcal{C})$ by $\mathcal{A}_0^{\text{-}}(C) := (\mathcal{A}_0^C, \omega)$ where $\mathcal{A}_0^C$ denotes the full subcategory of $\mathcal{A}^C$ of those C-comodules whose underlying object is in $\mathcal{A}_0$.

Now we address the question which properties of an algebra A or a coalgebra C in a monoidal category $\mathcal{A}$ can be recovered from the category of its modules $\mathcal{A}_A$ resp. comodules $\mathcal{A}^C$. Since reconstruction of coalgebras is somewhat simpler (see Theorem 4.7) we will perform the reconstruction of a coalgebra C from $\mathcal{A}^C$ explicitly and derive the reconstruction of an algebra by duality. As we remarked in the introduction C is not uniquely determined by $\mathcal{A}^C$. But if we use additional information about the forgetful $\mathcal{C}$-functor $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ we can reconstruct C up to isomorphism with its full structure.

In many cases we can actually "reconstruct" a coalgebra C from an fairly arbitrary $\mathcal{C}$-functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$. We will postpone the discussion of how to obtain the *object* $C \in \mathcal{A}$ from a functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ to the next section. In this section we will give the general definition and discuss the *structure* of such a reconstructed object C.

A different point of view is how to find a left adjoint functor to the functor $\mathcal{A}^{\text{-}} : \mathcal{A}\text{-coalg} \longrightarrow \mathfrak{A}(\mathcal{C})$. If such a left adjoint functor does not exist "globally", it might still exist "locally", i.e. a certain functor is representable.

Definition 3.2. Let $\mathcal{B}$ be a $\mathcal{C}$-category and $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ be a $\mathcal{C}$-functor. Then the sets $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes M)$ depend functorially on $M \in \mathcal{A}$, i.e. we have a functor

$$\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -) : \mathcal{A} \longrightarrow \mathrm{Set}.$$

If this functor is representable then the representing object will be denoted by $\mathrm{coend}_{\mathcal{C}}(\omega)$. It is unique up to isomorphism. (In the dual situation a representing object for $\mathrm{Nat}_{\mathcal{C}}(\omega \otimes -, \omega)$ will be denoted by $\mathrm{end}_{\mathcal{C}}(\omega)$.) So we have

$$\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes M) \cong \mathcal{A}(\mathrm{coend}_{\mathcal{C}}(\omega), M).$$

The universal arrow for this functor

$$\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}_{\mathcal{C}}(\omega)$$

is a $\mathcal{C}$-morphism, the image of the identity in $\mathcal{A}(\mathrm{coend}_{\mathcal{C}}(\omega), \mathrm{coend}_{\mathcal{C}}(\omega))$. It solves the following universal problem

- for every $M \in \mathcal{A}$ and every $\mathcal{C}$-morphism $\varphi : \omega \longrightarrow \omega \otimes M$ there is a unique morphism $f : C \longrightarrow M$ such that

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \omega \otimes C \arrow[d, "1 \otimes f"] \\ & \omega \otimes M \end{tikzcd}$$

  commutes.

This universal property is in fact equivalent to the representability of $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ and induces a universal factorization of ω through the category of comodules $\mathcal{A}^C$.

The study of $\mathcal{C}$-functors as conducted here has many properties in common with similar results for general functors. In fact general categories, functors and natural transformations may also be considered as $\mathcal{C}$-categories, $\mathcal{C}$-functors, resp. $\mathcal{C}$-morphisms for the monoidal category $\mathcal{C}$ with one object I and one morphism id_I.

Many of the following propositions are well known for the case of a monoidal category $\mathcal{C} = \{I\}$ and can be proved by standard universal abstract nonsense. So we only sketch the idea of the proofs. There are, however, subtle difficulties and restrictions with respect to braidings that do not occur in the case of a symmetric control category $\mathcal{C}$.

Proposition 3.3. *If* $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ *is representable, then the representing object* $C = \mathrm{coend}_{\mathcal{C}}(\omega)$ *is a coalgebra in* $\mathcal{A}$*. This coalgebra is uniquely determined up to isomorphisms of coalgebras.*

Furthermore every object $\omega(P) \in \mathcal{A}$ *with* $P \in \mathcal{B}$ *is a* C*-comodule via* $\delta : \omega(P) \longrightarrow \omega(P) \otimes C$ *and every morphism* $\omega(f)$ *is a morphism of* C*-comodules.*

Every object $\omega(X \otimes P) \in \mathcal{A}$ *with* $X \in \mathcal{C}$ *and* $P \in \mathcal{B}$ *is isomorphic as a* C*-comodule to* $X \otimes \omega(P)$ *with the structure induced by* $\omega(P)$*.*

Proof. The comultiplication Δ and counit ε are uniquely defined by $(1_\omega \otimes \Delta)\delta = (\delta \otimes 1_C)\delta$ and $(1_\omega \otimes \varepsilon)\delta = \rho_\omega^{-1}$. The last claim follows since δ is a $\mathcal{C}$-morphism. $\square$

We will encounter situations of comodules $(P, \vartheta : P \longrightarrow P \otimes C)$ in $\mathcal{A}$ where we want to know if this comodule comes about as in the previous Proposition. So we define

Definition 3.4. Let $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ be representable by $C \in \mathcal{A}$. Then a comodule $(P, \vartheta : P \longrightarrow P \otimes C)$ in $\mathcal{A}$ can be *lifted along* ω if there is an object $Q \in \mathcal{B}$ and a comodule isomorphism $(\omega(Q), \delta) \cong (P, \vartheta)$. In this case the comodule (P, ϑ) is called *liftable along* ω and $Q \in B$ a *lifting*.

3.2. Reconstruction of bialgebras. Assume now, that the base category $\mathcal{A}$ is a $\mathcal{C}$-braided $\mathcal{C}$-monoidal category. (It will be clear from the context which tensor product is being used, so we simply use $\otimes$ for the tensor product in $\mathcal{A}$.)

Consider a $\mathcal{C}$-functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$. Then the bifunctors $\omega \otimes \omega = \omega^2 : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{A}$ and $\omega^2 \otimes M : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{A}$ are $\mathcal{C}$-bifunctors as can be easily checked. The sets $\mathrm{Nat}_{\mathcal{C}}(\omega^2, \omega^2 \otimes M)$ of $\mathcal{C}$-bimorphisms depend functorially on M, i.e. we have a functor

$$\mathrm{Nat}_{\mathcal{C}}(\omega^2, \omega^2 \otimes -) : \mathcal{A} \longrightarrow \mathrm{Set}.$$

Let $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ be representable with universal $\mathcal{C}$-morphism $\delta : \omega \longrightarrow \omega \otimes C$. In general the morphism

$$\delta_2 := (1_\omega \otimes \sigma_{\mathrm{coend}_{\mathcal{C}}(\omega), \omega} \otimes 1_{\mathrm{coend}_{\mathcal{C}}(\omega)})(\delta \otimes \delta) : \omega^2 \longrightarrow \omega^2 \otimes \mathrm{coend}_{\mathcal{C}}(\omega)^2$$

will not be a $\mathcal{C}$-bimorphism. This is, however, the case if $C = \mathrm{coend}_{\mathcal{C}}(\omega)$ is $\mathcal{C}$-central (see Definition 2.9). Similarly δ^n is a $\mathcal{C}$-multimorphism if C is $\mathcal{C}$-central.

Definition 3.5. If the functor $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ is representable with universal $\mathcal{C}$-morphism $\delta : \omega \longrightarrow \omega \otimes C$, if C is $\mathcal{C}$-central and if $\mathrm{Nat}_{\mathcal{C}}(\omega^2, \omega^2 \otimes -)$ is also representable with the special universal $\mathcal{C}$-bimorphism

$$\delta_2 := (1_\omega \otimes \sigma_{C,\omega} \otimes 1_C)(\delta \otimes \delta) : \omega \otimes \omega \longrightarrow \omega \otimes \omega \otimes C \otimes C$$

then we say that $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ is *birepresentable*.

In a similar way we proceed for the multifunctor $\omega \otimes \ldots \otimes \omega = \omega^n$. If C is $\mathcal{C}$-central and the functor $\mathrm{Nat}_{\mathcal{C}}(\omega^n, \omega^n \otimes -)$ is representable with the universal morphism

$$\delta^{(n)} := \tau(\delta \otimes \ldots \otimes \delta) : \omega^n \longrightarrow \omega^n \otimes \mathrm{coend}_{\mathcal{C}}(\omega)^n$$

with the obvious choice of $\tau \in B_{2n}$, the Artin braid group, then we say that $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ is *n-representable*. If this holds for all $n \in \mathbb{N}$ we say that the functor $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ is *multirepresentable* (fully representable in [Mj93a]).

Proposition 3.6. *Let $\mathcal{A}$ be $\mathcal{C}$-braided $\mathcal{C}$-monoidal, $\mathcal{B}$ be $\mathcal{C}$-monoidal and $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ be a $\mathcal{C}$-monoidal functor. If $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ is multirepresentable, then $B := \mathrm{coend}_{\mathcal{C}}(\omega)$ is a bialgebra in $\mathcal{A}$. This bialgebra is uniquely determined up to isomorphisms of bialgebras.*

If in addition $\mathcal{B}$ is $\mathcal{C}$-braided (ω will usually not preserve the braiding), then $\mathrm{coend}_{\mathcal{C}}(\omega)$ is coquasitriangular in $\mathcal{A}$.

If ω factors through the full subcategory $\mathcal{A}_0$ of rigid objects in $\mathcal{A}$ then $\mathrm{coend}_{\mathcal{C}}(\omega)$ is a Hopf algebra in $\mathcal{A}$.

Furthermore for any objects $P, Q \in \mathcal{B}$ the B-comodule structure on $\omega'(P) \otimes \omega'(Q)$ is defined by the multiplication on B.

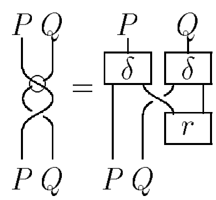
Proof. Similar to [Mj94a] Theorem 3.2. resp. 3.11. We check only that the relevant morphisms that are factorized through the universal morphisms are $\mathcal{C}$-morphisms.

The multiplication of B is defined by the $\mathcal{C}$-bimorphism $\delta'_{P\otimes Q} : \omega(P) \otimes \omega(Q) \cong \omega(P \otimes Q) \longrightarrow \omega(P \otimes Q) \otimes B$ as the uniquely determined morphism $\widetilde{m}_B : B \otimes B \longrightarrow B$ such that

$$(1_{\omega(P)} \otimes 1_{\omega(Q)} \otimes \widetilde{m})(1_{\omega(P)} \otimes \sigma_{B,Q} \otimes 1_B)(\delta_P \otimes \delta_Q) = \delta'_{P\otimes Q}.$$

The morphism $\delta'_{P\otimes Q\otimes R} : \omega(P) \otimes \omega(Q) \otimes \omega(R) \longrightarrow \omega(P \otimes Q \otimes R) \otimes B$ responsible for associativity is a $\mathcal{C}$-trimorphism.

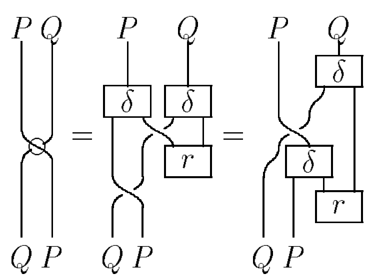
The coquasitriangular structure $r : B \otimes B \longrightarrow I$ is defined by the $\mathcal{C}$-bimorphism $\sigma_{\mathcal{A}}^{-1}(\omega(Q), \omega(P))\omega(\sigma_{\mathcal{B}}(P, Q)) : \omega(P) \otimes \omega(Q) \longrightarrow \omega(P) \otimes \omega(Q) \otimes I$ and the braid equation



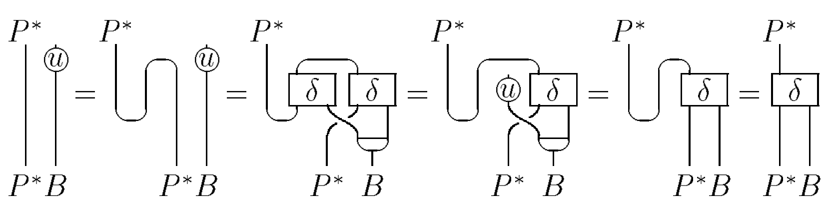
where the braid marked with a circle represents the braiding of the category $\mathcal{B}$. This diagram represents the equation

$$\sigma_{\mathcal{A}}^{-1}(\omega(Q), \omega(P))\omega(\sigma_{\mathcal{B}}(P, Q)) = (1_{\omega(P)} \otimes 1_{\omega(Q)} \otimes r)(1_{\omega(P)} \otimes \sigma_{\mathcal{A}}(B, \omega(Q)) \otimes 1_B)(\delta_P \otimes \delta_Q).$$

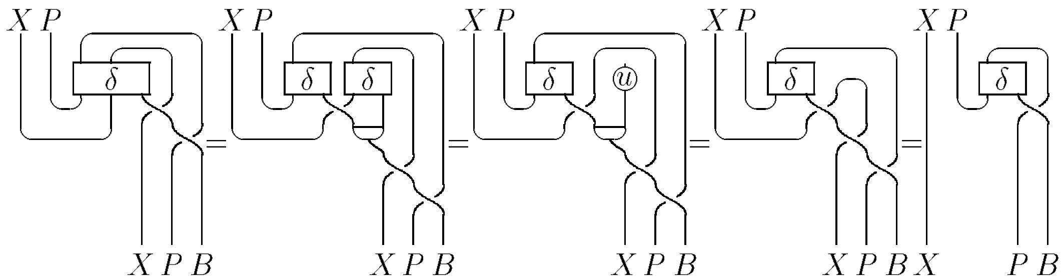
Observe that the braiding of $\mathcal{A}^B$ for a braided bialgebra B is described by the equation



Finally the antipode is defined by the morphism $(\omega(\mathrm{ev}_P) \otimes \sigma_{B,\omega(P)})(1_{\omega(P)} \otimes \delta_{\omega(P)^*} \otimes 1_{\omega(P)})(1_{\omega(P)} \otimes \omega(\mathrm{db}_P)) : \omega(P) \longrightarrow \omega(P) \otimes B$. To show that this is a $\mathcal{C}$-morphism, we first show the following claim. If $P \in \mathcal{B}$ defines a trivial B-comodule then $\omega(P)^*$ is also trivial. This follows from



Observe that we have $X \otimes P \cong X \otimes (I \mathbin{\widehat{\otimes}} P) \cong (X \otimes I) \mathbin{\widehat{\otimes}} P = \widetilde{X} \mathbin{\widehat{\otimes}} P$ in $\mathcal{B}$ for $\widetilde{X} := X \otimes I$ which gives trivial B-comodules $\omega(\widetilde{X}) \cong X \otimes \omega(I)$ and $\omega(\widetilde{X})^*$. So the diagram (where X denotes $\omega(\widetilde{X})$ and P denotes $\omega(P)$)



shows that the morphism $(\omega(\mathrm{ev}_P)\otimes\sigma_{B,\omega(P)})(1_{\omega(P)}\otimes\delta_{\omega(P)^*}\otimes 1_{\omega(P)})(1_{\omega(P)}\otimes\omega(\mathrm{db}_P))$: $\omega(P)\longrightarrow\omega(P)\otimes B$ is a $\mathcal{C}$-morphism. $\square$

Further interesting properties of $\mathcal{A}$ resp. ω for reconstruction may be found in [Dr89, KT92, Ye90]

3.3. Reconstruction of morphisms.

3.3.1. Let $(\mathcal{B},\omega)$ and $(\mathcal{B}',\omega')$ be objects in $\mathfrak{A}(\mathcal{C})$ and let $[\chi,\zeta]:(\mathcal{B},\omega)\longrightarrow(\mathcal{B}',\omega')$ be a morphism in $\mathfrak{A}(\mathcal{C})$. Then $\omega:\mathcal{B}\longrightarrow\mathcal{A}$, $\omega':\mathcal{B}'\longrightarrow\mathcal{A}$, and $\chi:\mathcal{B}\longrightarrow\mathcal{B}'$ are $\mathcal{C}$-functors, and $\zeta:\omega\cong\omega'\chi$ is a $\mathcal{C}$-isomorphism for the diagram:

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\chi"] \arrow[dr, "\omega"'] & & \mathcal{B}' \arrow[dl, "\omega'"] \\ & \mathcal{A}. & \end{tikzcd}$$

Let $\delta:\omega\longrightarrow\omega\otimes C$ and $\partial:\omega'\longrightarrow\omega'\otimes C'$ be universal $\mathcal{C}$-morphisms. Since $\zeta:\omega\longrightarrow\omega'\chi$ is a $\mathcal{C}$-isomorphism there is a unique morphism $\mathrm{coend}_{\mathcal{C}}([\chi,\zeta])=z:C\longrightarrow C'$ such that

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[d, "\zeta"'] & \omega\otimes C \arrow[d, "\zeta\otimes z"] \\ \omega'\chi \arrow[r, "\partial\chi"] & \omega'\chi\otimes C' \end{tikzcd}$$

commutes. If (χ,ζ) and (χ',ζ') are equivalent (representatives of $[\chi,\zeta]$) by $\varphi:\chi\longrightarrow\chi'$ with $\zeta'=\omega'\varphi\circ\zeta$ then the diagram

$$\begin{tikzcd} \omega \arrow[rrr, "\delta"] \arrow[dr, "\zeta"] \arrow[dd, "\zeta'"'] & & & \omega\otimes C \arrow[dl, "\zeta\otimes z"'] \arrow[dd, "\zeta'\otimes z'"] \\ & \omega'\chi \arrow[r, "\partial\chi"] \arrow[dl, "\omega'\varphi"] & \omega'\chi\otimes C' \arrow[dr, "\omega'\varphi\otimes 1"'] & \\ \omega'\chi' \arrow[rrr, "\partial\chi'"] & & & \omega'\chi'\otimes C' \end{tikzcd}$$

commutes. By the uniqueness of the induced morphism from C to C' we get $z=z'$, hence z is uniquely defined by the class $[\chi,\zeta]$.

It is easy to see that z is a coalgebra morphism.

Observe that $\omega(P)\xrightarrow{\delta}\omega(P)\otimes C\xrightarrow{1\otimes z}\omega(P)\otimes C'$ defines a C'-coaction on $\omega(P)$ for every $P\in\mathcal{B}$.

3.3.2. Let $\mathcal{A}$ be a $\mathcal{C}$-braided $\mathcal{C}$-monoidal category. Let $[\chi,\zeta] : (\mathcal{B},\omega) \longrightarrow (\mathcal{B}',\omega')$ and δ and ∂ be as before. Furthermore let $\mathcal{B}'$ be a $\mathcal{C}$-monoidal category and $\omega' : \mathcal{B}' \longrightarrow \mathcal{A}$ be a $\mathcal{C}$-monoidal functor. Assume that $\mathrm{Nat}_{\mathcal{C}}(\omega',\omega'\otimes -)$ is multirepresentable with the universal morphism $\partial : \omega' \longrightarrow \omega' \otimes B$. By Proposition 3.6 B is a bialgebra.

We call $\mathrm{Nat}_{\mathcal{C}}(\omega,\omega\otimes -) : \mathcal{B} \longrightarrow \mathrm{Set}$ *ω'-representable* if there is an object $\bar{C} \in \mathcal{B}'$ and a $\mathcal{C}$-morphism $d : \chi \longrightarrow \chi \otimes \bar{C}$ such that the induced morphism $y : C \longrightarrow \omega'\bar{C}$ in the commutative diagram

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\zeta"'] & \omega \otimes C \arrow[r, "1\otimes y"] & \omega \otimes \omega'\bar{C} \arrow[d, "\zeta\otimes 1"] \\
\omega'\chi \arrow[r, "\omega' d"] & \omega'(\chi\otimes\bar{C}) \arrow[r, "\upsilon"] & \omega'\chi \otimes \omega'\bar{C}
\end{tikzcd}$$

is an isomorphism. Equivalently the morphism

$$\omega \xrightarrow{\zeta} \omega'\chi \xrightarrow{\omega' d} \omega'(\chi\otimes\bar{C}) \xrightarrow{\upsilon} \omega'\chi\otimes\omega'\bar{C} \xrightarrow{\zeta^{-1}\otimes 1} \omega\otimes\omega'\bar{C}$$

is a universal $\mathcal{C}$-morphism.

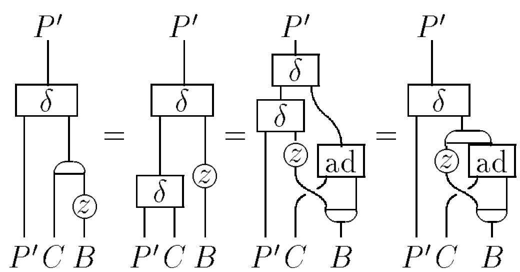
3.3.3. Observe that every morphism f in $\mathcal{B}$ induces a C-comodule morphism $\omega(f)$ in $\mathcal{A}$, and that every morphism g in $\mathcal{B}'$ induces a B-comodule morphism $\omega'(g)$ in $\mathcal{A}$. In particular $\omega'(d) \cong \delta$ is a B-comodule morphism in $\mathcal{A}$. Furthermore ζ is a B-comodule isomorphism with the B-comodule structure on ω as defined in 3.3.1.

Let $\mathrm{ad} : C \longrightarrow C \otimes B$ denote the coaction

$$C \xrightarrow{y} \omega'(\bar{C}) \xrightarrow{\partial(\bar{C})} \omega'(\bar{C}) \otimes B \xrightarrow{y^{-1}\otimes B} C \otimes B.$$

Proposition 3.7. *Under the conditions of* 3.3.2 $[\chi,\zeta]$ *induces a coalgebra morphism* $z : C \longrightarrow B$ *and the coaction of* B *on* C *is a right coadjoint coaction.*

Proof. The induced coalgebra morphism was defined in 3.3.1. For every $P \in \mathcal{B}$ and $P' := \omega(P)$ we get



where the first and last equalities arise from the coassociativity of the cooperation δ and the middle equality is the fact that (the left lower resp. right upper) δ is a B-comodule morphism, the coaction of B on P' as in 3.3.1 and on $P'\otimes C$ via the multiplication of B. Since the natural transformation $\omega'\chi \xrightarrow{\delta} \omega'\chi\otimes C \longrightarrow \omega'\chi\otimes C\otimes B$ given by the above graphic diagram induces precisely one morphism $C \longrightarrow C\otimes B$ we get (2.5.2) that $\mathrm{ad} : C \longrightarrow C\otimes B$ is a right coadjoint coaction.

Furthermore we know that $C \cong \omega'(\bar{C})$ is a B-comodule. $\square$

The previous Proposition says in particular that C with the given (coadjoint) B-comodule structure can be lifted along $\omega' : \mathcal{B}' \longrightarrow \mathcal{A}$.

Corollary 3.8. *Under the conditions of Proposition 3.7 if B is a Hopf algebra in $\mathcal{A}$ then C is a B-comodule coalgebra under the right coadjoint coaction defined by $z : C \longrightarrow B$.*

Proof. By Lemma 2.13 $\mathrm{ad} : C \longrightarrow C \otimes B$ is the uniquely defined coadjoint coaction which by Proposition 2.12 makes C a B-comodule coalgebra. $\square$

3.3.4. If $\mathrm{coend}_{\mathcal{C}}(\omega)$ exists for all objects $(\mathcal{B}, \omega)$ in $\mathfrak{A}(\mathcal{C})$ then we have a functor $\mathrm{coend}_{\mathcal{C}} : \mathfrak{A}(\mathcal{C}) \longrightarrow \mathcal{A}$-coalg. If furthermore all full comodule categories $\mathcal{A}^C$ together with the forgetful functor $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ are objects in $\mathfrak{A}(\mathcal{C})$ then $\mathrm{coend}_{\mathcal{C}} : \mathfrak{A}(\mathcal{C}) \longrightarrow \mathcal{A}$-coalg is left adjoint to $\mathcal{A}^{-} : \mathcal{A}$-coalg $\longrightarrow \mathfrak{A}(\mathcal{C})$ (from 3.1.1).

3.4. Applications.

3.4.1. We specialize $\mathcal{C}$ to the case of a one-element category. We call this case *full reconstruction*. Let $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ be a functor. Let $\delta : \omega \longrightarrow \omega \otimes C$ be a universal morphism. Then the propositions of this section specialize to:

If $\mathrm{Nat}(\omega, \omega \otimes -)$ is representable, then the representing object $C = \mathrm{coend}(\omega)$ is a coalgebra in $\mathcal{A}$. This coalgebra is uniquely determined up to isomorphisms of coalgebras.

Furthermore every object $\omega(P) \in \mathcal{A}$ with $P \in \mathcal{B}$ is a C-comodule via $\delta : \omega(P) \longrightarrow \omega(P) \otimes C$ and every morphism $\omega(f)$ is a C-comodule morphism.

Let $\mathcal{A}$ be braided monoidal, $\mathcal{B}$ be monoidal and $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ be a monoidal functor. If $\mathrm{Nat}(\omega, \omega \otimes -)$ is multirepresentable, then $B = \mathrm{coend}(\omega)$ is a bialgebra in $\mathcal{A}$. This bialgebra is uniquely determined up to isomorphisms of bialgebras.

If in addition $\mathcal{B}$ is braided, then $\mathrm{coend}(\omega)$ is coquasitriangular in $\mathcal{A}$.

If $\mathcal{B}$ is rigid then $\mathrm{coend}(\omega)$ is a Hopf algebra in $\mathcal{A}$.

Furthermore for any objects $P, Q \in \mathcal{B}$ the B-comodule structure on $\omega(P) \otimes \omega(Q)$ is defined by the multiplication on $\mathrm{coend}(\omega)$.

3.4.2. We specialize $\mathcal{C} = \mathcal{A}_0$ with $\mathcal{A}$ a braided monoidal category and $\mathcal{A}_0$ a full (braided) monoidal subcategory. We will call this case *restricted reconstruction*. Let $\omega : \mathcal{B} \longrightarrow \mathcal{A}_0$ be an $\mathcal{A}_0$-functor. Let $\delta : \omega \longrightarrow \omega \otimes C$ be a universal $\mathcal{A}_0$-morphism. Then the propositions of this section specialize to:

If $\mathrm{Nat}_{\mathcal{A}_0}(\omega, \omega \otimes -)$ is representable, then the representing object $C = \mathrm{coend}_{\mathcal{A}_0}(\omega)$ is a coalgebra in $\mathcal{A}$. This coalgebra is uniquely determined up to isomorphisms of coalgebras.

Furthermore every object $\omega(P) \in \mathcal{A}_0$ with $P \in \mathcal{B}$ is a C-comodule via $\delta : \omega(P) \longrightarrow \omega(P) \otimes C$ and every morphism $\omega(f)$ is a C-comodule morphism.

Let $\mathcal{B}$ be $\mathcal{A}_0$-monoidal and $\omega : \mathcal{B} \longrightarrow \mathcal{A}_0$ be an $\mathcal{A}_0$-monoidal functor. If $\mathrm{Nat}_{\mathcal{A}_0}(\omega, \omega \otimes -)$ is multirepresentable, then $B = \mathrm{coend}_{\mathcal{A}_0}(\omega)$ is a bialgebra in $\mathcal{A}$. This bialgebra is uniquely determined up to isomorphisms of bialgebras.

If in addition $\mathcal{B}$ is $\mathcal{A}_0$-braided, then $\mathrm{coend}_{\mathcal{A}_0}(\omega)$ is coquasitriangular in $\mathcal{A}$.

If $\mathcal{B}$ is rigid then $\mathrm{coend}_{\mathcal{A}_0}(\omega)$ is a Hopf algebra in $\mathcal{A}$.

Furthermore for any objects $P, Q \in \mathcal{B}$ the B-comodule structure on $\omega(P) \otimes \omega(Q)$ is defined by the multiplication on $\mathrm{coend}_{\mathcal{A}_0}(\omega)$.

4. Existence theorems in reconstruction theory

4.1. Restricted reconstruction. In this section we will study reconstruction of a given coalgebra C with help of the functor $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes -)$. We call this *restricted reconstruction.*

For this purpose let $\mathcal{C}$ be a braided monoidal category and $\mathcal{C}_0$ be a full (braided) monoidal subcategory of $\mathcal{C}$. For a Hopf algebra H in $\mathcal{C}$, a coalgebra C in $\mathcal{C}$ and a coalgebra morphism $z : C \longrightarrow H$ we know that C is a coalgebra in $\mathcal{C}^H$ with respect to the right coadjoint coaction $\mathrm{ad} : C \longrightarrow C \otimes H$ by Proposition 2.12. If C is a $\mathcal{C}_0$-generated coalgebra then the subcoalgebras C_i are also in $\mathcal{C}^H$ (actually in $\mathcal{C}_0^H$) by the coadjoint action. We consider the underlying functor $\omega : \mathcal{C}_0^C \longrightarrow \mathcal{C}^H$.

Theorem 4.1. *Let C be a $\mathcal{C}_0$-generated coalgebra in $\mathcal{C}$ and H be a Hopf algebra in $\mathcal{C}$. Let $z : C \longrightarrow H$ be a coalgebra morphism. Let $\omega := \mathcal{C}_0^z : \mathcal{C}_0^C \longrightarrow \mathcal{C}_0^H \subseteq \mathcal{C}^H$ be the functor induced by z. Then*

$$\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes -) : \mathcal{C}^H \longrightarrow \mathrm{Set}$$

is representable by the coalgebra $C = \mathrm{coend}_{\mathcal{C}_0}(\omega)$ in $\mathcal{C}^H$ with the canonical morphism $\delta : \omega \longrightarrow \omega \otimes C$.

Proof. We define maps

$$\Sigma : \mathcal{C}^H(C, M) \longrightarrow \mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes M)$$

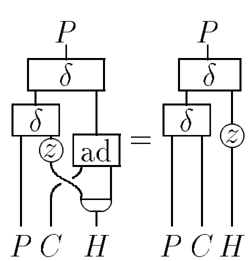
and

$$\Pi : \mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes M) \longrightarrow \mathcal{C}^H(C, M).$$

The first map is defined by $\Sigma(f)(P, \delta_P) := (1_P \otimes f)\delta_P : P \longrightarrow P \otimes C \longrightarrow P \otimes M$. Then $\Sigma(f)(P, \delta_P) : P \longrightarrow P \otimes M$ is an H-comodule morphism since the following diagram commutes

$$\begin{tikzcd}
P \arrow[r, "\delta_P"] \arrow[dd, "\delta_P"'] & P \otimes C \arrow[r, "1\otimes f"] \arrow[d, "\delta_P\otimes \mathrm{ad}"'] & P \otimes M \arrow[d, "\delta_P\otimes\delta_M"] \\
 & P \otimes C \otimes C \otimes H \arrow[r, "1\otimes 1\otimes f\otimes 1"] \arrow[d, "1\otimes z\otimes 1\otimes 1"'] & P \otimes C \otimes M \otimes H \arrow[d, "1\otimes z\otimes 1\otimes 1"] \\
P \otimes C \arrow[dd, "1\otimes z"'] & P \otimes H \otimes C \otimes H \arrow[r, "1\otimes 1\otimes f\otimes 1"] \arrow[d, "1\otimes\sigma\otimes 1"'] & P \otimes H \otimes M \otimes H \arrow[d, "1\otimes\sigma\otimes 1"] \\
 & P \otimes C \otimes H \otimes H \arrow[r, "1\otimes f\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes m_H"'] & P \otimes M \otimes H \otimes H \arrow[d, "1\otimes 1\otimes m_H"] \\
P \otimes H \arrow[r, "\delta_P\otimes 1"] & P \otimes C \otimes H \arrow[r, "1\otimes f\otimes 1"] & P \otimes M \otimes H
\end{tikzcd}$$

where the left hand side commutes by



which follows from Lemma 2.13. Hence $\Sigma(f) : \omega \longrightarrow \omega \otimes M$ is a natural transformation. Furthermore we have $\Sigma(f)(X \otimes (P, \delta_P)) = (1_{X \otimes P} \otimes f)\delta_{X \otimes P} = (1_X \otimes 1_P \otimes f)(1_X \otimes \delta_P) = 1_X \otimes \Sigma(f)(P, \delta_P)$, so $\Sigma(f)$ is a $\mathcal{C}_0$-morphism.

To define the map Π let $\varphi : \omega \longrightarrow \omega \otimes M$ be given. Define $\varinjlim \varphi(C_i) = \varphi(C) : C \longrightarrow C \otimes M$ as the uniquely determined morphism so that

$$\begin{array}{ccc} C_i & \xrightarrow{\varphi(C_i)} & C_i \otimes M \\ \downarrow{\scriptstyle \iota_i} & & \downarrow{\scriptstyle \iota_i \otimes 1_M} \\ C & \xrightarrow{\varphi(C)} & C \otimes M \end{array}$$

commutes and let

$$\Pi(\varphi) := (\epsilon \otimes 1_M)\varphi(C) : C \longrightarrow C \otimes M \longrightarrow M.$$

Since C_i is a subcoalgebra of C it is an H-subcomodule by the coadjoint coaction induced by the coalgebra morphism $z\iota_i : C_i \longrightarrow H$. Hence, C is a colimit of H-comodule coalgebras and $\varphi(C) : C \longrightarrow C \otimes M$ is an H-comodule morphism. Consequently $\Pi(\varphi)$ is in $\mathcal{C}^H$.

We have $\Pi\Sigma(f) = (\epsilon\otimes 1_M)\varinjlim((1_{C_i}\otimes f)\delta_i) = (\epsilon\otimes 1_M)(1_C\otimes f)\Delta = f(\epsilon\otimes 1_C)\Delta_C = f$.

Now observe that $\delta_P : P \longrightarrow P \otimes C$ is a C-comodule morphism with the C-structure on $P \otimes C$ coming from the one of C. Thus we get $(1 \otimes \delta_j)\delta_{P,j} = (\delta_{P,j} \otimes 1_C)\delta_P$ for the morphism $\delta_{P,j} : P \longrightarrow P \otimes C_j$ which exists by the assumptions about a $\mathcal{C}_0$-generated coalgebra. So $\delta_{P,j}$ is a C-comodule morphism as well, hence $\varphi(P \otimes C_j, 1_P \otimes \delta_j)\delta_{P,j} = (\delta_{P,j} \otimes 1_M)\varphi(P, \delta_P)$. From this we get

$$\begin{aligned} \Sigma\Pi(\varphi)(P, \delta_P) &= (1_P \otimes (\epsilon \otimes 1_M)\varphi(C))\delta_P \\ &= (1_P \otimes \epsilon \otimes 1_M)(1_P \otimes \varinjlim \varphi(C_i))(1_P \otimes \iota_j)\delta_{P,j} \\ &= (1_P \otimes \epsilon \otimes 1_M)(1_P \otimes \iota_j \otimes 1_M)(1_P \otimes \varphi(C_j))\delta_{P,j} \\ &= (1_P \otimes \epsilon \otimes 1_M)(1_P \otimes \iota_j \otimes 1_M)\varphi(P \otimes C_j)\delta_{P,j} \\ &= (1_P \otimes \epsilon \otimes 1_M)(1_P \otimes \iota_j \otimes 1_M)(\delta_{P,j} \otimes 1_M)\varphi(P) \\ &= \varphi(P, \delta_P). \end{aligned}$$

So Σ and Π are inverses of each other. The claim about the coalgebra structure is clear from the uniqueness of the reconstructed coalgebra. $\square$

4.1.1. Observe that in general the set of *all* natural transformations – not just the $\mathcal{C}$-morphisms – from ω to $\omega \otimes M$ is too large for the reconstruction of C as we have seen in 2.1.10. By Proposition 6.4, however, this difference does not occur if the base category $\mathcal{C}$ is the category of vector spaces Vec over $\mathbb{K}$. This explains the usual *full reconstruction* with the functor $\mathrm{Nat}(\omega, \omega \otimes -)$ like in [DM82, Ul89, Mj94a].

Examples of natural transformations which are not $\mathcal{C}$-morphisms can be derived from 2.1.10.

Now assume that $\mathcal{C}$ is a cocomplete braided monoidal category and $\mathcal{C}_0$ is a (braided) full monoidal subcategory of $\mathcal{C}$. Furthermore assume that the tensor product in $\mathcal{C}$ preserves arbitrary colimits in both variables.

Theorem 4.2. *Let B be a $\mathcal{C}_0$-generated coalgebra in $\mathcal{C}$ and a $\mathcal{C}_0$-central bialgebra, let H be a braided $\mathcal{C}_0$-central Hopf algebra in $\mathcal{C}$. Let $z : B \longrightarrow H$ be a bialgebra morphism. Let $\omega := \mathcal{C}_0^z : \mathcal{C}_0^B \longrightarrow \mathcal{C}_0^H \subseteq \mathcal{C}^H$ be the functor induced by z. Then* $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes -) : \mathcal{C}^H \longrightarrow \mathrm{Set}$ *is multirepresentable by the bialgebra* $\mathrm{coend}_{\mathcal{C}_0}(\omega)$ *in $\mathcal{C}^H$ which is equal to B as an H-comodule coalgebra under the coadjoint coaction, but carries a different multiplicative structure $\widetilde{m}_B : B \otimes B \longrightarrow B$, the transmuted multiplication.*

Proof. We extend the proof of 4.1 to ω^2. Define maps

$$\Sigma : \mathcal{C}^H(B \otimes B, M) \longrightarrow \mathrm{Nat}_{\mathcal{C}_0}(\omega \otimes \omega, \omega \otimes \omega \otimes M)$$

and

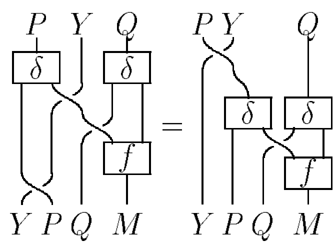
$$\Pi : \mathrm{Nat}_{\mathcal{C}_0}(\omega \otimes \omega, \omega \otimes \omega \otimes M) \longrightarrow \mathcal{C}^H(B \otimes B, M).$$

The first map is given by $\Sigma(f)((P, \delta_P), (Q, \delta_Q)) := (1_{P \otimes Q} \otimes f)(1_P \otimes \sigma_{B,Q} \otimes 1_B)(\delta_P \otimes \delta_Q) : P \otimes Q \longrightarrow P \otimes B \otimes Q \otimes B \longrightarrow P \otimes Q \otimes B \otimes B \longrightarrow P \otimes Q \otimes M$. Then $\Sigma(f)((P, \delta_P), (Q, \delta_Q)) : P \otimes Q \longrightarrow P \otimes Q \otimes M$ is an H-comodule morphism by a similar proof as in 4.1 replacing P by $P \otimes Q$. Hence $\Sigma(f) : \omega \otimes \omega \longrightarrow \omega \otimes \omega \otimes M$ is a natural transformation. To show that $\Sigma(f)$ is a $\mathcal{C}_0$-morphism we observe that $\xi : \omega(X \otimes P) \longrightarrow X \otimes \omega(P)$ is the identity. So we have

$$\begin{aligned}
&\Sigma(f)(X \otimes (P, \delta_P), (Q, \delta_Q)) \\
&\quad = (1_X \otimes 1_P \otimes 1_Q \otimes f)(1_X \otimes 1_P \otimes \sigma_{B,Q} \otimes 1_B)(1_X \otimes \delta_P \otimes \delta_Q) \\
&\quad = 1_X \otimes (1_P \otimes 1_Q \otimes f)(1_P \otimes \sigma_{B,Q} \otimes 1_B)(\delta_P \otimes \delta_Q) \\
&\quad = 1_X \otimes \Sigma(f)((P, \delta_P), (Q, \delta_Q)),
\end{aligned}$$

$$\begin{aligned}
&(\sigma_{P,Y} \otimes 1_{Q \otimes M})\Sigma(f)((P, \delta_P), Y \otimes (Q, \delta_Q)) \\
&\quad = (\sigma_{P,Y} \otimes 1_{Q \otimes M})(1_{P \otimes Y \otimes Q} \otimes f)(1_P \otimes \sigma_{B,Y \otimes Q} \otimes 1_B)(\delta_P \otimes \delta_{Y \otimes Q}) \\
&\quad = (1_{Y \otimes P \otimes Q} \otimes f)(1_{Y \otimes P} \otimes \sigma_{B,Q} \otimes 1_B)(1_Y \otimes \delta_P \otimes \delta_Q)(\sigma_{P,Y} \otimes 1_Q) \\
&\quad = (1_Y \otimes \Sigma(f)((P, \delta_P), (Q, \delta_Q)))(\sigma_{P,Y} \otimes 1_Q)
\end{aligned}$$

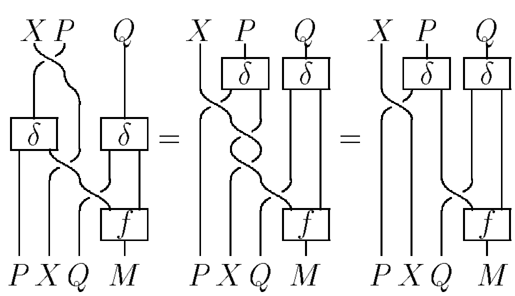
or as a braid diagram



and

$$\begin{aligned}
&\Sigma(f)((P, \delta_P), Y \otimes (Q, \delta_Q))(\sigma_{Y,P} \otimes 1_Q) \\
&\quad = (1_{P \otimes Y \otimes Q} \otimes f)(1_P \otimes \sigma_{B,Y \otimes Q} \otimes 1_B)(\delta_P \otimes \delta_{Y \otimes Q})(\sigma_{Y,P} \otimes 1_Q) \\
&\quad = (1_{P \otimes Y \otimes Q} \otimes f)(1_P \otimes \sigma_{B,Y \otimes Q} \otimes 1_B)(\sigma_{Y,P \otimes B} \otimes 1_{Q \otimes B})(1_Y \otimes \delta_P \otimes \delta_Q) \\
&\quad = (1_{P \otimes Y \otimes Q} \otimes f)(1_{P \otimes Y} \otimes \sigma_{B,Q} \otimes 1_B)(1_P \otimes (\sigma_{B,Y}\sigma_{Y,B} \otimes 1_{Q \otimes B})) \\
&\qquad\quad (\sigma_{Y,P} \otimes 1_{B \otimes Q \otimes B})(1_Y \otimes \delta_P \otimes \delta_Q) \\
&\quad = (1_{P \otimes Y \otimes Q} \otimes f)(1_{P \otimes Y} \otimes \sigma_{B,Q} \otimes 1_B)(\sigma_{Y,P} \otimes 1_{B \otimes Q \otimes B})(1_Y \otimes \delta_P \otimes \delta_Q) \\
&\quad = (1_{P \otimes Y \otimes Q} \otimes f)(\sigma_{Y,P} \otimes \sigma_{B,Q} \otimes 1_B)(1_Y \otimes \delta_P \otimes \delta_Q) \\
&\quad = (\sigma_{Y,P} \otimes 1_{Q \otimes M})(1_{Y \otimes P \otimes Q} \otimes f)(1_{Y \otimes P}\sigma_{B,Q} \otimes 1_B)(1_Y \otimes \delta_P \otimes \delta_Q) \\
&\quad = (\sigma_{Y,P} \otimes 1_{Q \otimes M})(1_Y \otimes \Sigma(f)((P, \delta_P), (Q, \delta_Q))
\end{aligned}$$

or as a braid diagram



since B is $\mathcal{C}_0$-central in $\mathcal{C}$. Thus $\Sigma(f)$ is a $\mathcal{C}_0$-morphism.

Now we construct the second map Π. Let $\varphi : \omega \otimes \omega \longrightarrow \omega \otimes \omega \otimes M$. Define $\varinjlim \varphi(B_i, B_j) = \varphi(B, B) : B \otimes B \longrightarrow B \otimes B \otimes M$ as the uniquely determined morphism so that

$$\begin{tikzcd} B_i \otimes B_j \arrow[r, "\varphi(B_i,B_j)"] \arrow[d, "\iota_i \otimes \iota_j"'] & B_i \otimes B_j \otimes M \arrow[d, "\iota_i \otimes \iota_j \otimes 1_M"] \\ B \otimes B \arrow[r, "\varphi(B,B)"] & B \otimes B \otimes M \end{tikzcd}$$

commutes. Observe that $B \otimes B$ is the colimit of the $B_i \otimes B_j$ since by assumption the tensor product preserves colimits. Let

$$\Pi(\varphi) := (\epsilon \otimes \epsilon \otimes 1_M)\varphi(B, B) : B \otimes B \longrightarrow B \otimes B \otimes M \longrightarrow M.$$

As in the proof of 4.1 $\Pi(\varphi)$ is an H-comodule morphism.

We have $\Pi\Sigma(f) = (\epsilon \otimes \epsilon \otimes 1_M)(1_{B\otimes B} \otimes f)(1_B \otimes \sigma_{B,B} \otimes 1_B)(\Delta_B \otimes \Delta_B) = f(\epsilon \otimes \epsilon \otimes 1_{B\otimes B})\Delta_{B\otimes B} = f$. Observe that $\delta_P : P \longrightarrow P \otimes B$ is a B-comodule morphism with the B-structure on $P \otimes B$ coming from the one of B. So we have

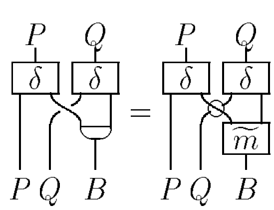
$$\begin{aligned}
&\Sigma\Pi(\varphi)((P, \delta_P), (Q, \delta_Q)) \\
&\quad = (1_{P\otimes Q} \otimes \epsilon \otimes \epsilon \otimes 1_M)(1_{P\otimes Q} \otimes \varphi(B, B))(1_P \otimes \sigma_{B,Q} \otimes 1_B)(\delta_P \otimes \delta_Q) \\
&\quad = (1_{P\otimes Q} \otimes \epsilon \otimes \epsilon \otimes 1_M)(1_{P\otimes Q} \otimes \varinjlim \varphi(B_k, B_l)) \\
&\qquad\quad (1_P \otimes \sigma_{B,Q} \otimes 1_B)(\delta_P \otimes \delta_Q) \\
&\quad = (1_{P\otimes Q} \otimes \epsilon \otimes \epsilon \otimes 1_M) \varinjlim[(1_{P\otimes Q} \otimes \varphi(B_k, B_l))(1_P \otimes \sigma_{B_k,Q} \otimes 1_{B_l})] \\
&\qquad\quad (1_P \otimes \iota_i \otimes 1_Q \otimes \iota_j)(\delta_{P,i} \otimes \delta_{Q,j}) \\
&\quad = (1_{P\otimes Q} \otimes \epsilon \otimes \epsilon \otimes 1_M)(1_{P\otimes Q} \otimes \iota_i \otimes \iota_j \otimes 1_M) \\
&\qquad\quad (1_{P\otimes Q} \otimes \varphi(B_i, B_j))(1_P \otimes \sigma_{B_i,Q} \otimes 1_{B_j})](\delta_{P,i} \otimes \delta_{Q,j}) \\
&\quad = (1_{P\otimes Q} \otimes \epsilon\iota_i \otimes \epsilon\iota_j \otimes 1_M)(1_P \otimes \sigma_{B_i,Q} \otimes 1_{B_j} \otimes 1_M) \\
&\qquad\quad (1_P \otimes \varphi(B_i, Q \otimes B_j))(\delta_{P,i} \otimes \delta_{Q,j}) \\
&\quad = (1_P \otimes \epsilon\iota_i \otimes 1_Q \otimes \epsilon\iota_j \otimes 1_M)\varphi(P \otimes B_i, Q \otimes B_j)(\delta_{P,i} \otimes \delta_{Q,j}) \\
&\quad = (1_P \otimes \epsilon\iota_i \otimes 1_Q \otimes \epsilon\iota_j \otimes 1_M)(\delta_{P,i} \otimes \delta_{Q,j} \otimes 1_M)\varphi(P, Q) \\
&\quad = \varphi((P, \delta_P), (Q, \delta_Q)).
\end{aligned}$$

Thus Σ and Π are inverses of each other.

The multiplicative structure $\widetilde{m}_B$ of B in $\mathcal{C}^H$ is now the uniquely determined morphism which makes the diagram

$$\begin{tikzcd} P \otimes Q \arrow[r, "\delta_2"] \arrow[d, "1_{P\otimes Q}"'] & P \otimes Q \otimes B \otimes B \arrow[d, "1_{P\otimes Q} \otimes \widetilde{m}_B"] \\ P \otimes Q \arrow[r, "\delta_{P\otimes Q}"] & P \otimes Q \otimes B \end{tikzcd}$$

commutative, since $\omega \otimes \omega = \omega(\text{-}\otimes\text{-}) \xrightarrow{\delta} \omega(\text{-}\otimes\text{-}) \otimes B$ is a $\mathcal{C}_0$-bimorphism of bifunctors. The unit is given by $\lambda : I \longrightarrow I \otimes B = B$. The new multiplication can be described by the braid diagram



where the braid morphism on the left is the one in $\mathcal{C}$ and the braid morphism one the right is the one in $\mathcal{C}^H$.

The proof shows that the universal morphism is $(1_P \otimes \sigma_{B,Q} \otimes 1_Q)(\delta_P \otimes \delta_Q) : \omega\otimes\omega \longrightarrow \omega\otimes\omega\otimes B\otimes B$. An analogous result holds for multifunctors $\omega\otimes\ldots\otimes\omega = \omega^n$ and $\mathcal{C}_0$-morphisms $\varphi : \omega^n \longrightarrow \omega^n \otimes M$. This proves that with a suitable element $\tau \in B_n$ in the Artin braid group the morphism

$$\delta_n := \tau\delta^n : \omega^n \longrightarrow \omega^n \otimes B^n$$

is the universal $\mathcal{C}_0$-morphism for all $n \in \mathbb{N}$, in particular $\mathrm{coend}_{\mathcal{C}_0}(\omega^n) = B^n$. □

4.1.2. The proof of 4.1 resp. 4.2 provides a proof for the "representability assumption for modules" (in [Mj90] 3.2) in a very general setting for the functor $\mathrm{Nat}_{\mathcal{C}_0}(\omega^n, \omega^n \otimes -)$ instead of the functor $\mathrm{Nat}(\omega^n, \omega^n \otimes -)$. A special case of Theorem 4.2 is [Mj93a] Proposition A.4 if one uses $\mathcal{C} = \mathrm{Vec}$.

We have reason to make a few additional comments on the existing literature. In [Mj93b] (Theorem 2.2) the following theorem is given: *Let $\mathcal{C}$ be a monoidal category and $\mathcal{V}$ a rigid braided monoidal category cocomplete over $\mathcal{C}$. Let $F : \mathcal{C} \longrightarrow \mathcal{V}$ be a monoidal functor. Then there exists a $\mathcal{V}$-bialgebra, $A = \mathrm{Aut}(\mathcal{C}, F, \mathcal{V})$, such that F factorizes monoidally as $\mathcal{C} \longrightarrow {}^A\mathcal{V} \longrightarrow \mathcal{V}$. ... A is universal with this property. ...*

Although Majid considers this theorem "a significant generalization of usual Tannaka-Krein ideas" (hep-th 9504007) it does not even specialize to other known cases of Tannaka-Krein reconstruction such as [Ul90] Proposition 2.4. Observe that the formation of big colimits will usually destroy rigidity. In fact the full subcategory vec of rigid objects in Vec, that is the category of finite dimensional vector spaces, is only cocomplete over finite categories $\mathcal{C}$. There are no nontrivial infinite direct sums in this category.

This is why in reconstruction theorems one usually assumes that the rigid category $\mathcal{V}$ is a subcategory of a bigger cocomplete monoidal category in which the universal bialgebra A is then reconstructed as a colimit of a diagram in $\mathcal{V}$.

Furthermore the theorem quoted above only allows to reconstruct rigid bialgebras, e.g. finite dimensional bialgebras in the case $\mathcal{V} = \mathrm{vec}$. The proof of this theorem cannot just simply be extended to the more general case because the rigidity of the reconstructed object is explicitly used in the proof of the algebra property, in particular in [Mj93b] Lemma 2.3. A hypothesis on the tensor product of $\mathcal{V}$, namely that it preserves colimits, in order to obtain a bialgebra structure in the more general case, does not become clear in the vague discussion that follows p.205 of [Mj93b].

Theorems 4.1 and 4.2 of [Mj93b] again are only proved under the restriction of finite dimensionality. The discussion of the infinite dimensional situation is not satisfactory (see p. 209). In fact in the dual situation Vec^{op} one reconstructs essentially linearly compact algebras, which are also ordinary algebras, but the bialgebra structure does not live in Vec any more, due to the fact that tensor products do not preserve limits.

So our Theorem 4.2 together with Proposition 3.6 remedy these shortcomings. Through the use of control categories they are clearly much more general.

The following is a generalization of [Pa78], Corollary 6.4.

Corollary 4.3. *Let C be a $\mathcal{C}_0$-generated coalgebra in the braided monoidal category $\mathcal{C}$ and let $\omega : \mathcal{C}_0^C \longrightarrow \mathcal{C}$ be the forgetful functor. Then $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes -) : \mathcal{C} \longrightarrow \mathrm{Set}$ is representable and $C = \mathrm{coend}_{\mathcal{C}_0}(\omega)$ as coalgebras in $\mathcal{C}$.*

Proof. Use $H = \mathbb{K}$ in the above theorem. $\square$

Corollary 4.4. *Let* $\omega : \mathcal{C}_0 \longrightarrow \mathcal{C}$ *be the embedding functor. Then* $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes -) : \mathcal{C} \longrightarrow \mathrm{Set}$ *is representable and* $\mathrm{coend}_{\mathcal{C}_0}(\omega) = I$.

In particular $\mathrm{coend}_{\mathcal{C}}(\mathrm{Id}_{\mathcal{C}}) \cong I$ *for any monoidal category* $\mathcal{C}$.

Proof. Use $C = \mathbb{K}$ in the above Corollary. $\square$

As we remarked after the definition of $\mathcal{C}_0$-generated coalgebras, the condition that C is $\mathcal{C}_0$-generated becomes vacuous in the case $\mathcal{C}_0 = \mathcal{C}$.

Corollary 4.5. *Let B be a $\mathcal{C}_0$-central bialgebra in $\mathcal{C}$ which is $\mathcal{C}_0$-generated as a coalgebra, let $\mathcal{A} := \mathcal{C}_0^B$, and let $\omega : \mathcal{C}_0^B \longrightarrow \mathcal{C}$ be the forgetful functor. Then* $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes -)$ *is multirepresentable and* $\mathrm{coend}_{\mathcal{C}_0}(\omega) = B$ *as bialgebras.*

By dualization of Proposition 4.1 one gets

Corollary 4.6. *Let A be a $\mathcal{C}_0$-generated algebra in $\mathcal{C}$, and let $\omega : \mathcal{C}_{0A} \longrightarrow \mathcal{C}$ be the forgetful functor. Then* $\mathrm{Nat}_{\mathcal{C}_0}(\omega \otimes -, \omega)$ *is representable and* $A = \mathrm{end}_{\mathcal{C}_0}(\omega)$ *as algebras in* $\mathcal{C}$.

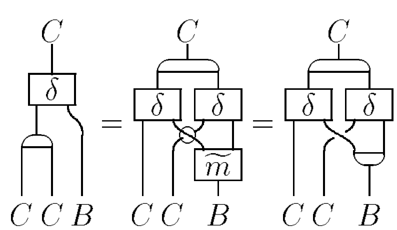
4.1.3. Later on we will need the notion of cosmash products in $\mathcal{A}$. So we define it here and show an important property. If B is a bialgebra and C is a B-comodule-coalgebra in $\mathcal{A}$ then we can construct a cosmash product $B\#^c C$ where the cosmash comultiplication is defined by

$$\Delta : B \otimes C \xrightarrow{\Delta_B \otimes \Delta_C} B \otimes B \otimes C \otimes C \xrightarrow{1 \otimes \delta'_C \otimes 1} B \otimes B \otimes C \otimes B \otimes C$$
$$\xrightarrow{1 \otimes \sigma \otimes 1} B \otimes C \otimes B \otimes B \otimes C \xrightarrow{1 \otimes m_B \otimes 1} B \otimes C \otimes B \otimes C.$$

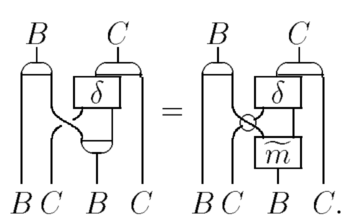
It is easy to see that $B\#^c C$ is a coalgebra in $\mathcal{A}$.

4.1.4. Let H be a braided Hopf algebra in $\mathcal{C}$, $z : B \longrightarrow H$ be a bialgebra morphism in $\mathcal{C}$ and $\widetilde{B}$ be the transmuted bialgebra in $\mathcal{C}^H$ (as in Theorem 4.2). Let C be a $\widetilde{B}$-comodule-coalgebra in $\mathcal{A} := \mathcal{C}^H$. Then we can form the cosmash product $\widetilde{B}\widetilde{\#}^c C$ in $\mathcal{C}^H$.

Since B is a bialgebra in $\mathcal{C}$ and C is a coalgebra in $\mathcal{C}$, C is also a B-comodule-coalgebra by



So there is a second way to define a cosmash product $B\#^c C$ this time in $\mathcal{C}$. These two coalgebra structures on $B \otimes C$, however, coincide as the following diagram shows



4.2. Finite reconstruction. In the previous section we started with an algebra A or a coalgebra C in $\mathcal{C}$ and reconstructed them from $\omega : \mathcal{C}_A \longrightarrow \mathcal{C}$ resp. $\omega : \mathcal{C}^C \longrightarrow \mathcal{C}$. If, however, an arbitrary $\mathcal{C}$-functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ is given it is not clear if $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ is a representable functor or if $\mathrm{coend}_{\mathcal{C}}(\omega)$ exists in $\mathcal{C}$. It is customary to call the construction of $\mathrm{coend}_{\mathcal{C}}(\omega)$ also in this situation "re"construction, although we do not start with an algebra or a coalgebra in $\mathcal{C}$ and then reconstruct is from its category of representations.

In one particular situation the (restricted) reconstruction is possible and well known, namely in the case of $\mathcal{C} = \mathrm{Vec}$ and a functor $\omega : \mathcal{B} \longrightarrow \mathrm{vec} \subseteq \mathrm{Vec}$ into the category of finite-dimensional vector spaces. Various generalizations of this result are known. We will lift this result to braided monoidal categories $\mathcal{C}$.

Let $\mathcal{C}$ be a cocomplete braided monoidal category and $\mathcal{C}_0$ be the full (rigid braided monoidal) subcategory of rigid objects. Furthermore assume that the tensor product in $\mathcal{C}$ preserves arbitrary colimits in both variables. Let $\mathcal{B}$ be a small category and $\omega : \mathcal{B} \longrightarrow \mathcal{C}_0 \subseteq \mathcal{C}$ be a functor.

Theorem 4.7. (1) *The functor* $\mathrm{Nat}(\omega, \omega \otimes M)$ *is multirepresentable:*

$$\mathrm{Nat}(\omega, \omega \otimes M) \cong \mathcal{C}(\mathrm{coend}(\omega), M),$$

(2) *If $\mathcal{B}$ is a $\mathcal{C}_0$-category and $\omega : \mathcal{B} \longrightarrow \mathcal{C}_0$ is a $\mathcal{C}_0$-functor, then the functor* $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes M)$ *is representable:*

$$\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes M) \cong \mathcal{C}(\mathrm{coend}_{\mathcal{C}_0}(\omega), M).$$

If $B := \mathrm{coend}_{\mathcal{C}_0}(\omega)$ *is $\mathcal{C}_0$-central then* $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes M)$ *is multirepresentable.*

Proof. (1) The existence of a representing object for the functor $\mathrm{Nat}(\omega, \omega \otimes M)$ is well known (see [Pa93]). We recall the main steps of its construction, since they play a role in the second part of the proof. The representing object $C = \mathrm{coend}(\omega)$ is obtained as colimit of the diagram consisting of all wedges for all morphisms $f : P \longrightarrow Q$ in $\mathcal{B}$:

$$\begin{tikzcd}[column sep=small] & \omega(P)^* \otimes \omega(P) \\ \omega(Q)^* \otimes \omega(P) \arrow[ur, "\omega(f)^* \otimes 1"] \arrow[dr, "1 \otimes \omega(f)"'] & \\ & \omega(Q)^* \otimes \omega(Q) \end{tikzcd}$$

Any (cone-) morphism from this diagram to an object $M \in \mathcal{C}$

$$\begin{tikzcd}[column sep=small] & \omega(P)^* \otimes \omega(P) \arrow[dr, "\psi(P)"] & \\ \omega(Q)^* \otimes \omega(P) \arrow[ur, "\omega(f)^* \otimes 1"] \arrow[dr, "1 \otimes \omega(f)"'] & & M \\ & \omega(Q)^* \otimes \omega(Q) \arrow[ur, "\psi(Q)"'] & \end{tikzcd} \tag{1}$$

is given by a natural transformation $\varphi : \omega \longrightarrow \omega \otimes M$:

$$\begin{tikzcd} \omega(P) \arrow[r, "\varphi(P)"] \arrow[d, "\omega(f)"] & \omega(P) \otimes M \arrow[d, "\omega(f) \otimes 1_M"] \\ \omega(Q) \arrow[r, "\varphi(Q)"] & \omega(Q) \otimes M \end{tikzcd}$$

The one-to-one correspondence between morphisms ψ and morphisms φ is given, using the evaluation $\mathrm{ev}_{\omega(P)} : \omega(P)^* \otimes \omega(P) \longrightarrow I$ and the dual basis $\mathrm{db}_{\omega(P)} : I \longrightarrow \omega(P) \otimes \omega(P)^*$, as

$$\psi(P) = (\mathrm{ev}_{\omega(P)} \otimes 1_M)(1_{\omega(P)^*} \otimes \varphi(P))$$

and

$$\varphi(P) := (1_{\omega(P)} \otimes \psi(P))(\mathrm{db}_{\omega(P)} \otimes 1_{\omega(P)}).$$

Given ψ the morphisms $\varphi(P)$ form a natural transformation since

$$\begin{tikzcd}
\omega(P) \arrow[rr, "\mathrm{db}_{\omega(P)}\otimes 1_{\omega(P)}"] \arrow[dr, "\mathrm{db}_{\omega(Q)}\otimes 1_{\omega(P)}"] \arrow[dd, "\omega(f)"'] & & \omega(P)\otimes\omega(P)^*\otimes\omega(P) \arrow[rr, "1_{\omega(P)}\otimes\psi(P)"] \arrow[dr, "\omega(f)\otimes 1_{\omega(P)^*}\otimes 1_{\omega(P)}"] & & \omega(P)\otimes M \arrow[dd, "\omega(f)\otimes 1_M"] \\
& \omega(Q)\otimes\omega(Q)^*\otimes\omega(P) \arrow[rr, "1_{\omega(Q)}\otimes\omega(f)^*\otimes 1_{\omega(P)}"] \arrow[dr, "1_{\omega(Q)}\otimes 1_{\omega(Q)^*}\otimes\omega(f)"'] & & \omega(Q)\otimes\omega(P)^*\otimes\omega(P) \arrow[dr, "1_{\omega(Q)}\otimes\psi(P)"'] & \\
\omega(Q) \arrow[rr, "\mathrm{db}_{\omega(Q)}\otimes 1_{\omega(Q)}"'] & & \omega(Q)\otimes\omega(Q)^*\otimes\omega(Q) \arrow[rr, "1_{\omega(Q)}\otimes\psi(Q)"'] & & \omega(Q)\otimes M
\end{tikzcd}$$

commutes where the left upper part of the diagram commutes by the definition of the adjoint morphism $\omega(f)^*$. Conversely given $\varphi : \omega \longrightarrow \omega \otimes M$ the diagram

$$\begin{tikzcd}
& \omega(P)^*\otimes\omega(P) \arrow[rr, "1_{\omega(P)^*}\otimes\varphi(P)"] & & \omega(P)^*\otimes\omega(P)\otimes M \arrow[dr, "\mathrm{ev}_{\omega(P)}\otimes 1_M"] & \\
\omega(Q)^*\otimes\omega(P) \arrow[ur, "\omega(f)^*\otimes 1_{\omega(P)}"] \arrow[rr, "1_{\omega(Q)^*}\otimes\varphi(P)"] \arrow[dr, "1_{\omega(Q)^*}\otimes\omega(f)"'] & & \omega(Q)^*\otimes\omega(P)\otimes M \arrow[ur, "\omega(f)^*\otimes 1_{\omega(P)}\otimes 1_M"] \arrow[dr, "1_{\omega(Q)^*}\otimes\omega(f)\otimes 1_M"'] & & M \\
& \omega(Q)^*\otimes\omega(Q) \arrow[rr, "1_{\omega(Q)^*}\otimes\varphi(Q)"] & & \omega(Q)^*\otimes\omega(Q)\otimes M \arrow[ur, "\mathrm{ev}_{\omega(Q)}\otimes 1_M"'] &
\end{tikzcd}$$

commutes.

Now we show that the functor $\mathrm{Nat}(\omega, \omega \otimes -)$ is multirepresentable. We restrict our attention just to the case $n = 2$. As before there is a bijective correspondence between the natural transformations $\varphi(P,Q) : \omega(P) \otimes \omega(Q) \longrightarrow \omega(P) \otimes \omega(Q) \otimes M$ and cones $\psi(P,Q) : \omega(Q)^* \otimes \omega(P)^* \otimes \omega(P) \otimes \omega(Q) \longrightarrow M$.

Let $\delta : \omega \longrightarrow \omega \otimes \mathrm{coend}(\omega)$ be the universal morphism and abbreviate $B := \mathrm{coend}(\omega)$. Let $\theta(P) : \omega(P)^* \otimes \omega(P) \longrightarrow B$ be the induced morphism. In the commutative diagram (colimit of a wedge in the sense used above) induced by morphisms $f : P \longrightarrow R$ and $g : Q \longrightarrow S$ in $\mathcal{B}$

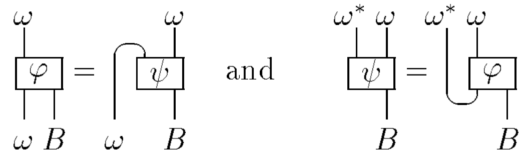
$$\begin{tikzcd}
& \omega(Q)^*\otimes\omega(P)^*\otimes\omega(P)\otimes\omega(Q) \arrow[r, "\sigma^{-1}_{\omega(Q)^*,\omega(P)^*\otimes\omega(P)}\otimes 1_{\omega(Q)}"] & \omega(P)^*\otimes\omega(P)\otimes\omega(Q)^*\otimes\omega(Q) \arrow[dr, "\theta(P)\otimes\theta(Q)"] & \\
\omega(S)^*\otimes\omega(R)^*\otimes\omega(P)\otimes\omega(Q) \arrow[ur, "\omega(g)^*\otimes\omega(f)^*\otimes 1_{\omega(P)\otimes\omega(Q)}"] \arrow[dr, "1_{\omega(S)^*\otimes\omega(R)^*}\otimes\omega(f)\otimes\omega(g)"'] & & & B\otimes B \\
& \omega(S)^*\otimes\omega(R)^*\otimes\omega(R)\otimes\omega(S) \arrow[r, "\sigma^{-1}_{\omega(S)^*,\omega(R)^*\otimes\omega(R)}\otimes 1_{\omega(S)}"] & \omega(R)^*\otimes\omega(R)\otimes\omega(S)^*\otimes\omega(S) \arrow[ur, "\theta(R)\otimes\theta(S)"'] &
\end{tikzcd} \tag{2}$$

$B \otimes B$ is a colimit with

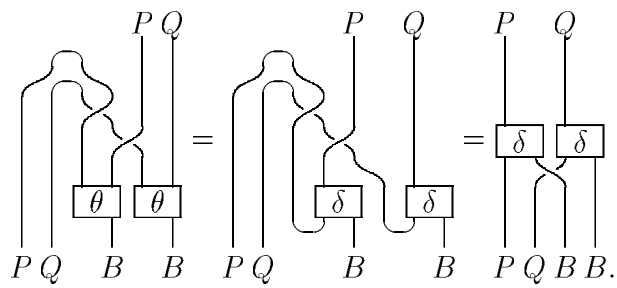
$$\theta(P,Q) = (\theta(P) \otimes \theta(Q))(\sigma^{-1}_{\omega(Q)^*,\omega(P)^*\otimes\omega(P)} \otimes 1_{\omega(Q)})$$

since tensor products preserve colimits. We have to show that the induced morphism $\delta(\text{-},\text{-}) : \omega\otimes\omega \longrightarrow \omega\otimes\omega\otimes B\otimes B$ is equal to $(1_\omega\otimes\sigma_{B,\omega}\otimes 1_B)(\delta\otimes\delta) : \omega^2 \longrightarrow \omega^2\otimes B\otimes B$.

We use graphic calculus and observe that the correspondence between the morphisms φ and ψ is given by



Then we get (writing P and Q instead of $\omega(P)$ resp. $\omega(Q)$)



(2) To describe the second isomorphism of the theorem the property of $\mathcal{C}_0$-transformation for $\varphi : \omega \longrightarrow \omega \otimes M$ is given by the commutative diagram

$$\begin{tikzcd}
\omega(X \otimes P) \arrow[r, "\varphi(X \otimes P)"] \arrow[d, "\zeta(X,P)"] & \omega(X \otimes P) \otimes M \arrow[d, "\zeta(X,P) \otimes 1_M"] \\
X \otimes \omega(P) \arrow[r, "1_X \otimes \varphi(P)"] & X \otimes \omega(P) \otimes M
\end{tikzcd}$$

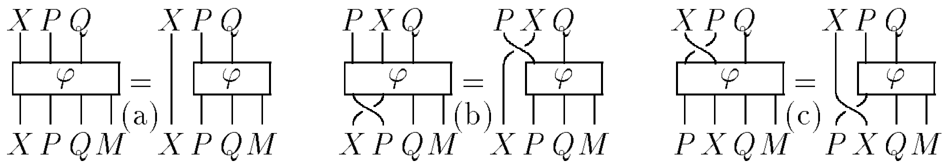
which translates into

$$\begin{tikzcd}
\omega(P)^* \otimes X^* \otimes \omega(X \otimes P) \arrow[r, "\zeta(X,P)^* \otimes 1", "\cong"'] \arrow[dd, "1_{\omega(P)^*} \otimes \zeta(X,P)"', "\cong"] & \omega(X \otimes P)^* \otimes \omega(X \otimes P) \arrow[d, "\psi(X \otimes P)"] \\
 & M \\
\omega(P)^* \otimes X^* \otimes X \otimes \omega(P) \arrow[r, "1_{\omega(P)^*} \otimes \mathrm{ev}_X \otimes 1_{\omega(P)}"] & \omega(P)^* \otimes \omega(P) \arrow[u, "\psi(P)"']
\end{tikzcd} \tag{3}$$

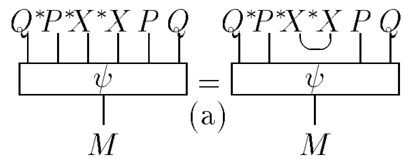
So the colimit $\mathrm{coend}_{\mathcal{C}_0}(\omega)$ exists and is described (similar to the way given in [Pa93] Definition 2.2) as $\coprod_{P \in \mathrm{Ob}(\mathcal{B})} \omega(P)^* \otimes \omega(P)$ modulo the relations given by all $f : P \longrightarrow Q$ as in the construction of $\mathrm{coend}(\omega)$ plus the relations for any pair (X, P) given above.

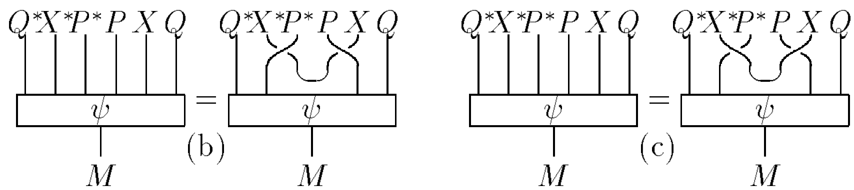
In particular $\delta : \omega \longrightarrow \omega \otimes B$ with $B := \mathrm{coend}_{\mathcal{C}_0}(\omega)$ is a universal $\mathcal{C}_0$-morphism.

Now we assume that $B = \mathrm{coend}_{\mathcal{C}_0}(\omega)$ is $\mathcal{C}_0$-central and show that the functor $\mathrm{Nat}_{\mathcal{C}_0}(\omega, \omega \otimes -)$ is multirepresentable. We restrict our attention again just to the case $n = 2$. Since $\delta_2 := (1_\omega \otimes \sigma_{B,\omega} \otimes 1_B)(\delta \otimes \delta) : \omega^2 \longrightarrow \omega^2 \otimes B^2$ is a $\mathcal{C}_0$-bimorphism – B is $\mathcal{C}_0$-central – we can show that $B \otimes B = \mathrm{coend}_{\mathcal{C}_0}(\omega \otimes \omega)$ with the universal $\mathcal{C}_0$-bimorphism δ_2. We first show that there is a one-to-one correspondence between $\mathcal{C}_0$-bimorphisms $\varphi(P, Q) : \omega(P) \otimes \omega(Q) \longrightarrow \omega(P) \otimes \omega(Q) \otimes M$ and cones $\psi(P, Q) : \omega(Q)^* \otimes \omega(P)^* \otimes \omega(P) \otimes \omega(Q) \longrightarrow M$ satisfying certain relations given below. We saw earlier for two variables (see diagram (2)) that φ is a natural transformation iff ψ is a cone. So we have to translate the conditions for the $\mathcal{C}_0$-structure. It is an easy exercise to show that under the correspondence between φ and ψ the conditions

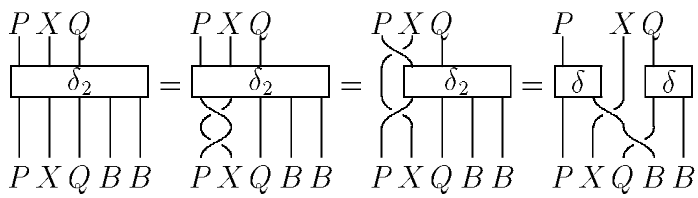


are equivalent to the following conditions

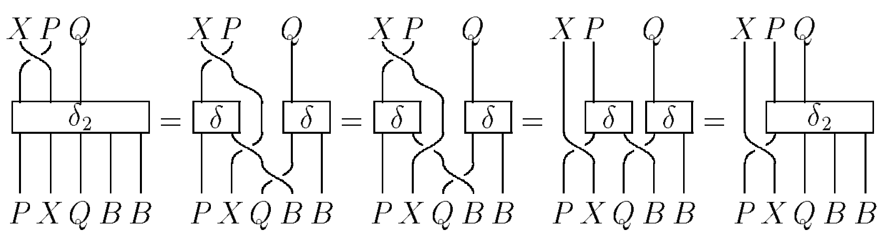




In particular the induced cone $\theta_2 : \omega(Q)^* \otimes \omega(P)^* \otimes \omega(P) \otimes \omega(Q) \longrightarrow B \otimes B$ satisfies the conditions (a), (b), and (c). Observe that condition (b) for $\varphi = \delta_2$ implies condition (c) for δ_2 since B is $\mathcal{C}_0$-central. In fact (b) implies

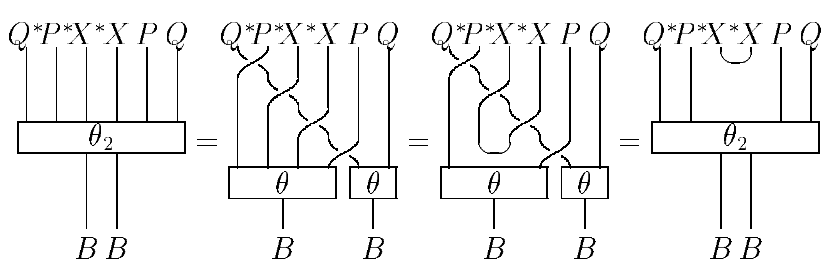


hence

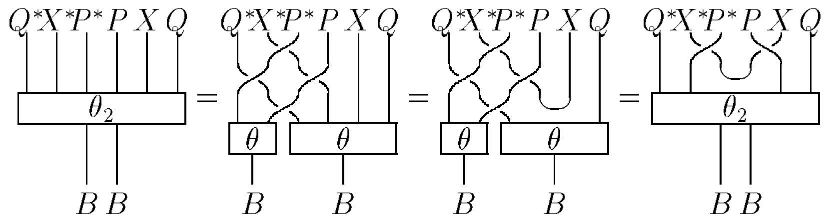


So condition (b) for $\psi = \theta_2$ implies condition (c) for θ_2. Now we show that $\theta_2 : \omega(Q)^* \otimes \omega(P)^* \otimes \omega(P) \otimes \omega(Q) \longrightarrow B \otimes B$ is the injection morphism of a colimit. Then $\delta_2 : \omega \otimes \omega \longrightarrow \omega \otimes \omega \otimes B \otimes B$ is a universal $\mathcal{C}_0$-morphism.

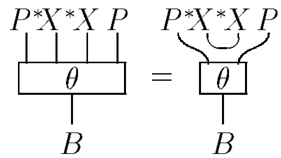
We have already seen that diagram (2) is a tensor product of wedges and cone morphisms of the type of diagram (1). Now we show that the relations (a) and (b) come about as tensor products of relations for B. We don't have to consider condition (c) which is automatically satisfied. The diagrams



and



show that relations (a) and (b) are tensor products with the relation



used in the construction of B (up to a preceding isomorphism). Since we are now considering a tensor product of two diagrams and the colimit thereof and since tensor products preserve colimits we have proved the claimed result. $\square$

4.2.1. Using the results of section 3 we obtain uniquely determined coalgebra, bialgebra, and Hopf algebra structures (depending on the given functor) on $\mathrm{coend}(\omega)$ and $\mathrm{coend}_{\mathcal{C}_0}(\omega)$.

5. Hidden symmetries

In section 4.1 we studied under which circumstances coalgebras and bialgebras (possibly with a transmuted multiplication) can be reconstructed from their categories of comodules and the functor $\omega : \mathcal{C}^C \longrightarrow \mathcal{C}$. We saw that they are obtained as the representing object $\mathrm{coend}_{\mathcal{C}}(\omega)$ of $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -) : \mathcal{C} \longrightarrow \mathrm{Set}$. In this section we will see that this reconstruction depends strongly on the choice of the control category $\mathcal{C}$. If $\mathcal{C}$ is decreased to a category $\mathcal{D}$ then the representing (reconstructed) object $\mathrm{coend}_{\mathcal{D}}(\omega)$ becomes larger. We will see that under certain conditions the reconstructed object decomposes into a cosmash product where one factor represents the "hidden symmetries".

5.1. Functors of control categories. We consider of a braided monoidal functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ of control categories.

5.1.1. If $\mathcal{B}$ is a $\mathcal{C}$-category via $\otimes : \mathcal{C} \times \mathcal{B} \longrightarrow \mathcal{B}$, then $\mathcal{B}$ becomes a $\mathcal{D}$-category by

$$\otimes : \mathcal{D} \times \mathcal{B} \xrightarrow{\mathcal{F}\times 1} \mathcal{C} \times \mathcal{B} \xrightarrow{\otimes} \mathcal{B}$$

with associativity morphism $\beta(v \otimes 1) : (X \otimes Y) \otimes P \longrightarrow X \otimes (Y \otimes P)$ and unary action $\pi(\varsigma \otimes 1) : I_{\mathcal{D}} \otimes P \longrightarrow P$ for $X, Y \in \mathcal{D}$ and $P \in \mathcal{B}$.

5.1.2. If $\chi : \mathcal{B} \longrightarrow \mathcal{B}'$ is a $\mathcal{C}$-functor, then χ becomes also a $\mathcal{D}$-functor. If $\chi : \mathcal{B} \times \mathcal{B}' \longrightarrow \mathcal{B}''$ is a $\mathcal{C}$-bifunctor, then χ becomes also a $\mathcal{D}$-bifunctor. In both cases the structure morphisms ζ resp. τ remain unchanged.

If $\chi, \chi' : \mathcal{B} \longrightarrow \mathcal{B}'$ are $\mathcal{C}$-functors and $\zeta : \chi \longrightarrow \chi'$ is a $\mathcal{C}$-morphism, then ζ is also a $\mathcal{D}$-morphism.

5.1.3. If $\mathcal{A}$ is a $\mathcal{C}$-monoidal category, then it becomes also a $\mathcal{D}$-monoidal category.

The above observations give immediately

Proposition 5.1. *If $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is a braided monoidal functor, then it induces an "underlying" functor $\mathfrak{A}(\mathcal{F}) : \mathfrak{A}(\mathcal{C}) \longrightarrow \mathfrak{A}(\mathcal{D})$.*

5.1.4. Let $(\mathcal{B},\omega) \in \mathfrak{A}(\mathcal{C})$ and consider $\mathfrak{A}(\mathcal{F})(\mathcal{B},\omega) = (\mathcal{B},\omega)$ with the induced structure morphisms. Assume that $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega\otimes -)$ and $\mathrm{Nat}_{\mathcal{D}}(\omega, \omega\otimes -)$ are representable by $\mathrm{coend}_{\mathcal{C}}(\omega)$ resp. $\mathrm{coend}_{\mathcal{D}}(\omega)$. Then $\mathcal{A}(\mathrm{coend}_{\mathcal{C}}(\omega), M) \cong \mathrm{Nat}_{\mathcal{C}}(\omega, \omega\otimes M) \subseteq \mathrm{Nat}_{\mathcal{D}}(\omega, \omega\otimes M) \cong \mathcal{A}(\mathrm{coend}_{\mathcal{D}}(\omega), M)$ as functors in $M \in \mathcal{A}$ hence there is an epimorphism of the representing objects $\mathrm{coend}_{\mathcal{F}}(\omega) : \mathrm{coend}_{\mathcal{D}}(\omega) \longrightarrow \mathrm{coend}_{\mathcal{C}}(\omega)$.

Theorem 5.2. *Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a braided monoidal functor. Let $\mathcal{A}$ be a $\mathcal{C}$-monoidal category, $\mathcal{B}$ a $\mathcal{C}$-category and $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ a $\mathcal{C}$-functor. Assume that* $\mathrm{coend}_{\mathcal{C}}(\omega)$ *and* $\mathrm{coend}_{\mathcal{D}}(\omega)$ *exist. Then there is an induced epimorphism of coalgebras* $\mathrm{coend}_{\mathcal{F}}(\omega) : \mathrm{coend}_{\mathcal{D}}(\omega) \longrightarrow \mathrm{coend}_{\mathcal{C}}(\omega)$ *in $\mathcal{C}$.*

If in addition the comodule $(\mathrm{coend}_{\mathcal{C}}(\omega), \Delta)$ *is liftable along ω then* $\mathrm{coend}_{\mathcal{F}}(\omega) : \mathrm{coend}_{\mathcal{D}}(\omega) \longrightarrow \mathrm{coend}_{\mathcal{C}}(\omega)$ *is a retraction of objects in $\mathcal{C}$.*

Proof. Let $C := \mathrm{coend}_{\mathcal{C}}(\omega)$ and $D := \mathrm{coend}_{\mathcal{D}}(\omega)$. Let $\delta : \omega \longrightarrow \omega \otimes C$ and $\partial : \omega \longrightarrow \omega \otimes D$ be the universal morphisms. Write $C^2 := C \otimes C$, $D^2 = D \otimes D$, and $f = \mathrm{coend}_{\mathcal{F}}(\omega)$. Then the commutativity of

$$\begin{tikzcd}
\omega \arrow[rr, "\partial"] \arrow[dr, "="] \arrow[dd, "\partial"'] & & \omega\otimes D \arrow[dr, "1\otimes f"] \arrow[dd, "1\otimes\Delta" near start] & \\
& \omega \arrow[rr, "\delta" near start] \arrow[dd, "\delta" near start] & & \omega\otimes C \arrow[dd, "1\otimes\Delta"] \\
\omega\otimes D \arrow[rr, "\partial\otimes 1" near start] \arrow[dr, "1\otimes f"'] & & \omega\otimes D^2 \arrow[dr, "1\otimes f\otimes f"] & \\
& \omega\otimes C \arrow[rr, "\delta\otimes 1"'] & & \omega\otimes C^2
\end{tikzcd}$$

and

$$\begin{tikzcd}
\omega \arrow[rr, "\partial"] \arrow[dr, "\delta"] \arrow[ddr, "\cong"'] & & \omega\otimes D \arrow[dl, "1\otimes f"'] \arrow[ddl, "1\otimes\varepsilon"] \\
& \omega\otimes C \arrow[d, "1\otimes\varepsilon"] & \\
& \omega\otimes I &
\end{tikzcd}$$

show that $f : D \longrightarrow C$ is a coalgebra morphism.

Let $g : \omega(\widetilde{C}) \longrightarrow C$ be a C-comodule isomorphism. Then the following diagram commutes

$$\begin{tikzcd}
\omega(\widetilde{C}) \arrow[r, "\partial"] \arrow[d, "g"'] \arrow[dr, "\delta"'] & \omega(\widetilde{C})\otimes D \arrow[r, "\varepsilon g\otimes 1"] \arrow[d, "1\otimes f"] & I\otimes D \arrow[r, phantom, "\cong"] \arrow[dd, "1\otimes f"'] & D \arrow[dd, "f"] \\
C \arrow[dr, "\Delta"'] & \omega(\widetilde{C})\otimes C \arrow[d, "g\otimes 1"] & & \\
& C\otimes C \arrow[r, "\varepsilon\otimes 1"] & I\otimes C \arrow[r, phantom, "\cong"] & C.
\end{tikzcd}$$

The lower morphism of the diagram is the identity hence f is a retraction in $\mathcal{A}$. $\square$

5.1.5. Observe that the morphism $\mathrm{coend}_{\mathcal{F}}(\omega) : \mathrm{coend}_{\mathcal{D}}(\omega) \longrightarrow \mathrm{coend}_{\mathcal{C}}(\omega)$ is the uniquely defined morphism such that $(1_\omega \otimes \mathrm{coend}_{\mathcal{F}}(\omega)) \circ \partial = \delta$.

5.1.6. The preceding theorem shows that the reconstructed coalgebra D w.r.t. $\mathcal{D}$ is larger than C. We consider the additional part in D as hidden symmetries in the sense described in the introduction. It is responsible for $\mathcal{D}$-morphisms $\varphi : \omega \longrightarrow \omega \otimes M$ which are not $\mathcal{C}$-morphisms or for certain elements in $\mathrm{Nat}_{\mathcal{D}}(\omega, \omega \otimes M)$ which are not contained in $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes M)$. As we have seen this part of D tends to split off.

5.1.7. An example of a hidden symmetry can be obtained for superalgebra representations. This is dual to the above considerations. Given an algebra A considered as a superalgebra $(A, 0)$. Consider $\mathcal{A} = \mathcal{C}$, the category of super vector spaces ($\mathbb{K}\mathbb{Z}_2$-Comod), the category $\mathcal{B} = \mathcal{C}_A$ of super A-modules, and the forgetful functor $\omega : \mathcal{C}_A \longrightarrow \mathcal{C}$. Let $\mathcal{F} : \mathcal{D} = \mathrm{Vec} \longrightarrow \mathcal{C}$ be the functor which sends each vector space V to the super vector space $(V, 0)$. Any $\mathcal{D}$-morphism $\varphi : \omega \longrightarrow \omega$ is described by its image under $\varphi \in \mathrm{Nat}_{\mathcal{D}}(\omega \otimes I, \omega) \cong \mathcal{A}(I, \mathrm{end}_{\mathcal{D}}(\omega))$.

The natural transformation $\varphi : P \longrightarrow P$ given by $(p_0, p_1) \mapsto (p_0, -p_1)$ is a symmetry for all representations of A (a natural automorphism of ω), which is not induced by the multiplication with any element of A. A multiplication with an element $a = (a, 0) \in A$ on A-modules (P_0, P_1) in $\mathcal{C}_A$ would have to satisfy $(p_0, p_1)a = (p_0 a, p_1 a) = (p_0, -p_1)$ for all choices of (p_0, p_1) which is not possible. The natural transformation φ is, however, a $\mathcal{D}$-morphism and thus comes from multiplication with an element $b \in \mathrm{end}_{\mathcal{D}}(\omega)$, in fact from the element $e_1 - e_t \in (\mathbb{K}\mathbb{Z}_2)^* \subseteq \mathrm{end}_{\mathcal{D}}(\omega)$ where e_1, e_t is the dual basis to $1, t \in \mathbb{K}[t]/(t^2 - 1) = \mathbb{K}\mathbb{Z}_2$.

5.1.8. We apply the preceding example in representation theory of groups. If we consider representations of a group G in vector spaces over a field $\mathbb{K}$, i.e. the category $\mathcal{M}_{\mathbb{K}G}$, then each element $g \in G$ induces a $\mathcal{C}$-monoidal automorphism $\varphi_g : \omega \longrightarrow \omega$ with $\varphi_g(p) := pg$, where $\omega : \mathcal{M}_{\mathbb{K}G} \longrightarrow \mathcal{M} = \mathrm{Vec} = \mathcal{C}$ is the forgetful functor. Observe that any natural transformation of functors into $\mathcal{C}$ is a $\mathcal{C}$-morphism by Theorem 6.4. Conversely given any $\mathcal{C}$-monoidal automorphism $\varphi : \omega \longrightarrow \omega$ there is precisely one $g \in G$ with $\varphi = \varphi_g$. Thus G can be reconstructed from its representations, i.e. from $\omega : \mathcal{M}_{\mathbb{K}G} \longrightarrow \mathcal{M}$.

Now consider representations of G in super vector spaces over $\mathbb{K}$, i.e. the category $\mathcal{A}$ of two-graded vector spaces. Let $\mathcal{C} = \mathcal{M}^{\mathbb{K}\mathbb{Z}_2} = \mathcal{A}$, $\mathcal{F} : \mathcal{D} = \mathrm{Vec} \longrightarrow \mathcal{C} = \mathcal{M}^{\mathbb{K}\mathbb{Z}_2}$, and $\omega : \mathcal{A}_{\mathbb{K}G} \longrightarrow \mathcal{A}$. We may consider $\mathbb{K}G$ as a Hopf algebra $(\mathbb{K}G, 0)$ in $\mathcal{A}$ and have $(p_0, p_1)g = (p_0 g, p_1 g)$ with a suitable G-structure on P_0 and P_1 separately. Then each element $g \in G$ induces a $\mathcal{C}$-monoidal automorphism $\varphi_g : \omega \longrightarrow \omega$. For any $\mathcal{C}$-monoidal automorphism there is precisely one $g \in G$ with $\varphi = \varphi_g$. For the $\mathcal{D}$-monoidal automorphism $\varphi : \omega \longrightarrow \omega$ with $\varphi(P_0, P_1)(p_0, p_1) := (p_0, -p_1)$ there is, however, no $g \in G$ with $\varphi = \varphi_g$. So in this case the group of symmetries (of monoidal automorphisms of ω) is a bigger group than the one we started out with. The given φ is an example of a hidden symmetry.

5.1.9. We consider now the situation of a morphism $[\chi, \zeta] : (\mathcal{B}, \omega) \longrightarrow (\mathcal{B}', \omega')$ in $\mathfrak{A}(\mathcal{C})$ together with a braided monoidal functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$. Assume the universal objects and morphisms $\delta : \omega \longrightarrow \omega \otimes C$, $\partial : \omega \longrightarrow \omega \otimes D$, $\delta' : \omega' \longrightarrow \omega' \otimes C'$, and $\partial' : \omega' \longrightarrow \omega' \otimes D'$ exist. Then by 3.3.1 we get induced morphisms $z : C \longrightarrow C'$ and $y : D \longrightarrow D'$ such that $(\zeta \otimes z) \circ \delta = \delta' \chi \circ \zeta$ and $(\zeta \otimes y) \circ \partial = \partial' \chi \circ \zeta$. Furthermore by Theorem 5.2 there are induced morphisms $f := \mathrm{coend}_{\mathcal{F}}(\omega) : D \longrightarrow C$ and

$f' := \mathrm{coend}_{\mathcal{F}}(\omega') : D' \longrightarrow C'$ such that $(1_\omega \otimes f) \circ \partial = \delta$ and $(1_{\omega'} \otimes f') \circ \partial' = \delta'$. Hence by the universal property of ∂ the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\partial"] \arrow[dd, "\zeta"'] \arrow[drr, "\delta"] & \omega \otimes D \arrow[dr, "1\otimes f"] \arrow[dd, "\zeta\otimes y"] & \\
& & \omega \otimes C \arrow[dd, "\zeta\otimes z"] \\
\omega'\chi \arrow[r, "\partial'\chi"] \arrow[drr, "\delta'\chi"] & \omega'\chi \otimes D' \arrow[dr, "1\otimes f'"] & \\
& & \omega'\chi \otimes C'
\end{tikzcd}$$

commutes and from $\zeta \otimes zf = (\zeta \otimes z)(1 \otimes f) = (1 \otimes f')(\zeta \otimes y) = \zeta \otimes f'y$ and the uniqueness of induced morphisms we get a commutative diagram of coalgebra morphisms

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, "y"'] & C \arrow[d, "z"] \\
D' \arrow[r, "f'"] & C'
\end{tikzcd}$$

or

$$\mathrm{coend}_{\mathcal{C}}([\chi,\zeta])\mathrm{coend}_{\mathcal{F}}(\omega) = \mathrm{coend}_{\mathcal{F}}(\omega')\mathrm{coend}_{\mathcal{D}}([\chi,\zeta]).$$

In particular we have proved

Theorem 5.3. *Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a braided monoidal functor. Assume that the functors* $\mathrm{coend}_{\mathcal{C}} : \mathfrak{A}(\mathcal{C}) \longrightarrow \mathcal{A}$-coalg *and* $\mathrm{coend}_{\mathcal{D}} : \mathfrak{A}(\mathcal{C}) \longrightarrow \mathcal{A}$-coalg *exist. Then* $\mathrm{coend}_{\mathcal{F}} : \mathrm{coend}_{\mathcal{D}} \longrightarrow \mathrm{coend}_{\mathcal{C}}$ *is a natural epimorphism of functors from* $\mathfrak{A}(\mathcal{C})$ *to* $\mathcal{A}$-coalg*.*

5.1.10. Assume now that $\mathcal{B}$ is $\mathcal{C}$-monoidal, that $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ is a $\mathcal{C}$-monoidal functor and that $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega \otimes -)$ and $\mathrm{Nat}_{\mathcal{D}}(\omega, \omega \otimes -)$ are multirepresentable. Then $z = \mathrm{coend}_{\mathcal{C}}([\chi,\zeta]) : D \longrightarrow C$ is a bialgebra morphism. The multiplicativity follows from the commutative diagram

$$\begin{tikzcd}
\omega\otimes\omega \arrow[r, "\partial_2"] \arrow[dddd, "\upsilon^{-1}"'] \arrow[drr, "\delta_2"] & \omega\otimes\omega\otimes D\otimes D \arrow[dr, "1\otimes f\otimes f"] \arrow[dd, "1\otimes m_D"'] & \\
& & \omega\otimes\omega\otimes C\otimes C \arrow[dd, "1\otimes m_C"] \\
& \omega\otimes\omega\otimes D \arrow[dr, "1\otimes f"] \arrow[dd, "\upsilon^{-1}\otimes 1"'] & \\
& & \omega\otimes\omega\otimes C \arrow[dd, "\upsilon^{-1}\otimes 1"] \\
\omega(\otimes) \arrow[r, "\partial"] \arrow[drr, "\delta"] & \omega(\otimes)\otimes D \arrow[dr, "1\otimes f"] & \\
& & \omega\otimes C
\end{tikzcd}$$

(where $\omega(\otimes)(P,Q) := \omega(P \otimes Q)$) and the unary property is proved similarly.

If the bialgebras C' and D' are Hopf algebras then by Corollary 3.8 C is a C'-comodule coalgebra by the coadjoint coaction w.r.t. the induced coalgebra morphism $z : C \longrightarrow C'$ and D is a D'-comodule coalgebra by the coadjoint coaction w.r.t. the induced coalgebra morphism $y : D \longrightarrow D'$. Furthermore $f' : D' \longrightarrow C'$ is an epimorphism and a bialgebra (Hopf algebra) morphism.

5.2. Hidden symmetries of the base category. Consider a braided monoidal functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ and a morphism $[\chi, \zeta] : (\mathcal{B}, \omega) \longrightarrow (\mathcal{B}', \omega')$ in $\mathfrak{A}(\mathcal{C})$. Assume that $\mathcal{B}'$ is a $\mathcal{C}$-monoidal category and ω' is a $\mathcal{C}$-monoidal functor. Let $\delta : \omega \longrightarrow \omega \otimes D$ be a universal $\mathcal{D}$-morphism. Let D' be a bialgebra in $\mathcal{A}$ and let $\delta' : \omega' \longrightarrow \omega' \otimes D'$ be a $\mathcal{D}$-morphism compatible with the bialgebra structure of D' (e.g. δ' is a universal $\mathcal{D}$-morphism). Let E be a coalgebra in $\mathcal{B}'$ and let $\mu : \chi \longrightarrow \chi \otimes E$ be a $\mathcal{C}$-morphism compatible with the structure of E (e.g. a universal $\mathcal{C}$-morphism).

Theorem 5.4. *In the setup given above $D' \otimes \omega' E$ carries the structure of a cosmash product $D' \#^c \omega' E$ and there is a canonical coalgebra morphism $f : D \longrightarrow D' \#^c \omega' E$.*

Proof. We first observe that $\omega' E$ is a coalgebra in $\mathcal{A}$ since ω' is a monoidal functor. Furthermore $\omega' E$ is a D'-comodule coalgebra by the morphism $\delta' E : \omega' E \longrightarrow \omega' E \otimes D'$.

Consider the following induced morphism $f : D \longrightarrow D' \#^c \omega' E$ defined by

$$\begin{tikzcd}
\omega \arrow[rr, "\delta"] \arrow[d, "\zeta"'] & & \omega \otimes D \arrow[d, "1 \otimes f"] \\
\omega' \chi \arrow[d, "\omega' \mu"'] & & \omega \otimes D' \otimes \omega' E \arrow[d, "\zeta \otimes 1"] \\
\omega'(\chi \otimes E) \arrow[r, "\upsilon"] & \omega' \chi \otimes \omega' E \arrow[r, "\delta' \chi \otimes 1"] & \omega' \chi \otimes D' \otimes \omega' E.
\end{tikzcd}$$

Since the composition along the lower edge of the square is a $\mathcal{D}$-morphism $\omega \longrightarrow \omega \otimes D' \otimes \omega' E$ the morphism f is uniquely determined. We show that f is a coalgebra morphism with $D' \otimes \omega' E = D' \#^c \omega' E$ the cosmash product. For this purpose define a morphism

$$\tau_0 : D' \otimes \omega' E \xrightarrow{1 \otimes \delta' E} D' \otimes \omega' E \otimes D' \xrightarrow{\sigma \otimes 1} \omega' E \otimes D' \otimes D' \xrightarrow{1 \otimes m_{D'}} \omega' E \otimes D'.$$

Then the following diagram of $\mathcal{D}$-morphisms commutes

$$\begin{tikzcd}
\omega' \chi \arrow[r, "\omega' \mu"] \arrow[dddd, "\delta' \chi"'] & \omega'(\chi \otimes E) \arrow[r, "\upsilon"] \arrow[dddd, "\delta'(\chi \otimes E)"] & \omega' \chi \otimes \omega' E \arrow[r, "\delta' \chi \otimes 1"] & \omega' \chi \otimes D' \otimes \omega' E \arrow[d, "1 \otimes \delta' E"] \\
 & & & \omega' \chi \otimes D' \otimes \omega' E \otimes D' \arrow[d, "1 \otimes \sigma \otimes 1"] \\
 & & & \omega' \chi \otimes \omega' E \otimes D' \otimes D' \arrow[d, "1 \otimes m_{D'}"] \\
 & & & \\
\omega' \chi \otimes D' \arrow[r, "\omega' \mu \otimes 1"] & \omega'(\chi \otimes E) \otimes D' \arrow[rr, "\upsilon \otimes 1"] & & \omega' \chi \otimes \omega' E \otimes D'
\end{tikzcd}$$

by the very fact that the bialgebra structure of D' is compatible with δ'. Hence with suitable identifications we get the commutative diagram

$$\begin{tikzcd}
\omega'\chi \arrow[r, "\omega'\mu"] \arrow[dd, "\delta'\chi"'] & \omega'\chi\otimes\omega'E \arrow[d, "\delta'\chi\otimes 1"] \\
& \omega'\chi\otimes D'\otimes\omega'E \arrow[d, "1\otimes\tau_0"] \\
\omega'\chi\otimes D' \arrow[r, "\omega'\mu\otimes 1"] & \omega'\chi\otimes\omega'E\otimes D'.
\end{tikzcd}$$

Now consider the commutative diagram

$$\begin{tikzcd}
\omega'\chi \arrow[r, "\delta"] \arrow[d, "\omega'\mu"'] & \omega'\chi\otimes D \arrow[rr, "\delta\otimes 1"] \arrow[d, "1\otimes f"'] \arrow[dr, "\omega'\mu\otimes 1"] & & \omega'\chi\otimes D\otimes D \arrow[d, "1\otimes f\otimes 1"] \\
\omega'\chi\otimes\omega'E \arrow[r, "\delta'\chi\otimes 1"] \arrow[ddd, "\omega'\mu\otimes 1"] & \omega'\chi\otimes D'\otimes\omega'E \arrow[ddr, "\omega'\mu\otimes 1"'] & \omega'\chi\otimes\omega'E\otimes D \arrow[r, "\delta'\chi\otimes 1"] \arrow[d, "1\otimes f"] & \omega'\chi\otimes D'\otimes\omega'E\otimes D \arrow[d, "1\otimes 1\otimes f"] \\
& & \omega'\chi\otimes\omega'E\otimes D'\otimes\omega'E \arrow[r, "\delta'\chi\otimes 1"] & \omega'\chi\otimes D'\otimes\omega'E\otimes D'\otimes\omega'E \\
\omega'\chi\otimes\omega'E\otimes\omega'E \arrow[r, "\delta'\chi\otimes 1"] & \omega'\chi\otimes D'\otimes\omega'E\otimes\omega'E \arrow[ur, "1\otimes\tau_0\otimes 1"'] \arrow[r, "\delta'\chi\otimes 1"] & \omega'\chi\otimes D'\otimes D'\otimes\omega'E\otimes\omega'E \arrow[ur, "1\otimes\tau_0\otimes 1"'] &
\end{tikzcd}$$

From it we get that all morphisms $\omega'\chi \longrightarrow \omega'\chi\otimes D'\otimes\omega'E\otimes D'\otimes\omega'E$ in the following diagram are equal

$$\begin{tikzcd}
\omega'\chi \arrow[r, "\delta"] \arrow[d, "\omega'\mu"'] & \omega'\chi\otimes D \arrow[rr, shift left, "1\otimes\Delta_D"] \arrow[rr, shift right, "\delta\otimes 1"'] \arrow[d, "1\otimes f"'] \arrow[dr, "\omega'\mu\otimes 1"] & & \omega'\chi\otimes D\otimes D \arrow[d, "1\otimes f\otimes 1"] \\
\omega'\chi\otimes\omega'E \arrow[r, "\delta'\chi\otimes 1"] \arrow[ddd, shift right, "1\otimes\Delta_{\omega'E}"'] \arrow[ddd, shift left, "\omega'\mu\otimes 1"] & \omega'\chi\otimes D'\otimes\omega'E \arrow[ddr, "\omega'\mu\otimes 1"'] \arrow[ddd, "1\otimes\Delta_{\omega'E}"'] & \omega'\chi\otimes\omega'E\otimes D \arrow[r, "\delta'\chi\otimes 1"] \arrow[d, "1\otimes f"] & \omega'\chi\otimes D'\otimes\omega'E\otimes D \arrow[d, "1\otimes 1\otimes f"] \\
& & \omega'\chi\otimes\omega'E\otimes D'\otimes\omega'E \arrow[r, "\delta'\chi\otimes 1"] & \omega'\chi\otimes D'\otimes\omega'E\otimes D'\otimes\omega'E \\
\omega'\chi\otimes\omega'E\otimes\omega'E \arrow[r, "\delta'\chi\otimes 1"] & \omega'\chi\otimes D'\otimes\omega'E\otimes\omega'E \arrow[ur, "1\otimes\tau_0\otimes 1"'] \arrow[r, shift left, "\delta'\chi\otimes 1"] \arrow[r, shift right, "1\otimes\Delta_{D'}\otimes 1"'] & \omega'\chi\otimes D'\otimes D'\otimes\omega'E\otimes\omega'E \arrow[ur, "1\otimes\tau_0\otimes 1"'] &
\end{tikzcd}$$

We define

$$\Delta_{D'\otimes\omega'E} : D'\otimes\omega'E \overset{\Delta_{D'}\otimes\Delta_{\omega'E}}{\longrightarrow} D'\otimes D'\otimes\omega'E\otimes\omega'E \overset{1\otimes\tau_0\otimes 1}{\longrightarrow} D'\otimes\omega'E\otimes D'\otimes\omega'E$$

and observe that $\delta : \omega'\chi \longrightarrow \omega'\chi\otimes D$ is a universal $\mathcal{D}$-morphism. Hence the diagram of induced morphisms

$$\begin{tikzcd}
D \arrow[r, "\Delta_D"] \arrow[d, "f"] & D\otimes D \arrow[d, "f\otimes f"] \\
D'\otimes\omega'E \arrow[r, "\Delta_{D'\otimes\omega'E}"] & D'\otimes\omega'E\otimes D'\otimes\omega'E
\end{tikzcd}$$

commutes. It is now easy to see that $D' \otimes \omega' E$ with $\Delta_{D' \otimes \omega' E}$ and $\varepsilon : D' \otimes \omega' E \xrightarrow{\varepsilon \otimes \omega' \varepsilon} I$ is a coalgebra, the cosmash product $D' \#^c \omega' E$, and that $f : D \longrightarrow D' \otimes \omega' E$ is a coalgebra morphism. $\square$

A special application of the theorem is the following

Corollary 5.5. *Let $(\mathcal{B}, \omega)$ be in $\mathfrak{A}(\mathcal{C})$ and let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a braided monoidal functor. If $\mathrm{coend}_{\mathcal{D}}(\omega)$, $\mathrm{coend}_{\mathcal{C}}(\omega)$, and $\mathrm{coend}_{\mathcal{D}}(\mathrm{id}_{\mathcal{A}})$ (with $\mathrm{Nat}_{\mathcal{D}}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}} \otimes -)$ multirepresentable) exist then there is a canonical coalgebra morphism*

$$f : \mathrm{coend}_{\mathcal{D}}(\omega) \longrightarrow \mathrm{coend}_{\mathcal{D}}(\mathrm{id}_{\mathcal{A}}) \#^c \mathrm{coend}_{\mathcal{C}}(\omega).$$

In certain cases the canonical morphism of the preceding corollary is an isomorphism. If this is the case then we have identified the hidden symmetries of the functor $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ as the component $D' = \mathrm{coend}_{\mathcal{D}}(\mathrm{id}_{\mathcal{A}})$ in the cosmash product.

Theorem 5.6. *Let $\mathcal{D}$ be a braided monoidal category and H be a braided Hopf algebra in $\mathcal{D}$. Let $\mathcal{A} = \mathcal{C} = \mathcal{D}^H$. Let C be a coalgebra in $\mathcal{A}$ and $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ be the forgetful functor. Then with $\mathrm{coend}_{\mathcal{D}}(\mathrm{id}_{\mathcal{A}}) \widetilde{\#}^c \mathrm{coend}_{\mathcal{C}}(\omega)$ the cosmash product in $\mathcal{A}$*

$$f : \mathrm{coend}_{\mathcal{D}}(\omega) \longrightarrow \mathrm{coend}_{\mathcal{D}}(\mathrm{id}_{\mathcal{A}}) \widetilde{\#}^c \mathrm{coend}_{\mathcal{C}}(\omega)$$

is an isomorphism of coalgebras in $\mathcal{A}$.

Proof. We consider the diagram

$$\begin{tikzcd} \mathcal{A}^C \arrow[rr, "\omega"] \arrow[dr, "\omega"'] & & \mathcal{A} \arrow[dl, "\mathrm{id}"] \\ & \mathcal{A} & \end{tikzcd}$$

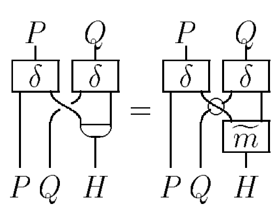
where $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$ is the forgetful functor together with the functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ induced by $u : I \longrightarrow H$. It is easy to check that $\mathcal{A}^C = (\mathcal{D}^H)^C$ and $\mathcal{D}^{H \#^c C}$ are isomorphic $\mathcal{D}$-categories with the cosmash product formed in $\mathcal{D}$ (for cosmash products and transmutation see also 4.1.4) by sending each object $(P, \mu : P \longrightarrow P \otimes C)$ in $\mathcal{A}^C$ to the object $(P, \delta : P \xrightarrow{\mu} P \otimes C \xrightarrow{\delta' \otimes 1} P \otimes H \otimes C)$ in $\mathcal{D}^{H \#^c C}$ (see 2.1.7), where the cosmash comultiplication is defined by

$$\begin{aligned} \Delta : H \otimes C &\xrightarrow{\Delta_H \otimes \Delta_C} H \otimes H \otimes C \otimes C \xrightarrow{1 \otimes \delta'_C \otimes 1} H \otimes H \otimes C \otimes H \otimes C \\ &\xrightarrow{1 \otimes \sigma \otimes 1} H \otimes C \otimes H \otimes H \otimes C \xrightarrow{1 \otimes m_H \otimes 1} H \otimes C \otimes H \otimes C. \end{aligned}$$

The morphism $z = 1 \otimes \varepsilon_C : H \#^c C \longrightarrow H$ is a coalgebra morphism. It induces a $\mathcal{D}$-functor $\mathcal{D}^z : \mathcal{D}^{H \#^c C} \longrightarrow \mathcal{D}^H$ which can be identified with $\omega : \mathcal{A}^C \longrightarrow \mathcal{A}$.

By Theorem 4.1 we get $\mathrm{coend}_{\mathcal{D}}(\omega) = H \#^c C$, the cosmash product defined in $\mathcal{D}$ as above.

From Theorem 4.2 we get $\mathrm{coend}_{\mathcal{D}}(\mathrm{id}_{\mathcal{A}}) = H$ as a coalgebra but with the new multiplication $\widetilde{m}_H : H \otimes H \longrightarrow H$ as defined in Proposition 3.6 by the braid diagram



In particular $\widetilde{m}_H : H \otimes H \longrightarrow H$ is a morphism of H-comodules under the coadjoint coaction. (This morphism has been studied in [Mj93b] under the notion of transmutation.)

Furthermore we have $\mathrm{coend}_{\mathcal{C}}(\omega) = C$ as coalgebras in $\mathcal{A}$ again by Theorem 4.1.

By Theorem 5.4 we thus get a canonical coalgebra morphism $f : H\#^c C \longrightarrow H\widetilde{\#}^c C$ where the first cosmash product was described above and the second cosmash product comes from the transmutation multiplication on H and the braiding in $\mathcal{A} = \mathcal{D}^H$. As observed in 4.1.4 these two cosmash products are the same.

With these coalgebras the canonical morphism f is defined as in the proof of Theorem 5.4 by

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[d, "\mu"'] & \omega \otimes H\#^c C \arrow[d, "1\otimes f"] \\ \omega \otimes C \arrow[r, "\delta'\omega\otimes 1"] & \omega \otimes H \otimes C \end{tikzcd}$$

with $\delta = (\delta'\omega \otimes 1)\mu$ as above, so that the uniquely determined morphism f under the given identifications is the identity. $\square$

Example 5.7. Let H be a coquasitriangular Hopf algebra over the field $\mathbb{K}$ and let C be an H-comodule coalgebra. Let $\omega : (\mathcal{M}^H)^C \longrightarrow \mathcal{M}^H$ be the forgetful functor. Then

$$\mathrm{coend}(\omega) \cong H\#^c C.$$

Proof. Use Proposition 6.4 to show $\mathrm{coend}(\omega) = \mathrm{coend}_{\mathcal{M}}(\omega)$. This special case can also be derived from [Mj94b] without the use of control categories. $\square$

The last example shows that the hidden symmetries as given in 5.1.8 are represented by the Hopf algebra H (with the transmutation multiplication), i.e. $H = \mathrm{coend}_{\mathcal{M}}(\mathrm{id} : \mathcal{M}^H \longrightarrow \mathcal{M}^H)$.

6. Appendix on $\mathbb{K}$-additive categories and $\mathcal{C}$-categories

Let $\mathbb{K}$ be a commutative ring and let $\mathcal{C} := \mathbb{K}$-mod be the category of finitely generated projective $\mathbb{K}$-modules. Then $\mathcal{C}$ is a symmetric monoidal $\mathbb{K}$-abelian category.

Let $\mathcal{A}$ be a category with splitting idempotents, i.e. for a morphism $f : X \longrightarrow X$ with $f^2 = f$ there are morphisms $q : X \longrightarrow P$ and $j : P \longrightarrow X$ with $jq = f$ and $qj = 1_P$. Then $q : X \longrightarrow P$ is a coequalizer of $(1_X, f)$ and q and j are unique up to isomorphisms of P.

Lemma 6.1. *Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be idempotents with $fg = gf$ and splittings (P, q, j) of f and (P', q', j') of g. Let (R, q'', j'') be a splitting of the idempotent jgq. Then $(R, q''q, jj'')$ is a splitting of $fg : X \longrightarrow X$.*

Proof. qgj is idempotent since $qgjqgj = qgfgj = qggfj = qgjqj = qgj$, so a splitting (R, q'', j'') exists with $qgj = j''q''$ and $q''j'' = 1$. Then $q''qjj'' = q''j'' = 1_R$ and $jj''q''q = jqgjq = fgf = ffg = fg$. $\square$

The following theorem is a generalization of [Sch92a] Lemma 2.2.2.

Theorem 6.2. *Let $\mathcal{A}$ be a $\mathbb{K}$-additive category with splitting idempotents. Then $\mathcal{A}$ is a $\mathcal{C}$-category.*

Proof. Let $\mathcal{A}$ be a $\mathbb{K}$-additive category. Then the following hold for $\alpha \in \mathbb{K}$ and f, g morphisms in $\mathcal{A}$: $\alpha(f+g) = \alpha f + \alpha g$, $\alpha(fg) = (\alpha f)g = f(\alpha g)$ and $\alpha(f \oplus g) = (\alpha f) \oplus (\alpha g)$.

We define the functor $(\mathbb{K} \otimes - : \mathcal{A} \longrightarrow \mathcal{A}) := (\mathrm{Id}_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A})$. This is a $\mathbb{K}$-additive functor. For $n \geq 0$ we define $(\mathbb{K}^n \otimes - : \mathcal{A} \longrightarrow \mathcal{A}) := (\mathrm{Id}^n : \mathcal{A} \longrightarrow \mathcal{A})$ (so for an object $P \in \mathcal{A}$ we have $\mathbb{K}^n \otimes P = P^n$) which again is a $\mathbb{K}$-additive functor.

Let $f : \mathbb{K}^m \longrightarrow \mathbb{K}^n$ be a morphism in $\mathcal{C}$. Then $f = \sum_{ij} \alpha_{ij} e_{ij}$ where the e_{ij} are given by the compositions $e_{ij} := \mathbb{K}^m \xrightarrow{p_i} \mathbb{K} \xrightarrow{\iota_j} \mathbb{K}^n$, the canonical basis of the matrix space $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^m, \mathbb{K}^n)$.

Since $\mathcal{A}$ is additive, we have corresponding natural transformations

$$\bar{e}_{ij}(P) : P^m \xrightarrow{\bar{p}_i} P \xrightarrow{\bar{\iota}_j} P^n.$$

For $f : \mathbb{K}^m \longrightarrow \mathbb{K}^n$ we define $f \otimes P : P^m \longrightarrow P^n$ by $f \otimes P := \sum_{ij} \alpha_{ij} \bar{e}_{ij}$. Then it is easy to verify, that $f \otimes \text{-} : \mathbb{K}^m \otimes - \longrightarrow \mathbb{K}^n \otimes -$ is a natural transformation. Furthermore it is easy to see that $fg \otimes \text{-} = (f \otimes \text{-})(g \otimes \text{-})$ and $\mathrm{id} \otimes \text{-} = \mathrm{id}$. If $\mathcal{C}^-$ is the full subcategory of $\mathcal{C}$ with the objects $\mathbb{K}^n$, then $\otimes : \mathcal{C}^- \times \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathbb{K}$-bilinear bifunctor.

Now let X be a finitely generated projective $\mathbb{K}$-module. Then there are homomorphisms $j : X \longrightarrow \mathbb{K}^n$ and $q : \mathbb{K}^n \longrightarrow X$ for some $n \in \mathbb{N}$ with $qj = 1_X$. For $P \in \mathcal{A}$ the morphism $jq \otimes P : \mathbb{K}^n \otimes P \longrightarrow \mathbb{K}^n \otimes P$ is an idempotent $(jq \otimes P)^2 = jq \otimes P$ so there is a splitting $q \otimes P : \mathbb{K}^n \otimes P \longrightarrow X \otimes P$ and $j \otimes P : X \otimes P \longrightarrow \mathbb{K}^n \otimes P$ (thus defining $q \otimes P$, $j \otimes P$ and $X \otimes P$) with

$$(j \otimes P)(q \otimes P) = jq \otimes P \text{ and } (q \otimes P)(j \otimes P) = 1_{X \otimes P}.$$

In particular $q \otimes P : \mathbb{K}^n \otimes P \longrightarrow X \otimes P$ is a cokernel of $(1 - jq) \otimes P : \mathbb{K}^n \otimes P \longrightarrow \mathbb{K}^n \otimes P$ and we have a commutative diagram

$$\begin{tikzcd}
\mathbb{K}^n \otimes P \arrow[r, "(1-jq)\otimes P"] \arrow[d, "jq\otimes P"] & \mathbb{K}^n \otimes P \arrow[r, "q\otimes P"] \arrow[d, "jq\otimes P"'] & X \otimes P \arrow[d, "="] \arrow[dl, "j\otimes P"] \\
\mathbb{K}^n \otimes P \arrow[r, "(1-jq)\otimes P"] & \mathbb{K}^n \otimes P \arrow[r, "q\otimes P"] & X \otimes P
\end{tikzcd}$$

Let $j : X \longrightarrow \mathbb{K}^n$, $q : \mathbb{K}^n \longrightarrow X$ and $j' : X \longrightarrow \mathbb{K}^m$ and $q' : \mathbb{K}^m \longrightarrow X$ be two choices of a representation of X as a direct summand of a free $\mathbb{K}$-modules. Then the following diagram commutes

$$\begin{tikzcd}
 & \mathbb{K}^n \otimes P \arrow[rr, "(1-jq)\otimes P"] \arrow[dl, "j'q\otimes P"'] \arrow[dd, "jq\otimes P" near start] & & \mathbb{K}^n \otimes P \arrow[rr, "q\otimes P"] \arrow[dl, "j'q\otimes P"'] \arrow[dd, "jq\otimes P" near start] & & X \otimes P \arrow[dl, "\cong"] \arrow[dd, "1_{X\otimes P}"] \\
\mathbb{K}^m \otimes P \arrow[rr, "(1-j'q')\otimes P"] \arrow[dr, "jq'\otimes P"'] & & \mathbb{K}^m \otimes P \arrow[r, "q'\otimes P"] \arrow[dr, "jq'\otimes P"'] & {[X \otimes P]} \arrow[drr, "\cong"] & & \\
 & \mathbb{K}^n \otimes P \arrow[rr, "(1-jq)\otimes P"] & & \mathbb{K}^n \otimes P \arrow[rr, "q\otimes P"] & & X \otimes P
\end{tikzcd}$$

and the isomorphisms between $X \otimes P$ and $[X \otimes P]$ (the corresponding object for the second representation of X) arise from this and the symmetric diagram with $(j, q, \mathbb{K}^n)$ and $(j', q', \mathbb{K}^m)$ interchanged.

Since the morphism $(1 - jq) \otimes P$ is a natural transformation it commutes with morphisms $f : P \longrightarrow Q$ and thus induces uniquely determined morphisms on the cokernels $X \otimes f : X \otimes P \longrightarrow X \otimes Q$, so that the following diagram commutes:

$$\begin{tikzcd} \mathbb{K}^n \otimes P \arrow[r, "(1-jq)\otimes P"] \arrow[d, "\mathbb{K}^n\otimes f"] & \mathbb{K}^n \otimes P \arrow[r] \arrow[d, "\mathbb{K}^n\otimes f"] & X \otimes P \arrow[d, "X\otimes f"] \\ \mathbb{K}^n \otimes Q \arrow[r, "(1-jq)\otimes Q"] & \mathbb{K}^n \otimes Q \arrow[r] & X \otimes Q. \end{tikzcd}$$

In particular one sees that $X \otimes - : \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathbb{K}$-additive functor.

Now let $f : X \longrightarrow X'$ be a homomorphism. Choose $j : X \longrightarrow \mathbb{K}^n$, $q : \mathbb{K}^n \longrightarrow X$, $j' : X' \longrightarrow \mathbb{K}^m$, and $q' : \mathbb{K}^m \longrightarrow X'$ with $qj = 1_X$ and $q'j' = 1_{X'}$. Then by the cokernel property there is a unique morphism $f \otimes P$ which makes the following diagram commutative:

$$\begin{tikzcd} \mathbb{K}^n \otimes P \arrow[r, "(1-jq)\otimes P"] \arrow[d, "j'fq\otimes P"] & \mathbb{K}^n \otimes P \arrow[r] \arrow[d, "j'fq\otimes P"] & X \otimes P \arrow[d, "f\otimes P"] \\ \mathbb{K}^m \otimes P \arrow[r, "(1-j'q')\otimes P"] & \mathbb{K}^m \otimes P \arrow[r] & X' \otimes P. \end{tikzcd}$$

By the universal property of the cokernel we get, that $f \otimes P : X \otimes P \longrightarrow X' \otimes P$ is a natural transformation. Furthermore we get $fg \otimes \text{-} = (f \otimes \text{-})(g \otimes \text{-})$ and $\mathrm{id} \otimes \text{-} = \mathrm{id}$. In particular we have that $\otimes : \mathcal{C} \times \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathbb{K}$-bilinear bifunctor.

We sketch the construction of the associativity morphism $\beta : (X \otimes Y) \otimes P \longrightarrow X \otimes (Y \otimes P)$. We first observe that $\mathbb{K}^n \otimes (\mathbb{K}^m \otimes P) \cong P^{nm} \cong (\mathbb{K}^n \otimes \mathbb{K}^m) \otimes P$, which defines β in the free case. Now consider representations $q : \mathbb{K}^n \longrightarrow X$, $j : X \longrightarrow \mathbb{K}^n$ and $q' : \mathbb{K}^m \longrightarrow Y$, $j' : Y \longrightarrow \mathbb{K}^m$. Then

$$f := jq \otimes 1 \otimes 1 : \mathbb{K}^n \otimes \mathbb{K}^m \otimes P \longrightarrow X \otimes (\mathbb{K}^n \otimes P) \longrightarrow \mathbb{K}^n \otimes \mathbb{K}^m \otimes P$$

$$g := 1 \otimes j'q' \otimes 1 : \mathbb{K}^n \otimes \mathbb{K}^m \otimes P \longrightarrow \mathbb{K}^n \otimes (Y \otimes P) \longrightarrow \mathbb{K}^n \otimes \mathbb{K}^m \otimes P$$

are idempotents with $fg = (jq \otimes 1 \otimes 1)(1 \otimes j'q' \otimes 1) = jq \otimes j'q' \otimes 1 = gf$. So we can apply Lemma 6.1. Since $(1 \otimes q' \otimes 1)(q \otimes 1 \otimes 1) : \mathbb{K}^n \otimes \mathbb{K}^m \otimes P \longrightarrow X \otimes (\mathbb{K}^m \otimes P) \longrightarrow X \otimes (Y \otimes P)$ and $(j \otimes 1 \otimes 1)(1 \otimes j' \otimes 1) : X \otimes (Y \otimes P) \longrightarrow X \otimes (\mathbb{K}^m \otimes P) \longrightarrow \mathbb{K}^n \otimes \mathbb{K}^m \otimes P$ is a splitting of $(j \otimes 1 \otimes 1)g(q \otimes 1 \otimes 1)$, we get a splitting $\mathbb{K}^n \otimes \mathbb{K}^m \otimes P \longrightarrow X \otimes (Y \otimes P) \longrightarrow \mathbb{K}^n \otimes \mathbb{K}^m \otimes P$ which defines up to unique isomorphism the object $(X \otimes Y) \otimes P$, so $\beta : (X \otimes Y) \otimes P \cong X \otimes (Y \otimes P)$. By the uniqueness this isomorphism is a natural transformation and coherent.

Hence $\mathcal{A}$ is a $\mathcal{C}$-category. $\square$

Proposition 6.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{K}$-additive categories with splitting idempotents with the $\mathcal{C}$-structure derived in Theorem 6.2 and let $\omega : \mathcal{B} \longrightarrow \mathcal{A}$ be a $\mathbb{K}$-additive functor. Then ω is a $\mathcal{C}$-functor.*

If ω and ω' are $\mathbb{K}$-additive functors from $\mathcal{B}$ to $\mathcal{A}$ and $\varphi : \omega \longrightarrow \omega'$ is a natural transformation then φ is a $\mathcal{C}$-morphism.

Proof. An object in $\mathcal{C}$ is given as above by the splitting $j : X \longrightarrow \mathbb{K}^n$ and $q : \mathbb{K}^n \longrightarrow X$. Then the tensor product with X is given as the splitting $j \otimes 1 : X \otimes P \longrightarrow \mathbb{K}^n \otimes P$ with $q \otimes 1 : \mathbb{K}^n \otimes P \longrightarrow X \otimes P$. We define $\xi : \omega(X \otimes P) \cong X \otimes \omega(P)$ as the uniquely defined morphism by the isomorphic splittings

$$\begin{tikzcd} \omega(X \otimes P) \arrow[r, "\xi"] \arrow[d, shift right] & X \otimes \omega(P) \arrow[d, shift right] \\ \omega(\mathbb{K}^n \otimes P) \arrow[r, "\cong"] \arrow[u, shift right] & \mathbb{K}^n \otimes \omega(P) \arrow[u, shift right] \end{tikzcd}$$

We leave it to the reader to check naturality and coherence of ξ.

The following commutative diagram shows that φ is a $\mathcal{C}$-morphism

$$\begin{tikzcd}[column sep=small]
& \omega(X\otimes P) \arrow[rr, "\xi"] \arrow[dr, "\varphi(X\otimes P)"] \arrow[dd, shift right] & & X\otimes\omega(P) \arrow[dr, "X\otimes\varphi(P)"] \arrow[dd, shift right] & & & \\
& & \omega'(X\otimes P) \arrow[rr] \arrow[dd, shift right] & & X\otimes\omega'(P) \arrow[dd, shift right] & & \\
\omega(P^n) = \arrow[dr, "\varphi(P^n)"'] & \omega(\mathbb{K}^n\otimes P) \arrow[uu, shift right] \arrow[rr, "\cong"] \arrow[dr] & & \mathbb{K}^n\otimes\omega(P) \arrow[uu, shift right] \arrow[dr] & & = \omega(P)^n \arrow[dr, "\varphi(P)^n"] & \\
& \omega'(P^n) = & \omega'(\mathbb{K}^n\otimes P) \arrow[uu, shift right] \arrow[rr] & & \mathbb{K}^n\otimes\omega'(P) \arrow[uu, shift right] & & = \omega'(P)^n.
\end{tikzcd}$$

□

Theorem 6.4. *Let $\mathcal{C}$ be a full monoidal subcategory of $\mathcal{M} = \mathbb{K}$-Mod and $\mathcal{A}$ be a $\mathcal{C}$-category. Let $\omega, \omega' : \mathcal{A} \longrightarrow \mathcal{C}$ be $\mathcal{C}$-functors. Then every natural transformation $\varphi : \omega \longrightarrow \omega'$ is a $\mathcal{C}$-morphism.*

Proof. Since

$$\begin{tikzcd}
\omega(\mathbb{K}\otimes P) \arrow[r, "\varphi(\mathbb{K}\otimes P)"] \arrow[d, "\omega(\pi_P)"] & \omega'(\mathbb{K}\otimes P) \arrow[d, "\omega'(\pi_P)"] \\
\omega(P) \arrow[r, "\varphi(P)"] \arrow[d, "\lambda_P^{-1}"] & \omega'(P) \arrow[d, "\lambda_P^{-1}"] \\
\mathbb{K}\otimes\omega(P) \arrow[r, "1\otimes\varphi(P)"] & \mathbb{K}\otimes\omega'(P)
\end{tikzcd}$$

commutes and the vertical morphisms are by coherence the canonical morphisms $\xi : \omega(\mathbb{K}\otimes P) \longrightarrow \mathbb{K}\otimes\omega(P)$ resp. $\xi' : \omega'(\mathbb{K}\otimes P) \longrightarrow \mathbb{K}\otimes\omega'(P)$. For $x \in X \in \mathcal{C}$ let $f_x : \mathbb{K} \longrightarrow X$ be the homomorphism with $f_x(1) = x$. Then the following diagram commutes, possibly with the exception of the front face

$$\begin{tikzcd}
\omega(\mathbb{K}\otimes P) \arrow[rr, "\varphi(\mathbb{K}\otimes P)"] \arrow[dr, "\omega(f_x\otimes 1)"] \arrow[dd, "\xi"'] & & \omega'(\mathbb{K}\otimes P) \arrow[dr, "\omega'(f_x\otimes 1)"] \arrow[dd, "\xi'"'] & \\
& \omega(X\otimes P) \arrow[rr, "\varphi(X\otimes P)"] \arrow[dd, "\xi"] & & \omega'(X\otimes P) \arrow[dd, "\xi'"] \\
\mathbb{K}\otimes\omega(P) \arrow[rr, "1_{\mathbb{K}}\otimes\varphi(P)"'] \arrow[dr, "f_x\otimes 1"'] & & \mathbb{K}\otimes\omega'(P) \arrow[dr, "f_x\otimes 1"'] & \\
& X\otimes\omega(P) \arrow[rr, "1_X\otimes\varphi(P)"'] & & X\otimes\omega'(P).
\end{tikzcd}$$

So for $q \in \omega(P)$ we get $\varphi(X\otimes P)\xi^{-1}(x\otimes q) = \varphi(X\otimes P)\xi^{-1}(f_x\otimes 1)(1\otimes q) = {\xi'}^{-1}(1_X\otimes\varphi(P))(f_x\otimes 1)(1\otimes q) = {\xi'}^{-1}(1_X\otimes\varphi(P))(x\otimes q)$. This holds for all $x \in X$ and all $q \in \omega(P)$ so that $\varphi(X\otimes P)\xi^{-1} = ({\xi'}^{-1}\otimes 1)(1_X\otimes\varphi(P))$ and φ is a $\mathcal{C}$-morphism. □

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