Algebra 2

Exercises Tutorium 8

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Exercise 1. Let A be a ring. Consider $Spec\, A$ with the Zariski topology. Recall that every closed subset in $Spec\, A$ is of the form $V(I) := \{\mathfrak{p} \in Spec\, A \mid I \subset \mathfrak{p}\}$ for some ideal I in A.
(1) Show that

$$V(I) \cap V(J) = V(I + J) \text{ and } V(I) \cup V(J) = V(IJ)\,.$$

(2) Show that if there exists a non-trivial idempotent $e \in A$, that is $e^2 = e$, $e \neq 0, 1$, then $Spec\, A$ is not connected.

Exercise 2. Let $C[0, 1]$ denote the ring of continuous real-valued functions on the closed interval $[0, 1]$. Show that $C[0, 1]$ is not Noetherian.

Exercise 3. For any ring R, denote by $R[[x_1, \ldots , x_n]]$ the ring of formal power series, i.e. formal expressions of the form

$$\sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$$

where the sum need *not* be finite.
(1) Show that $R[[x_1, \ldots , x_{n-1}]][[x_n]] = R[[x_1, \ldots , x_n]]$.
(2) Show that $R[[x_1, \ldots , x_n]]$ is Noetherian if R is.
Hint: replace “leading term” with “lowest term” in the usual proof of the Hilbert basis theorem.