Algebra 2

Exercises Tutorium 7

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Summer Semester 2020
08.06 - 12.06.2020

Exercise 1. Let R be a ring and S a multiplicative subset in R. Denote by φ the ring homomorphism $R \to S^{-1}R$, $r \mapsto \frac{r}{1}$. We want to find a condition (*) on S, such that the following equivalence holds for all R and S

$$\varphi : R \to S^{-1}R \text{ is a bijection} \quad \Longleftrightarrow \quad S \text{ satisfies condition (*)}.$$

(1) What condition (*) should be? Write your guess in the chat (without verification)!
(2) Show the above equivalence for the chosen condition.
Hint: Use without proof that

$$\text{Ker } \varphi = \{x \in R \mid \exists s \in S, \text{ such that } sx = 0\}.$$

Exercise 2. (1) Let n be a positive integer. Let $\mathbb{Z}_{(n)} := S^{-1}\mathbb{Z}$, where $S = \{s \in \mathbb{Z} \mid gcd(s, n) = 1\}$. Determine the prime spectra of $\mathbb{Z}[1/n], \mathbb{Z}_{(n)}$ as well as their maps to $Spec(\mathbb{Z})$.

(2) Determine the prime spectrum of $\mathbb{Z}[i]$ and its map to $Spec(\mathbb{Z})$.
Hint: Use Aufgabe 4, Übungsblatt 10 from Algebra 1.

Exercise 3. Let A be a ring. Consider $Spec\,A$ with the Zariski topology. Recall that every closed subset in $Spec\,A$ is of the form $V(I) := \{\mathfrak{p} \in Spec\,A \mid I \subset \mathfrak{p}\}$ for some ideal I in A.
(1) Show that

$$V(I) \cap V(J) = V(I + J) \text{ and } V(I) \cup V(J) = V(IJ).$$

(2) Show that if there exists a non-trivial idempotent $e \in A$, that is $e^2 = e$, $e \neq 0, 1$, then $Spec\,A$ is not connected.