Algebra 2

Exercises Tutorium 5

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The goal of the next exercise is to find an irreducible polynomial over $\mathbb{Z}$, which is reducible modulo p for every prime p.

Exercise 1. Let $L = \mathbb{Q}[\sqrt{2}, i]$. Recall, that L is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\alpha = \sqrt{2} + i \in L$.
(1) Show that $L = \mathbb{Q}[\alpha]$.
(2) Find the minimal polynomial F of α over $\mathbb{Q}$.
(3) Show that F is irreducible over $\mathbb{Q}$, but the reduction of F modulo p is reducible in $\mathbb{F}_p[X]$ for every prime p.
Remark: See also Aufgabe 2, Tutoriumsblatt 11 from the last semester.

Exercise 2. Let K be a field and P be an irreducible separable polynomial in $K[X]$. Let L be a splitting field of P over K. Show that if $\mathrm{Gal}(L/K)$ is abelian then $L = K[\alpha]$ for every root $\alpha \in L$ of P.

Exercise 3. Show that $\mathrm{PSL}_2(\mathbb{F}_3) \simeq A_4$.
Hint: Use the action of $\mathrm{PSL}_2(\mathbb{F}_3)$ on $\mathbb{P}^1(\mathbb{F}_3)$.