Algebra 2

Exercises Tutorium 4

Dr. Maksim Zhykhovich
Dr. Tom Bachmann

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Exercise 1. Let G be a subgroup of S_n, $n \geq 3$.
(1) Assume that G acts transitively on the set $\{1, 2, ..., n\}$. Show that, if G contains an $(n-1)$-cycle and a transposition, then $G = S_n$.
(2) Assume that G contains an n-cycle, an $(n-1)$-cycle and a transposition. Show that $G = S_n$.

Exercise 2. Let G be the Galois group of $F(X) = X^4 + 3X^2 - 3X - 2$ over $\mathbb{Q}$.
(1) Using the reduction modulo 2 show that $G \subset S_4$ containes a 3-cycle and that G acts transitively on the set of roots $R_f(\mathbb{C})$.
(2) Using the reduction modulo 5, conclude that $G = S_4$. Alternatively, use without verification that $disc(F) = -20183$ and deduce from (1) that $G = S_4$. .

The goal of the next exercise is to find an irreducible polynomial over $\mathbb{Z}$, which is reducible modulo p for every prime p.

Exercise 3. Let $L = \mathbb{Q}[\sqrt{2}, i]$. Recall, that L is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\alpha = \sqrt{2} + i \in L$.
(1) Show that $L = \mathbb{Q}[\alpha]$.
(2) Find the minimal polynomial F of α over $\mathbb{Q}$.
(3) Show that F is irreducible over $\mathbb{Q}$, but the reduction of F modulo p is reducible in $\mathbb{F}_p[X]$ for every prime p.
Remark: See also Aufgabe 2, Tutoriumsblatt 11 from the last semester.