Algebra 2

Exercises Tutorium 3

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Exercise 1. Let K be a field with $char(K) = 2$.
(1) Show that every separable quadratic extension of K is isomorphic over K to

$$K_a := K[X]/(X^2 + X + a)\,,$$

where $a \in K$ and $X^2 + X + a$ is irreducible over K.
(2) Show that $\varphi : (K, +) \to (K, +)$, $x \mapsto x^2 + x$, is a group homomorphism. Find Ker φ.
(3) Show that K_a and K_b are isomorphic over K if and only if $a = b$ in the group $(K, +)/\mathrm{Im}\,\varphi$.
(4) Assume K is finite. How many quadratic extensions of K are there up to isomorphism?

Exercise 2. (1) Let p be an odd prime number and let $n = p^2$. Denote by ζ_n the primitive n-th root of unity in $\mathbb{C}$. Show that $\mathrm{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$.
Hint: Show that $(X^{p^2} - 1)/(X^p - 1)$ is the minimal polynomial of ζ_n over $\mathbb{Q}$.
(2) Deduce that there exists a cyclic extension of degree p of $\mathbb{Q}$ for every prime p.

Exercise 3. Determine the Galois group of $f(x) = x^4 + 3x^2 - 3x - 2$ over $\mathbb{Q}$. Does there exist a solution of f by radicals? *Hint:* You may use without verification that $disc(f) = -20183$.