Algebra 2

Exercises Tutorium 2

Dr. Maksim Zhykhovich — Summer Semester 2020
Dr. Tom Bachmann — 04.05 - 08.05.2020

Exercise 1. Let $L \subset \mathbb{C}$ be the splitting field of the polynomial $X^3 - 2$ over $\mathbb{Q}$.
(1) Show that $L = \mathbb{Q}[j, \sqrt[3]{2}]$, where $j = \frac{-1+\sqrt{3}i}{2} \in \mathbb{C}$.
(2) Show that $\mathrm{Gal}(L/\mathbb{Q}) \simeq S_3$.
(3) Using Galois correspondence, find all intermediate fields of the extension $\mathbb{Q} \subset L$.

Exercise 2. Let K be a field with $\operatorname{char} K \neq 2$ and $f \in K[X]$ a separable polynomial of degree $n \geq 1$. Let L be a spliting field of f over K.
The *discriminant* of f is defined as follows:

$$disc\, f = \prod_{i<j} (\alpha_j - \alpha_i)^2 \,,$$

where $\alpha_1, ..., \alpha_n$ are the roots of f in L.
(1) Show that $disc\, f \in K$.
(2) Recall that the Galois group $\mathrm{Gal}(L/K)$ acts on the set of n roots of f in L and therefore can be identified with the subgroup of S_n. Show that $\mathrm{Gal}(L/K)$ is a subgroup of A_n if and only if $disc\, f$ is a square in K.
Hint: Consider the action of $\mathrm{Gal}(L/K)$ on $\prod_{i<j}(\alpha_j - \alpha_i)$.

Exercise 3. Let k be a field of characteristic 2. Classify quadratic extensions of k.