Algebra 2

Exercises Tutorium 11

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Exercise 1. Let A be a ring and M be an A-module. We say that a filtration by A-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \,,$$

is a *composition series* if for every $i = 1, ..., n$ the quotient M_i/M_{i-1} is isomorphic to $A/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$ of A. We call n the length of the composition series.

Show that any two composition series of M have the same length.

Remark: Assume that M is of finite length. It follows from the exercise that $l(M)$ from the lecture is well-defined.

Exercise 2. (1) Let A be a ring and M a finitely generated A-module. Let I be an ideal in A. Show that if $M = IM$ then there exists an element $i \in I$, such that $m = im$ for all $m \in M$.

Hint: Let $m_1, ..., m_n$ be generators of M. Consider $X = (m_1, ..., m_n)^t$ and show that $X = BX$ for some $n \times n$ matrix B with coefficients in I. Deduce that $\det(Id_n - B)\, X = 0$.

(2) Using (1) give another proof of Nakayama's Lemma from the lecture.

Exercise 3. Let A be a ring and M be a finitely generated A-module. Show that every surjective morphism $\varphi : M \to M$ of A-modules is an isomorphism.

Hint: Consider M as an $A[X]$-module via $P(X).m := P(\varphi)(m)$, $m \in M$, $P \in A[X]$, and apply Exercise 2(1).

Remark: Compare to Exercise 1, Tutorium 9.