Algebra 2

Exercises Tutorium 1

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Exercise 1. Let $\alpha = \sqrt{1+\sqrt{3}} \in \mathbb{R}$ and $\beta = \sqrt{3+2\sqrt{2}} \in \mathbb{R}$. Show that α and β are algebraic over $\mathbb{Q}$ and find their minimal polynomials over $\mathbb{Q}$.

Exercise 2. Let $P = X^3 - 3X - 1 \in \mathbb{Q}[X]$. Let $\alpha \in \mathbb{C}$ be a root of P.
(1) Show that P is irreducible over $\mathbb{Q}$.
(2) Find the minimal polynomial of α^{-1} over $\mathbb{Q}$.
(3) Show that $-\alpha^{-1} - 1$ is also a root of P.
(4) Conclude that $\mathbb{Q} \subset \mathbb{Q}[\alpha]$ is a Galois field extension.

Exercise 3. (1) (**Liouville's theorem**) Let $\alpha \in \mathbb{R}$ be an irrational algebraic number satisfying $f(\alpha) = 0$ with non-zero irreducible $f \in \mathbb{Z}[X]$ of degree d. Then there is a non-zero constant C such that for every fraction $p/q \in \mathbb{Q}$

$$|\alpha - p/q| \geq \frac{C}{q^d}\,.$$

Hint: You may wish to first show that $|f(p/q)| \geq 1/q^d$, and use the mean value theorem.

(2) Show that $\beta = \sum_{n=1}^{\infty} \frac{1}{2^{n!}}$ is not algebraic over $\mathbb{Q}$.