Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 9

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Exercise 1. Let F be a field, $A = \left(\frac{-1,1}{F}\right)$ and let $\varphi : A \to M_2(F)$ be the F-algebra isomorphism (from Example 1.5, Chapter III) with

$$\varphi(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varphi(j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that $N(v) = \det(\varphi(v))$ and $T(v) = \mathrm{Tr}(\varphi(v))$ for any $v \in A$, where $N(v) = v\bar{v}$ and $T(v) = v + \bar{v}$.

Exercise 2. Let A be a 4-dimensional central division algebra over a field F. The goal of this exercise is to show that A is isomorphic to a quaternion algebra.
(1) Show that there exists $v \in A$, such that $v^2 \in F$ but $v \notin F$.
Hint: Take any $u \in A \backslash F$ and consider an F-subalgebra $F[u] \subseteq A$.
(2) Show that there exists $w \in A$, such that $vw = -wv$.
Hint: Consider the map $\varphi : A \to A$, $x \mapsto vxv^{-1}$. What one can say about φ^2?
(3) Show that $w^2 \in F$ and conclude that A is isomorphic to a quaternion algebra.

Exercise 3. Let F be a field and $A = (\frac{a,b}{F}) \otimes_F (\frac{c,d}{F})$ a biquaternion algebra over F, where $a, b, c, d \in F^\times$. Let $q = \langle -a, -b, ab, c, d, -cd\rangle$ (this form is called the *Albert form* of A). Show the following implications:
(1) q is hyperbolic $\Longrightarrow$ A is split (that is $A \simeq M_4(F)$).
(2) q is isotropic $\Longrightarrow$ A is not division.
Hint: Use the following formula, which will be proven on Monday:

$$\left(\frac{x,y}{F}\right) \otimes_F \left(\frac{x,z}{F}\right) \simeq \left(\frac{x,yz}{F}\right) \otimes_F M_2(F)\,.$$