# Algebraic theory of quadratic forms and Kaplansky's problem

## Exercise Sheet 8

PD Dr. Maksim Zhykhovich Summer Semester 2025, 27.06.2025

**Exercise 1.** Find the higher Witt indices of the form $11\langle 1\rangle$ over $\mathbb{R}$.

**Exercise 2.** Let $F$ be a field and let $IF$ be the fundamental ideal in $W(F)$.
(1) (**Arason-Pfister Theorem**) Let $q$ be an anisotropic quadratic form over $F$, such that $q \in I^nF$, $n \in \mathbb{N}$. Show that $\dim q \geq 2^n$.
*Hint:* The ideal $I^nF$ is additively generated by $n$-fold Pfister forms. Hence, $q = a_1\pi_1 + ... + a_r\pi_r$ in $W(F)$ for some $n$-fold Pfister forms $\pi_1, ..., \pi_r$ , $r \in \mathbb{N}$ and $a_i = \pm 1$. Proceed by induction on $r$. For the induction step use the function fields of quadratic forms.
(2) Deduce from (1) that $\cap_{n\in\mathbb{N}} I^nF = 0$ in $W(F)$.
(3) Let $f$ be an odd dimensional quadratic form over $F$. Show that $f$ is not a zero divisor in $W(F)$.
*Hint:* Use the question (2).
*Remark:* If $s(F) < +\infty$, then it was proven in the lecture, that $f$ is a unit in $W(F)$.

**Exercise 3.** Let $F$ be a field and $n$ a positive integer. Show that the matrix $F$-algebra $M_n(F)$ is central and simple.

**Exercise 4.** Let $F$ be a field and let $A$ be a quaternion algebra over $F$ with the usual basis $1, i, j, k$. Set $A_0 = \{yi + zj + tk \mid y, z, t \in F\}$.
Let $v \in A$. Show that $v \in A_0$ if and only if $v^2 \in F$ but $v \notin F$.