Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 7

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Exercise 1. (1) Let (F, P) be an ordered field. Recall that P is a proper subset of F ($P \neq F$) satisfying: $P + P \subseteq P$, $P \cdot P \subseteq P$ and $P \cup (-P) = F$.
Show that $s(F) = +\infty$ (i.e. F is formally real).
Hint: First prove $F^2 \subseteq P$. Assuming $s(F) < +\infty$, show that $P = F$ and get a contradiction.
(2) Let F be a formally real field with $|F^\times/F^{\times 2}| = 2$. Show that $P = F^2$ defines an ordering in F.
(3) Let F be a formally real field, let $\bar{F}$ be an algebraic closure of F. Show that there exists a formally real field K, $F \subseteq K \subseteq \bar{F}$, such that no proper algebraic extension of K is formally real. Show that $K^\times/K^{\times 2} = \{\pm 1\}$ and that K posseses an ordering.
Hint: Use Zorn's Lemma to show the existence of K.
(4) (**Artin-Schreier's Theorem**) Show that a field F is formally real if and only if F posseses at least one ordering.
Hint: Use questions (1) and (3).

Exercise 2. The level $s(A)$ of of a ring A is defined in the same way as for fields.

$$s(A) := \min\{n \in \mathbb{N} \mid -1 \text{ is a sum of } n \text{ squares in } A\}\,.$$

Let $A_n = \mathbb{R}[x_1, ..., x_n]/(x_1^2 + ... + x_n^2 + 1)$, where $n \in \mathbb{N}$.
Show that $s(A_n) = n$.
Hint: Assume that $s(A_n) < n$ and get a polynomial equation

$$-1 = f_1^2 + ... + f_{n-1}^2 + f_0(1 + x_1^2 + ... + x_n^2)\,, \tag{1}$$

where $f_j \in \mathbb{R}[x_1, ..., x_n]$. Write $f_j(ix) = p_j(x) + iq_j(x)$, where $p_j, q_j \in \mathbb{R}[x] = \mathbb{R}[x_1, ..., x_n]$ and $i = \sqrt{-1} \in \mathbb{C}$. Then replace x by ix in the equation (1) and compare the real parts. Finally apply the following topological result to the map $(q_1, ..., q_{n-1}) : \mathbb{R}^n \to \mathbb{R}^{n-1}$.
Borsuk-Ulam Theorem: Let $Q : S^{n-1} \to \mathbb{R}^{n-1}$ be a continious map, where S^{n-1} is a unit sphere in $\mathbb{R}^n$. Then there exist two opposite points on S^{n-1} with the same image in $\mathbb{R}^{n-1}$.

Exercise 3. Let φ be a quadratic form over a field F and let $K = F(x_1, ..., x_n)$. Let $e \in F^n$ and $\frac{f}{g} \in D_K(\varphi)$, where $f, g \in F[x_1, ..., x_n]$ are such that $f(e) \neq 0$ and

$g(e) \neq 0$. Show that $\frac{f(e)}{g(e)} \in D_F(\varphi)$.
Hint: Use Theorem 4.2 and induction on n.

Exercise 4. Let q be an anisotropic quadratic form over a field F. Show that the following conditions are equivalent:

(1) q is a Pfister form.
(2) For every field extension K/F the set $D_K(q)$ is a subgroup of $K^\times$

Hint: To prove that (2) implies (1) consider a Pfister subform φ of q of maximal dimension. Assume $\varphi \not\simeq q$, then $q \simeq \varphi \perp q'$ for some q' of dimension > 0. Let $a \in D_F(q')$. Using the "Subform Theorem" (Theorem 4.6) show that $\varphi \perp a\varphi$ is also a subform of q and get a contradiction.