Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 6

PD Dr. Maksim Zhykhovich Summer Semester 2025, 05.06.2025

Exercise 1. Show that if two Pfister forms over a field F are similar then they are isometric. Recall that two quadratic forms φ_1 and φ_2 are called *similar* if $\varphi_1 \simeq a\varphi_2$ for some $a \in F^\times$.
Hint: Use Theorem 2.4 (Chapter II) from the lecture.

Exercise 2. Let F be a field and let $a, b, c \in F^\times$.
Show that the quadratic form

$$\langle a, b, c, ab, ac, bc \rangle$$

is hyperbolic if and only if $-abc$ is a square in $F^\times$.
Hint: Observe that the above quadratic form is a subform of a Pfister form.

Exercise 3. Let

$$A_3 = \mathbb{R}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 + 1)$$

and let F_3 be the field of fractions of A_3. Since $-1 = x_1^2 + x_2^2 + x_3^2$ in F_3, we have $s(F_3) \leq 3$ and, hence, $s(F_3) \leq 2$ (by Theorem 3.2, Chapter II, the level of a field is always a power of 2).
(1) Write explicitly -1 as a sum of two squares in F_3.
Hint: Show that the form $4\langle 1 \rangle$ is hyperbolic over F_3 and find explicitly a maximal totally isotropic subspace U of this form, then intersect U with the underlying vector space of the subform $3\langle 1 \rangle$.
(2) Show that $s(F_3) = 2$.

Exercise 4. Let F be a field. Let q be a quadratic form of dimension > 1 over F and let φ be a n-fold Pfister form over F, $n \geq 1$.
(1) Assume that φ is anisotropic and $\varphi \otimes q$ is isotropic. Show that

- (a) There exists an isotropic quadratic form q' over F, such that $\varphi \otimes q' \simeq \varphi \otimes q$.
- (b) The anisotropic part of $\varphi \otimes q$ is of the form $\varphi \otimes \rho$ for some quadratic form ρ.
- (c) If φ is anisotropic, then the Witt index of $\varphi \otimes q$ is divisible by 2^n.

(2) Show that if q has odd dimension and $\varphi \otimes q$ is hyperbolic, then φ is hyperbolic.